The Small World Theorem

Ulrich Meierfrankenfeld, Bernd Stellmacher

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Assume $\mathcal{M}(S) \geq 2$ and $Q!$. We investigate the Structure of $E/O_p(E)$.
For $L \in \mathcal{L}$ define $L_\circ = L^\circ O_p(L)$. In this section we assume

Hypothesis 0.1 [**hypothesis e structure theorem**]

(ES1) $\mathcal{M}(S) \geq 2$ *and* $Q!$.

(ES2) $P \in \mathcal{P}(S)$ *and* $P \not\leq \widetilde{C}$.

(ES3) $P_\circ/O_p(P) \cong SL_2(q)$, *$q$ a power of* 2.

(ES4) Y_P *is a natural module for* $P/O_p(P)$.

(ES5) $N_P(S \cap P_\circ) \leq \tilde{C}$.

(ES6) $\langle Y_P^E \rangle$ *is abelian.*

(ES7) $0_p(\langle P, E \rangle) = 1$.

Some remarks on this assumptions. (ES7) follows from $E!$ but not from $Q!$ (example $(L_n(q)$. (ES2) to (ES5) follow from $P!$ theorem. But $\neg P!$ have currently been treated only for $Y_M \leq Q$.

Let $L = N_G(P^\circ)$ and $H = E(L \cap \tilde{C})$.

By (ES7) $O_p(\rangle H, L\langle) = 1$. By part (a) of the preceeding lemma we can apply the amalgam method to (H, L). Let $\Gamma_0 = \Gamma(G; L, H)$ and Γ the connected component of Γ containing L and H. For $\alpha \in \Gamma$. If $\alpha = Lg$ define $E_\alpha = P_\circ^g$, if $\alpha = Hg$ define $E_\alpha = (EQ)^g$, $\widetilde{C}_\alpha = \widetilde{C}^g$ and $Q_\alpha^* = Q^g$.

Lemma 0.2 [**basic es**] *Let (α, β) be adjacent vertices with $\alpha \sim H$*

(a) $G_\alpha = E_\alpha G_{\alpha\beta}$ *and* $G_\beta = E_\beta G_{\alpha\beta}$.

(c) $C_U(Y_\beta) = Q_\alpha Q_\beta = Q_a Q_\beta^* = Q_{\alpha\beta} \in \mathrm{Syl}_p(L_\alpha)$, *where* $U \in Syl_p(G_{\alpha\beta})$.

(d) $G_{\alpha\beta} = N_{G_\alpha}(Q_{\alpha\beta})$

(e) $Y_{\alpha\beta} = Y_\beta = [Y_\alpha, x]$ *for all* $x \in Q_{\alpha\beta} \setminus Q_\alpha$.

(f) *Let* $\widetilde{V_\beta} = V_\beta/Y_\beta$. *Then* $C_{G_\beta}(\widetilde{V_\beta}) \cap C_{G_\beta}(Y_\beta) = Q_\beta$ *In particular* $G_{\alpha\beta}$ *contains a point stablizer for* G_β *on* $\widetilde{V_\beta}$.

Proof: By edge transitivity we may assume that $\alpha = L$ and $\beta = H$. By (ES5), and the Frattini Argument, $L = P_\circ(L \cap \widetilde{C}) = P_\circ(L \cap H)$. By definition of H, $H = (H \cap L)E$. Thus (a) holds.

Note that $Y_{H\cap L} \leq Y_H$ and so the definition of E implies $[Y_H, E] = 1$. Since $L = (L \cap H)E$ we get $Y_{H\cap L} = Y_H$. Let $T = S \cap P^\circ$. Since $N_P(T)$ is a maximal subgroup of L we get $N_P(T) = H \cap L$ and $O_p(H \cap L) = T$. Since $Y_{H\cap L} \leq Y_L$ and Y_P is a natutal module for $P_\circ$ we see that $Y_H = Y_{H\cap L} = C_{Y_L}(T) = \Omega_1 \mathrm{Z}(T)$ and $C_S(Y_H) = T$. Also since $Q \nleq O_p(P)$, $O_p(H)O_p(L)/O_p(L)$ is a non-trivial subgroup of $T/O_p(L)$ normalized by $H \cap L$. Thus $O_p(H)O_p(L) = T$.

For (f) let $D = C_H(\widetilde{V}) \cap C_H(Y_H)$. Since $[Y_L, O_p(H)] = Y_H$, $O_p(H) \leq D$. Note that $O^p(D)$ centralizes Y_P and so $[P^\circ, O^p(D) \leq O_p(P)$. Since $O^p(D) = O^p(O^p(D)O_p(P)]$ we get $P^\circ \leq N_G(O^p(D))$. Since also H normalizes $O^p(D)$ we conclude that $O^p(D) = 1$, D is a p-group and $D = O_p(H)$

$G_{\alpha\beta} = N_G(Q_{\alpha\beta})$

Suppose that $N_G(T) \nleq L$. Put $M = \langle N_G(T), P_\circ T\rangle$. If $O_p(M) = 1$ then pushing up $SL_2(q)$ and $\Omega_1 \mathrm{Z}(P^\circ) = 1$ gives $[O_p(P), O^p(P)] \leq Y_P$. By $ES6$, $V = \langle Z_P^H \rangle \leq O_p(L)$. Thus V is normal in H and L, a contradiction. The last equality in (e) follows since Y_P is a the natural module. □

The following is not needed in the FF-module argumnet:

$C_{G_\beta}(\widetilde{x}) \leq G_{\alpha\beta}$ for all $x \in Y_\alpha \setminus Y_\beta$.

For (g) let $D^* = C_H(\tilde{x})$ and $D = C_H(x)$, T acts transitively on the coset $Y_H x$, $D^* = DT$. Let $x \in Y_H^l$ for some $l \in L$. Then $D \leq C_G(x)$ and by $Q!$, $D \leq N_G(Q^l)$. Thus $D \leq N_G(\langle Q, Q^l \rangle = N_G(P^\circ) = L$. □ □

Let (α, α') be a critical pair. Let $\beta = \alpha + 1$ and $\alpha - 1 \in \Delta(a)$ with $\alpha - 1 \neq \beta$.

Lemma 0.3 **[b¿2]**

(a) $b > 2$.

(b) $\alpha \sim H$.

(b) *b is odd.*

$b > 2$ follows from $(ES6)$ and $\alpha \sim H$ follows from $Y_H \leq Y_L$. Suppose that b is even. The by 0.2 $Y_\beta = [Y_\alpha, Y_\alpha] = Y_{\alpha'-1}$. Hence by **??**(c), $E_\beta = E_{\alpha'-1}$. Since $b > 3$, $V_{\alpha-1} \leq Q_\beta$ and $V_{\alpha-1} \leq N_G(Q_\beta^*) = N_G(Q_{\alpha'-1}^*)$. Thus

$$V_{\alpha-1} \leq N_{G_{\alpha-2}}(Q_{\alpha'-2} Q_{\alpha'-1}^*) = N_{G_{\alpha'-2}}(Q_{\alpha-2\alpha-1}) = G_{\alpha-2\alpha-1}$$

As $V_{\alpha-1} \leq Q_\beta$, $[V_{\alpha-1}, E_\beta]$ is a p-group and so $V_{\alpha-1} \leq Q_{\alpha'-1} \leq Q_{\alpha'-1\alpha'}$ and hence $[V_{\alpha-1}, Y_{\alpha'}] \leq Y_{\alpha'-1} = Y_\beta \leq Y_\alpha$. Hence $Y_{\alpha'}$ normalizes $V_{\alpha-1}$, a contradiction. Thus b is odd. □

Lemma 0.4 [**offender on Vbeta**] *One of the following holds*

1. *There exists* $1 \neq A \leq Q_{\alpha\beta}/Q_\beta$ *such* A *is an offender on* $\widetilde{V_\beta}$.
2. $b = 3$ *and there exists an non-trivial* $G_{\alpha\beta}$ *invariant subgroup* A *of* $Q_{\alpha\beta}/Q_\beta$ *such that* A *is a quadratic* $2F$*-offender on* $\widetilde{V_\beta}$.

Proof: If $Y_\beta \cap Y_{\alpha'} \neq 1$, then **??**(c) implies $E_\beta = E_{\alpha'}$, a contradiction since $[Y_\beta, E_\beta]$ is a p-group, but $[V_\beta, E_{\alpha'}]$ is not. Hence

Step 1 [**zbza**] $Y_\beta \cap Y_{\alpha'} = 1$

Hence by 0.2 $[V_\beta \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta] \leq Y_\beta \cap Y_{\alpha'} = 1$ and we proved

Step 2 [**vqvq**] $[V_\beta \cap Q_{\alpha'}, V_{\alpha'} \cap Q_\beta] = 1$

Suppose now that $b = 3$. Since $Q_{\alpha'}$ acts transitively on $\triangle(\alpha+2) \setminus \{\alpha'\}$ get $G_{\alpha+2\alpha'} = G_{\beta\alpha+2\alpha'}Q_{\alpha'}$. Hence $V_\beta Q_{\alpha'}$ is normal in $G_{\alpha+2\alpha'}$. Also $[V_{\alpha'}, V_\beta, V_\beta] \leq [V_\beta, V_\beta = 1$. Since $G_{\alpha+2}$ is doubly transitive on $\triangle(\alpha+2)$,

$$V_\beta Q_{\alpha'}/Q_{\alpha'} = |V_{\alpha'}/V_{\alpha'} \cap Q_\beta)$$

Let $\delta \in \triangle(\beta)$. Then no subgroup of Q_β is an over-offender on Z_δ. This together with Step 2 implies

$$V_{\alpha'} \cap Q_\beta / C_{V_{\alpha'} \cap Q_{\alpha'-1}}(V_\beta) \leq |V_\beta / C_{V_\beta}(V_{\alpha'} \cap Q_\beta)| \leq |V_\beta / V_\beta \cap Q_{\alpha'}| = |V_\beta Q_{\alpha'} / Q_{\alpha'}|$$

By the lasy two displayed equations, V_β is a $2F$ offender on $\widetilde{V}_{\alpha'}$. So Case 2. of the lemma holds.

So we may assume from now on

Step 3 $b > 3$.

Suppose that $V_{\alpha'} \leq Q_\beta$.

Then by Step 2 $[V_{\alpha'}, Y_\alpha \cap Q_{\alpha'} = 1$. By 0.2f $[V_{\alpha'}, Z_\alpha] \neq 1$ and since Y_α is a natural module for E_α and since $V_{\alpha'} \leq E_\alpha$ we get

$$|V_{\alpha'} / C_{V_{\alpha'}}(Y_{\alpha'}) \leq q = |Y_\alpha / C_{Y_\alpha} V_{\alpha'}|$$

Thus 1. holds in this case.

So we may assume that for all critcial pairs:

Step 4 [**sym**] *$V_{\alpha'} \not\leq Q_\beta$ and the situation is symmetric in β and α'.*

If $[V_\beta, V_{\alpha'} \cap Q_\beta] = 1 = [V_{\alpha'}, V_\beta \cap Q_{\alpha'}]$, then again (1) holds. So we may assume

Step 5 [**vvqa**] *$Y_\beta = [Y_\beta, Y_{\alpha'} \cap Q_\beta] \leq V_{\alpha'}$ or $Y_{\alpha'} \leq [Y_{\alpha'}, Y_\beta \cap Q_{\alpha'}] \leq Y_\beta$*

By symmetry in α, α' we may assume

Step 6 [**vvq**] *$Y_\beta = [V_\beta, V_{\alpha'} \cap Q_\beta] \leq V_{\alpha'}$.*

Pick $\mu \in \triangle(\beta)$ and $t \in V_{\alpha'} \cap Q_\beta$ with $[Z_\mu, t] \neq 1$. Then $\mu \neq \alpha + 2$ and by Step 2 , $Z_\mu \not\leq Q_{\alpha'}$ and we may assume that $\mu = \alpha$. Hence

Step 7 [**vbq**] *There exists $t \in V_{\alpha'} \cap Q_\beta$ with $[Z_\alpha, t] \neq 1$. In particular, $t \notin Q_\alpha$*

Note that

Step 8 [**O2G**] *$O^p(E_\alpha) \leq \langle Q_{\alpha-1}, t \rangle$.*

By Step 2 and Step 7 we have $|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq |Y_\alpha Q_{\alpha'}/Q_{\alpha'}| = |Y_\alpha / C_{Y_\alpha}(t)| \geq q$. We record

Step 9 [**vbqa**] *$|V_\beta Q_{\alpha'}/Q_{\alpha'}| \geq q$.*

We next show:

Step 10 *If $[V_{\alpha-1}, V_{\alpha'-2}] = 1$ then 1. holds.*

Suppose $[V_{\alpha-1}, V_{\alpha'-2}] = 1$. Then $V_{\alpha-1} \leq Q_{\alpha'-2} \cap Q_{\alpha'-1}$. Put $A = V_{\alpha-1} \cap (V_\beta Q_{\alpha'})$. Then $A \leq V_\beta(V_\beta V_{\alpha-1} \cap Q_{\alpha'}) \leq V_\beta(Q_{\alpha'-1} \cap Q_{\alpha'})$. Thus by 0.2

$$[A, t] \leq [V_\beta, t][Q_{\alpha'-1} \cap Q_{\alpha'}, t] \leq Y_\beta Y_{\alpha'}.$$

Let X be maximal in A with $[X, t] \leq Y_\beta$. As $|Y_{\alpha'}| = q$ we have $|A/X| \leq q$. Since $Y_\beta^* \leq X$, t normalizes X. By 0.2, $[XZ_a, Q_{\alpha-1}] \leq [V_{\alpha-1}, Q_{\alpha-1}] = Y_{\alpha-1} \leq XZ_\alpha$. So by Step 8 , $O^p(E_\alpha)$ normalizes XZ_α. Since $O^p(E_\alpha)$ is transitively on $\triangle(\alpha)$ we conclude that $XZ_\alpha \leq D_\alpha := \bigcap_{\delta \in \triangle(\alpha)} V_\delta$. Put $a = |V_{\alpha-1}/A|$. Then $|V_{\alpha-1} D_a / D_a| \leq |V_{\alpha-1}/A||A/X| \leq aq$. Hence

$$|V_\beta D_a / D_\alpha| \leq aq.$$

Note that $V_{\alpha-1} \leq Q_{\alpha'-2} \cap Q_{\alpha'-1} \leq G_{\alpha'}$. Since $D_{\alpha'-1} \leq V_{\alpha'-2}$ we conclude from $|V_\beta D_a / Y D_\alpha| \leq qa$ and edge-transitivity that

$$|V_{\alpha'} / C_{V_{\alpha'}}(V_{\alpha-1} V_\beta)| \leq |V_{\alpha'} D_{\alpha'-1} / D_{\alpha'-1}| = |V_\beta D_a / D_\alpha| \leq aq.$$

On the otherhand by definition of a, an isomorphism theorem and Step 9

$$|V_{\alpha-1} V_\beta Q_{\alpha'} / Q_{\alpha'}| = |V_{\alpha-1} V_\beta Q_{\alpha'} / V_\beta Q_{\alpha'}||V_\beta Q_{\alpha'} / Q_{\alpha'}| \geq aq.$$

By the last two equations 1. holds. So we may assume from now on that

Step 11 [**va-1va-2**] $[V_{\alpha-1}, V_{\alpha'-2}] \neq 1$

Suppose that $V_{\alpha'-2} \leq Q_{a-1}$. Then by Step 4 , $V_{\alpha-1} \leq Q_{\alpha'-2}$. Note that by Step 8 , $C_{Y_{\alpha-1}}(t) = 1$. Thus

$$1 \neq [V_{\alpha-1}, V_{\alpha'-2}] \leq Y_{\alpha-1} \cap Y_{\alpha'-2} \leq C_{Y_{\alpha-1}}(t) = 1$$

a contradiction to Step 11 . Thus

Step 12 [**va-1qa-1**] $V_{\alpha'-2} \not\leq Q_{\alpha-1}$

By Step 4 we get

Step 13 [**va-1qa-2**] $V_{\alpha-1} \not\leq Q_{\alpha'-2}$

Since $b > 3$, t centralizes $[V_{\alpha'-2} \cap Q_{\alpha-1}, V_{\alpha-1}]$. and so

$$[V_{\alpha'} - 2 \cap Q_{\alpha-1}, V_{\alpha-1}] = C_{Y_{\alpha-1}}(t) = 1$$

Thus by Step 5 and **??** that $Y_{\alpha'-2} = [V_{\alpha-1} \cap Q_{\alpha'-2} V_{\alpha'-1}] \leq V_{\alpha-1}$. Hence there exists $1 \neq x \leq Y_{\alpha'-2} \cap V_{\alpha-1}$.

Note that t centralizes x and $[x, Q_{\alpha-1} \leq Y_{\alpha-1} \leq Y_\alpha$. So by Step 8 , $O^p(E_\alpha)$ normalizes the coset xY_α.

Suppose that $[x, Q_\alpha] \neq 1$. Let $R = O^p(E_\alpha)$ and $D = [Q_\alpha, R]$. Since $C_{Y_\alpha}(R) = 1$, the Three Subgroup Lemma implies $[x, D] \neq 1$. Since R normalizes $[x, D]$ we get $[x, D] = Y_\alpha$. Thus D acts transitively on $Y_\alpha x$ and so by the Frattini argument, $R = C_R(x)D$. Since $x \in Y_{\alpha'-2}$, $Q!$ implies $C_R(x) \leq \widetilde{C}_{\alpha'-2}$. Also since $[E_{\alpha'-2}, Q_{\alpha'-2}] \leq Q^*_{\alpha'-2}$ and $E_{\alpha'-2}$ acts transitively on $\triangle\alpha' - 2$ we have

$$t \in V^{(2)}_{\alpha'-1} \cap Q_{\alpha'-2} \leq (V^{(2)}_{\alpha'-1} \cap Q_{\alpha'-2})Q^*_{\alpha'-2} = (V^{(2)}_{\alpha'-3} \cap Q_{\alpha'-2})Q^*_{\alpha'-2} \leq (Q_\alpha \cap \widetilde{C}_{\alpha'-2})Q^*_{\alpha'-2}$$

The right hand side of this equation is p-group normalized by $C_R(x)$ and so $\langle t^{C_R(x)} \rangle Q_\alpha / Q_\alpha$ is a p-group. But this contradicts $t \in Q_{\alpha\beta} \setminus Q_\alpha$ and $O^p(E_\alpha) \leq C_R(x)Q_\alpha$.

Thus $[x, Q_\alpha] = 1$, and so $x \in \Omega_1 \mathrm{Z}(Q_\alpha) = Y_\alpha$. Since $[x, t] = 1$ we conclude $x \in Y_\beta$. Since also $x \in Y_{\alpha'-2}$ we conclude that $E_{\alpha'-2} \leq \widetilde{C}_\beta$

Since $b > 3$,

$$V_{\alpha-1} \leq V^{(2)}_\alpha Q^*_\beta = V^{(2)}_{\alpha+2} Q^*_\beta \leq (Q_{\alpha'-2} \cap \widetilde{C}_\beta)Q^*_\beta$$

The right hand side is a p-group normalized by $E_{\alpha'-2}$ and we obtain a contradiction to Step 13 . □

Theorem 0.5 (The abelian E-Structure Theorem) [**abelian es**] *Assume Hypothesis 0.1 (and maybe that there exists a unique $\widetilde{P} \in \mathcal{P}(ES)$ with $\widetilde{P} \not\leq N_G(P^\circ)$).) Let $V = \langle Y_P^E \rangle$ and $\tilde{V} = V/[V, O_p(E)]$. Then*

Proof:

$G_{\alpha\beta} = N_G(Q_{\alpha\beta})$

Suppose that $N_G(T) \not\leq L$. Put $M = \langle N_G(T), P_\circ T\rangle$. If $O_p(M) = 1$ then pushing up $SL_2(q)$ and $\Omega_1 Z(P^\circ) = 1$ gives $[O_p(P), O^p(P)] \leq Y_P$. By $ES6$, $V = \langle Z_P^H\rangle \leq O_p(L)$. Thus V is normal in H and L, a contradiction. The last equality in (e) follows since Y_P is a the natural module.

$C_{G_\beta}(\widetilde{x}) \leq G_{\alpha\beta}$ for all $x \in Y_\alpha \setminus Y_\beta$.

For (g) let $D^* = C_H(\tilde{x})$ and $D = C_H(x)$, T acts transitively on the coset $Y_H x$, $D^* = DT$. Let $x \in Y_H^l$ for some $l \in L$. Then $D \leq C_G(x)$ and by $Q!$, $D \leq N_G(Q^l)$. Thus $D \leq N_G(\langle Q, Q^l\rangle = N_G(P^\circ) = L$. □

Let G be a group of local characteristic p. We say that G has rank 2, provided that there exists $P, \widetilde{P} \in \mathcal{P}(S)$ such that $\widetilde{P} \leq ES$ and $\langle P, \widetilde{P}\rangle \notin \mathcal{L}$. We say that H has rank at least three if G has neither rank 1 nor rank 2.

The next lemma shows how $\widetilde{P}!$ can be used to obtain information about $E/O_p(E)$.

Lemma 0.6 [**unique component in e**] *Suppose $E!$, $P!$, $\widetilde{P}$ uniqueness and that G has rank at least three. Let $L = N_G(P^\circ)$ and $H = (L \cap \widetilde{C})E$.*

(a) *$N_G(T) \leq L$ for all $O_p(H \cap L) \leq T \trianglelefteq S$.*

(b) *There exists a unique $\widetilde{P} \in \mathcal{P}_H(S)$ with $\widetilde{P} \not\leq L$. Moreover, $\widetilde{P} \leq ES$.*

(c) *$\widetilde{P}/O_p(\widetilde{P}) \sim SL_2(q).p^k$.*

(d) *$H = K(L \cap H)$, $L \cap H$ is a maximal subgroup of H and $O_p(H \cap L) \neq O_p(H)$.*

(e) *H has a unique p-component K.*

(f) *Let $Z_0 = C_{Y_P}(S \cap P^\circ)$ and $V = \langle Y_P^H\rangle$. Then $Z_0 \trianglelefteq V$ and $V \leq Q \leq O_p(H)$.*

(g) *Let $D = C_H(K/O_p(K)$. Then D is the largest normal subgroup of H contained in L and $D/O_p(H)$ is isomorphic to a section of the Borel subgroup of $\mathrm{Aut}(SL_2(q)$*

(h) *Let $\overline{V} = V/Z_0$. Then*

  (ha) $[\overline{V}, O_p(H)] = 1$

  (hb) *$C_H(\overline{V}) \leq D$ and $C_H(\overline{V}) \cap C_H(Z_0) = O_p(H)$.*

  (hc) *Let $1 \neq X \leq Y_P/Z_0$. Then $N_H(X) \leq H \cap L$.*

  (hd) *$H \cap L$ contains a point-stabilizer for H on $\overline{V}$.*

Proof:

By $E!$, $O_p(\langle L, H\rangle) = 1$ and so $E \not\leq L$. Since $\{P\} = \{\mathcal{P}^\circ(S)$ and $N_G(S) \leq \widetilde{C}$, $N_G(S) \leq N_G(P) \leq L$. Since $E = \langle \mathcal{P}_{ES}, N_E(S) >$ there exists $\widetilde{P} \in \mathcal{P}_{ES}(S)$ with $\widetilde{P} \not\leq L$. Since rank G is at least three, $\langle P, \widetilde{P}\rangle \in \mathcal{L}$ and so by $\widetilde{P}!$, $\widetilde{P}$ is uniquely determined.

Let T be as in (a). It follows easily from $P!$ that $QO_p(L)$ is a Sylow p subgroup of $P^\circ O_p(L)$. Since $QO_p(L) \leq O_p(H \cap L) \leq T$ we conclude that T is a Sylow p-subgroup of TP°. Suppose that $M := \langle P^\circ T, N_G(T)\rangle \not in \mathcal{L}$. Then by Pushing up **??** and $Q!$ $P \sim q^2SL_2(q)$. Since $\langle P, \widetilde{P}\rangle \in \mathcal{L}$ we get $\widetilde{P} \leq N_G(Y_P)$ and **??**(dd) gives the contradiction $\widetilde{P} \leq N_G(P^\circ) = L$. Thus $M \in \mathcal{L}$. If $\widetilde{P} \in M$, then $T \trianglelefteq \widetilde{P}$ and so $O_p(P^\circ) \leq O_p(\widetilde{P})$, a contradiction to $(\widetilde{P} - 2b)$. Hence $\widetilde{P} \not\leq M$. By the uniqueness of $\widetilde{P}$ and since $S \leq M$, $M \leq L$. Thus (a) holds.

Let $P^* \in \mathcal{P}_H(S)$ with $P^* \not\leq L$. Suppose that $\widetilde{P} \not{\widetilde{P}}$. Then $P^* \not\leq ES$ and so $S \cap E \leq O_p(P^*)$ Since $EO_p(H \cap L)$ is normalized by $E(L \cap H) = H$ we get $O_p(H \cap L) \leq O_p(P^*)$. Thus by (a) $P^* \leq L$, a contradiction. So (b) holds. (a) is obvious.

Let $H \cap L < M \leq H$. Then $M \not\leq L$ and so $\widetilde{P} \leq M$. Let $R \in \mathcal{P}(H)(S)$. If $R = \widetilde{P}$, then $R \leq M$, if $R \neq \widetilde{P}$, then $R \leq L$ and again $R \leq M$. Since also $N_H(S) \leq H \cap L \leq M$ we get $M = H$. By (a) $O_p(H \cap L) \neq O_p(H)$.

Let N be a normal subgroup of H minimal with respect to $N \not\leq L$. By the uniqeness of $\widetilde{P}$, $\widetilde{P} \leq NS$. Hence $O^p(\tilde{P}) \leq N$ and since $O^p(\widetilde{P}) \not\leq L$, $N = \langle O^p(\widetilde{P})^H\rangle \leq E$. Next let F be the largest normal subgroup of H contained in L. Then $[O_p(H \cap L), F] \leq O_p(H \cap L) \cap F \leq O_p(F) \leq O_p(H)$. Note that $[O_p(H), N]$ is normal in $N(H \cap L) = N$ and so $[O_p(H), N] = N$ and we conclude that $[N, F] \leq O_p(H)$. In particular $F \cap N/O_p(N) \leq Z(N/O_p(N)$.

Suppose that N is solvable. Then the mimimality of N implies that $N/O_p(N)$ is a r-group for some prime $r \neq p$. In particular $H \cap N < N_N(H \cap N)$ and the maximality of $H \cap N$ implies $H \cap N \trianglelefteq H$. Thus $H \cap N \leq F$. Suppose that S does not act irreducible on $N/H \cap N$. The by coprime action there exists an S-invarinat $R \leq N$ with $R \not\leq L$ and $O^p(\widetilde{P}) \not\leq R$. Then by (b) $RS \leq L$, a contradiction. So S acts irreducible on $N/H \cap N$. Thus $N = (H \cap N)O^p(\widetilde{P})$ and $N = [N, O_p(H)] \leq O^p(\widetilde{P})$. Note that $\langle P^\circ, N\rangle$ is normalizes by P°, $H \cap L$ and N and so by $\langle L, H\rangle$, it follows that $O_p\langle P^\circ, N\rangle = 1$ and so also $O_p(\langle P, \widetilde{P}\rangle = 1$, a contradiction.

Thus N is not solvable and so the product of p-components.

Let K_1 be a p-component of N. By minimality of N, $N = \langle K_1^{H \cap L}\rangle$ If S does not act tranisitively on the p-components of N, we can choose K_1 such that $O^p(\widetilde{P}) \not\leq K_1^S\rangle$. But then $K_1 \leq L$, a contradiction. Thus $N = \langle K_1^S\rangle$. Suppose that $K_1 \cap \widetilde{P}$ lies in the unique maximal subgroup of $\widetilde{P}$ containing S. Since $K_1 \cap \widetilde{P}$ is subnormal in $\widetilde{P}$ the structure of $\widetilde{P}$ implies $[K_1 \cap \widetilde{P}, S]$ is a p-group. Thus $K_1 \neq N$ and $K_1 \cap \widetilde{P}$ is a p-group. Hence $N \cap S \trianglelefteq O_p(\widetilde{P})$. Since $N_N(N \cap S)/N \cap S$ is a p-group we conclude from coprimes action that $N \cap L$ projects onto $N_{K_1}(N \cap S)F/F$. Thus $[K_1 \cap L)F, N_{K_1}(N \cap S)] \leq [K_1 \cap L)F, N \cap L] \leq L$. Conjugation with S yields $[N \cap L, N_N(N \cap S) \leq N \cap L$. Thus $H = \langle \widetilde{P}, L\rangle \leq N_H(N \cap L)$ and so $N \cap L = F \cap L$. Since L contains a Sylow p-subgroup of N, we conclude that NF/F is a p'-group. Let r be a prime dividing the order of NF/F and R/F an S-invariant Sylow p-subgroup of NF/F. Then RS is not contained in L and so $\widetilde{P} \leq RS$. Thus $\widetilde{P} =$ is a $\{r, p\}$ group and r is unique. Thus NF/F is a r-group, a contradiction.

We proved that $K_1 \cap \widetilde{P}$ is not contained in the unique maximal subgroup of $\widetilde{P}$ containing S. Since $[K_1 \cap \widetilde{P}, (K_1 \cap \widetilde{P})^g$ is a p-group for all $g \in \widetilde{P} \setminus N_H(K_1)$ the structure of $\widetilde{P}$ implies $O^p(\widetilde{P}) \not\leq K_1$. Thus $S \leq N_H(K_1)$ and so $N = K_1$. Thus (e) is proved.

By $P!$ uniqueness Z_0 is normal in $\widetilde{C}$ and $[Z_0, Q] = 1$. Since Y_P is a natural module for P° we get $[Y_P, Q] \leq [V, O_p(H) \leq [V, O_P(L \cap H)] \leq Z_0$. So Y_P acts trivial an all factors of $1 \leq Z_0 \leq Q$ and since $\mathbb{C}$ is of characteristic p, $Y_P \leq Q$. This proves (f) and (ha).

To prove (g) note that $N \not\leq D$ and so by uniqueness of N, $D \leq L$ and so $D \leq F$. But as seen above $F \leq D$ and so $D = F$. Let D_0 be maximal in D with $[P^\circ, D_0] \leq O_p(P^\circ)]$. Then H and P° normalize $O^p(D_0)$ and so $O^p(D_0) = 1$. Thus D_0 is a p-group and (g) holds.

Note that $C_H(\overline{V}) \leq N_H(Y_P) \cap H \cap L$ and So $C_H(\overline{V}) \leq D$. Let $R = O^p(C_H(\overline{V}) \cap C_H(Z_0))$. Then $R)$ centralizes Y_P and $[R, P^\circ] \leq C_{P^\circ}(Y_P) = O_p(P^\circ) \leq O_p(H \cap L)$. But R is normal in $H \cap L$ and so $R = O^p(R) = O^p(RO_p(H \cap L)$. Thus H and P° both normalizes R and so $R = 1$. Hence (hb) holds.

Let $e \in Y_P \setminus Z_0$ with $eZ_0 \in X$. Let $g \in N_H(X)$. Since $H \cap L$ acts transitively on $Y_P \setminus Z_0$, there exists $h \in H \cap L$ with $e^{gh} = e$. Let $t \in P^\circ$ with $e \in Z_0^t$. Then $[e, Q^t] = 1$ and so $gh \leq \widetilde{C}^t$. Thus $gh \in N_G(\langle Q, Q^t \rangle = N_G(P^\circ) = L$. Hence $g \in L$ and (hc) holds.

(hd) follows from (hc). □

1 The Small World Theorem

Given $Q!$ and $P \in \mathcal{P}^\circ(S)$. We say that $b = 2$ for P if $b > 1$ for P and $\langle Y_P^{E\rangle}$ is not abelian. If neither $b = 1$ nor $b = 2$ for P we say that b is at least three for P.

Theorem 1.1 (The Small World Theorem) [**the small world theorem**] *Suppose $E!$ and let $P \in \mathcal{P}^\circ(S)$. Then one of the following holds:*

1. *G has rank 1 or 2.*
2. *$b = 1$ or $b = 2$ for P.*
3. *A rank three sitiation described below.*

Proof: Assume that G has rank at least three and that b is at least three. In the exceptional case of the $P!$-theorems (**??** and **??** one easily sees that $b = 2$ for P. Thus $P!$ holds. Also in the exceptionell case of the $\widetilde{P}!$ Theorem **??** one gets $b = 2$ for P. Thus (strong) $\widetilde{P}!$ holds. We proved

Step 1 [**P! and wP!**] *$P!$ and $\widetilde{P}!$ hold.*

0.6 gives us a good amount of information about E. We use the notation introduced in 0.6.

Since $\langle H, L \rangle \notin \mathcal{L}$, we can apply the amalgam method to the pair (H, L). A non-trivial argument shows

Step 2 [**offender on V**] *One of the following holds:*

1. *$O_p(H \cap L)/O_p(H)$ contains a non-trivial quadratic offender on $\widetilde{V}$.*

2. *There exists a non-trivial normal subgroup A of $H \cap L/O_p(H \cap L)$ and normal subgroups $Y_P \le Z_1 \le Z_2 \le Z_3 \le V$ of $H \cap L$ such that:*

   (a) *A and V/Z_3 are isomorphic as $\mathbb{F}_p C_{H \cap L}(Y_P)$-modules.*

   (b) $|Z_3/Z_2| \le |A|$.

   (c) $[\overline{V}, A] \le \overline{Z_2} \le C_{\overline{V}}(A)$. *In particular, A is a quadratic $2F$-offender.*

   (d) $[\overline{x}, A] = \overline{Y_P}$ *for all $x \in Z_3 \setminus Z_2$.*

   (e) $\overline{Z_1}$ *is a natural $SL_2(q)$-module for $\widetilde{P} \cap C_H(Z_0)$.*

Using **??** and **??** (and the Z^*-theorem to deal with the case $|A| = 2$) it is not too difficult to derive

Step 3 [**e-structure**] *$K/O_p(K) \cong SL_n(q)$, ($n \ge 3$), $Sp_{2n}(q)'$,($n \ge 2$) or $G_2(q)'$,($p = 2$). Moreover, if $\overline{W}$ is a maximal submodule of $\overline{V}$, then V/W is the natural module for $K/O_p(K)$ and $H \cap L$ contains a point-stabilizer on V/W.*

Let $M \in \mathcal{M}(\langle P, \widetilde{P} \rangle)$ with M° maximal.

Suppose first that $M^\circ \not\le \langle P, \widetilde{P} \rangle$. Then by the Structure Theorem **??** $M^\circ/O_p(M^\circ) \cong SL_n(q), n \ge 4$ or $Sp_{2n}(q)$, $n \ge 3$. Moreover,

$$R := O^p(M^\circ \cap \tilde{C} \le O^p(\widetilde{P})^{M \cap \widetilde{C}} \le K$$

In the case of $Sp_{2n}(q)$ we have $R/O_p(R) \cong Sp_{2n-2}(q)'$. So Step 3 implies $K/O_p(K) \cong Sp_{2m}(q)$ and $R \le H \cap L$, a contradiction.

Thus $M^\circ/O_p(M^\circ) \cong SL_n(q)$ and $R/O_p(R) \cong SL_{n-1}(q)$. Let R^* be a parabolic subgroup of KS minimal with $RS < R^*$. Then $R^* \cap K/O_p(R^* \cap K) \cong SL_n(q)$ or $Sp_{2n-2}(q)$. Let $M^* = \langle M^\circ S, R^* \langle$. Since

$R^* \le \langle O^p(\widetilde{P})^{R^*} \rangle S \le (M^*)^\circ S$

the maximality of M° implies that $M^* \notin \mathcal{L}$.

Now M^* has a geometry of type A_n or C_{2n} with all its residues classical we conclude that M^* has a normal subgroup $SL_{n+1}(q)$ or $Sp_{2n}(q)$. But this contradicts the assumption that V is abelian.

(Remark: One does not have to identify M^* first to obtain this contradiction. Indeed an easy geometric argument shows that M^* has rank at most three on $M^*/M^* \cap \widetilde{C}$. But then V abelian gives the contradiction $\langle Z_0^{M^*} \rangle$ abelian .)

We conclude that $M^\circ \le \langle P, \widetilde{P} \rangle$. Let P^* be the unique elemenent of $\mathcal{P}KS(S)$ with $P^* \not\le N_G(O^p(\widetilde{P}))$. By maximality of M° we obtain $\langle P, \widetilde{P}, P^* \rangle \notin \mathcal{L}$.

This is rank three situation eluded to in (c). □

Lemma 1.2 **[quadratic normal point stabilizer theorem]** *Let H be a finite group and V a faithful irreducible $\mathbb{F}_p H$-module. Let P be a point stabilizer for H on V and $A \leq P$. Suppose that*

(i) *$F^*(H)$ is quasi simple and $H = \langle A^H \rangle$*

(ii) *$A \trianglelefteq P$ and $|A| > 2$.*

(iii) *A acts quadratically on V.*

Then one of the following holds:

1. *$H \cong SL_n(q)$, $Sp_{2n}(q)$,or $G_2(q)$ and V is the natural module. Moreover,*

2. *$p = 2$, H is a group of Lie Typ in char p, and H is contained in a long root subgroup of H.*

3. *Who knows*

References