A Characteristic Subgroup for Pushing Up in Finite Groups

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1 Introduction

2 The Kieler Lemma and Points stabilizers

An elementary abelian normal subgroup $V$ of a finite group $L$ is called $p$-reduced if any subnormal subgroup of $L$ which acts unipotently on $V$ has to act trivially. Note that this is equivalent to $O_p(L/C_L(V)) = 1$. Here are the basic properties of $p$-reduced normal subgroups.

Comment: due to Thompson? check history

Lemma 2.1 [YL] Let $L$ be a finite group of characteristic $p$ and $T \in \text{Syl}_p(L)$

(a) [a] There exists a unique maximal $p$-reduced normal subgroup $Y_L$ of $L$.

(b) [b] Let $T \leq R \leq L$ and $X$ a $p$-reduced normal subgroup of $R$. Then $\langle X^L \rangle$ is a $p$-reduced normal subgroup of $L$. In particular, $Y_R \leq Y_L$.

(c) [c] Let $T_L = C_T(Y_L)$ and $L^f = N_G(T_L)$. Then $L = L^f C_L(Y_L)$, $T_L = O_p(L^f)$ and $Y_L = \Omega_1 Z(T_L)$.

(d) [d] $Y_T = \Omega_1 Z(T)$, $Z_L := \langle \Omega_1 Z(T)^L \rangle$ is $p$-reduced for $L$ and $\Omega_1 Z(T) \leq Z_L \leq Y_L$.

Now let $L$ be any finite group and $T \in \text{Syl}_p(L)$. definitionine $P_L(T) := O_p'(C_L(\Omega_1 Z(T)))$. Then $P_L(T)$ is called a point stabilizer of $L$. The following lemma ist the principal tool for working with point stabilizers.

Lemma 2.2 [kieler lemma] Let $H$ be a finite group of local characteristic $p$, $T \in \text{Syl}_p(H)$ and $L$ a subnormal subgroup of $H$. Then
(a) [KIILER Lemma] $C_L(\Omega_1Z(T)) = C_L(\Omega_1Z(T \cap L))$

(b) $P_L(T \cap L) = O^{\prime}(P_H(T) \cap L)$

(c) $C_L(Y_L) = C_L(Y_H)$

(d) Suppose $L = \langle L_1, L_2 \rangle$ for some subnormal subgroups $L_1, L_2$ of $H$. Then

(a) $P_L(T \cap L) = \langle P_{L_1}(T \cap L_1), P_{L_2}(T \cap L_2) \rangle$.

(b) For $i = 1, 2$ let $P_i$ be a point stabilizer of $L_i$. Then $\langle P_1, P_2 \rangle$ contains a point stabilizer of $L$.

The proof of the above lemma is elementary and does not require any $K$-group assumption assumption.

Comment: not all parts of this lemma are really needed

**Lemma 2.3** [minimal overgroups] Let $H$ be a finite group and $F < H$.

(a) Let $\mathcal{I}_H(F)$ be the set of all $I$ with $F < I \leq H$ such that $F$ lies in a unique maximal subgroup of $I$. Then $H = \langle \mathcal{I}_H(F) \rangle$.

(b) Let $\mathcal{J}_H(F) = \{I \in \mathcal{I}_G(F) \mid F \not\trianglelefteq I \}$. Then $H = \langle \mathcal{J}_H(F) \rangle N_H(F)$.

**Proof:** By induction on $|H|$. Suppose that $F$ lies in two different maximal subgroups $M_1, M_2$ of $H$. By induction, $M_i = \langle \mathcal{I}_{M_i}(F) \rangle = \langle \mathcal{J}_{M_i}(F) \rangle N_{M_i}(F)$.

Thus $H = \langle M_1, M_2 \rangle = \langle \mathcal{I}_H(F) \rangle = \langle \mathcal{J}_H(F) \rangle N_H(F)$.

So suppose $F$ lies in a unique maximal subgroup of $H$. Then $H \in \mathcal{I}$ and $H = \langle \mathcal{I} \rangle$. Moreover either $F$ is normal in $H$ or $H \in \mathcal{J}$. In any case $H = \langle \mathcal{J} \rangle N_H(F)$.

**Lemma 2.4** (Schur multipliers) [schur multipliers]

**Proof:** [Schur]
3 Modules

Lemma 3.1 (Point Stabilizer Theorem) [the point stabilizer theorem]
Let $H$ be a finite group, $V$ a $\mathbb{F}_p H$-module, $L$ a point stabilizer for $H$ on $V$ and $A \leq O_p(L)$.

(a) [a] If $V$ is $p$-reduced, then $|V/C_V(A)| \geq |A/C_A(V)|$.

(b) [b] If $V$ is irreducible, $F^*(H)$ is quasi-simple, $H = \langle A^H \rangle$ and $A$ is a non-trivial offender on $V$, then $M \cong SL_n(q), Sp_{2n}(q), G_2(q)$ or $Sym(n)$, where $p = 2$ in the last two cases.

Proof: [BBSM]

Lemma 3.2 (FF-modules for minimal parabolics) [ff-modules for minimal parabolics]

Proof: [BBSM]

Lemma 3.3 [spin module] Let $H = Sp_{2n}(q)$, $V$ a $\mathbb{F}_p H$-module, $P$ a point stabilizer for $H$ on the natural module, $T = O_p(P)$, $Z = Z(P)$ and $W$ an $\mathbb{F}_p T$ submodule of $V$. Suppose that

(i) [i] $V = \langle W^H \rangle$.
(ii) [ii] $[V,T,T] = 1$.
(iii) [iii] $[V,Z] \leq W \leq C_V(T)$.

Let $U = \bigcap_{h \in H} W^h[V,T]^h$ and $\overline{V} = V/U$. Let $h \in H$ with $Z \not< P^h$. Then

(a) [a] $V = [V,Z]C_V(T^h) = W[V,T]^h$, $\overline{W} = [\overline{W},T] = C_{\overline{V}}(T) = C_{\overline{V}}(Z)$
and $\overline{V} = \overline{W} \times \overline{W}^h$.

(b) [b] If $[W,H] \neq 1$, then $|\overline{V}| \geq q^{2^n}$ and $|V/C_V(T)| \geq q^{2^{n-1}}$.

Proof: Let $Y = W[V,T]$. Then $Y \leq C_V(T)$. Note that $H = \langle Z,T^h \rangle$. Since $[V,Z] \leq W$ we conclude that $H$ normalizes $W[V,T]^h$ and so by (i), $V = W[V,T]^h$. Also $H$ also normalizes $[V,Z]Y^h$ and since $W^h \leq Y^h$ we conclude $V = [V,Z]^h = [V,Z]C_V(T^h)$. Let $X/U = C_{\overline{V}}(Z)$. Then $U \leq X \cap Y^h$. Thus $H = \langle Z,T^h \rangle$ normalizes $X \cap Y^h$ and so $X \cap Y^h = U$. Thus
\[ V = X \times Y^h. \] Since \( V = [V, Z] Y^h \) we also get \( V = [V, Z] \times Y^h \). This implies \([V, Z] = X = C_V(Z)\).

Note that

\[
[V, Z] \leq [V, T] \leq Y \leq C_V(T) \leq C_V(Z)
\]

Now all the inequalities in the preceding inequalities have to be equalities. So (a) is proved.

To prove (b) suppose that \( [W, H] \neq 1 \). By (a) also \( [W, H] = 1 \) and so we may assume that \( U = 1 \).

Suppose first that \( n = 1 \) and \( 1 \neq z \in Z \). Since \( H = \langle z, T^h \rangle \), \( C_{Y^h}(z) \leq U = 1 \). Let \( 1 \neq y \in Y^h \). We conclude that \( |[y, Z]| \geq |Z| = q \) and so \( |W| \geq q \) and \( |V| \geq q^2 \).

Suppose next that \( n > 1 \) and let \( H^* = C_H(\langle Z, Z^h \rangle) \). Then \( H^* \cong Sp_{2n-2}(q) \) and \( Z^* := Z^k \leq H^* \) for some \( k \in H \). Then \( P^* := P^k \cap H^* \) is a point stabilizer for \( H^* \) on its natural module, \( T^* := T^k \cap H^* = O_p(P^*) \) and \( Z^* = Z(P^*) \). Since \( W = C_V(Z) \) and \( H^* \leq C_G(Z) \), \( W \) is a \( \mathbb{F}_pH^* \) submodule of \( W \). Suppose that \( [W, Z^*, H^*] = 1 \). Let \( h^* \in H^* \) with \( Z^{*h^*} \not\leq P^* \). Then \( [W, Z^*] \leq [V, Z^*] \cap [V, Z^{*h^*}] = 1 \) and so \( [W, Z^*] = 1 \). Thus \( C_V(Z) = W = C_W(Z^*) \) and so \( P \) and \( P^* \) normalize \( W \), a contradiction since \( H = \langle P, P^* \rangle \). Thus \( [W, Z^*, H^*] \neq 1 \). Let \( V^* := ([W, Z^*] H^*) \). Then by induction \( |V^*| \geq q^{2n-1} \). Since \( V^* \leq W \) and \( |V| = |W|^2 \) we get \( |V| \geq q^{2n} \).

We remark that (for example by [BBSM]), \( V \) from the preceding lemma must be a direct sum of spin-modules for \( H \).

**Lemma 3.4 (H1 of natural modules) [h1]**

**Proof:** [BBSM] \( \square \)

### 4 The Baumann subgroup

For a \( p \)-group \( R \) we let \( \mathcal{PU}_1(R) \) be the class of all finite groups \( L \) containing \( R \) such

(a) \([a]\) \( L \) is of characteristic \( p \),

(b) \([b]\) \( R = O_p(N_L(R)) \)

(c) \([c]\) \( N_L(R) \) contains a point stabilizer of \( L \).
Let $\mathcal{P}U_2(R)$ be the class of all finite groups $L$ containing $R$ such that $L$ is of characteristic $p$ and

$$L = \langle N_L(R), H \mid R \leq H \leq L, H \in \mathcal{P}U_1(R) \rangle.$$ 

Let $\mathcal{P}U_3(R)$ be the class of all finite groups $L$ such that

(a) [a] $L$ is of characteristic $p$.

(b) [b] $R \leq L$ and $L = \langle R^L \rangle$

(c) [c] $L/C_L(Y_L) \cong SL_n(q), Sp_{2n}(q)$ or $G_2(q)$, where $q$ is a power of $p$ and $p = 2$ in the last case.

(d) [d] $Y_L/C_Y(L)$ is the corresponding natural module.

(e) [e] $O_p(L) \leq R$ and $N_L(R)$ contains a point stabilizer of $L$.

(f) [f] If $L/C_L(Y_L) \neq G_2(q)$ then $R = O_p(N_L(R))$.

Let $\mathcal{P}U_4(R)$ be the class of all finite groups $L$ containing $R$ such that $L$ is of characteristic $p$ and

$$L = \langle N_L(R), H \mid R \leq H \leq L, H \in \mathcal{P}U_3(R) \rangle.$$ 

Let $B(R) = C_R(\Omega_1 Z(J(R)))$, the Baumann subgroup of $R$. Recall that a finite group $F$ is $p$-closed if $O^F_p = O_p(F)$.

**Lemma 4.1 (Baumann Argument)** [baumann argument] Let $L$ be a finite group, $R$ a $p$-subgroup of $L$, $V := \Omega_1 Z(O_p(L))$, $K := \langle B(R)^L \rangle$, $\tilde{V} = V/C_V(O_p(K))$, and suppose that each of the following holds:

(i) [i] $O_p(L) \leq R$ and $L = \langle J(R)^L \rangle N_L(J(R))$.

(ii) [ii] $C_K(\tilde{V})$ is $p$-closed.

(iii) [iii] $|\tilde{V}/C_V(A)| \geq |A/C_A(\tilde{V})|$ for all elementary abelian subgroups $A$ of $R$.

(iv) [iv] If $U$ is $L/O_p(L)$ module with $\tilde{V} \leq U$ and $U = C_U(O_p(K))\tilde{V}$, then $U = C_U(O_p(K))\tilde{V}$.

Then $O_p(K) \leq B(R)$. 

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Proof: Let \( T = O_p(L) \), \( \mathcal{L} = L/C_L(V) \) and \( Y = \Omega_1 Z(J(R)) \). Let \( A \in \mathcal{A}(R) \). Since \( A \in \mathcal{A}(R) \) and \( V \leq T \leq R \), \( |V/C_V(A)| \leq |A/C_A(V)| \). By (ii) \( C_A(V) = A \cap T \) and so also \( C_A(V) = A \cap T \). Thus (iii) implies \( |V/C_V(A)| = |\overline{A}| = |A/A \cap T| \) and so \( V(A \cap T) \in \mathcal{A}(R) \cap \mathcal{A}(T) \). Thus \( Y \leq V(A \cap T) \leq T \). Put \( W = \langle Y^L \rangle V \). We conclude that \( W \leq \Omega_1 Z(J(T)) \) and so \( W \) is elementary abelian and \( (A \cap T)V \) centralizes \( W \). Hence \( W \leq (A \cap T)V \) and \( W = V(A \cap W) = V C_W(A) \). It follows that \( A \) centralizes \( W/V \). Since \( A \) was arbitrary in \( \mathcal{A}(R) \), \( \langle J(R)^L \rangle \) centralizes \( W/V \). Since \( Y = \Omega_1 Z(J(R)) \), \( N_L(J(R)) \) normalizes \( Y \). So by (i) also \( L \) normalizes \( YV \). Thus \( W = YU \) and \( [W, T] = [Y, T] \leq Y \). Since \( L \) normalizes \( [W, T] \) we get \( [W, T] \leq C_W(K) \). Let \( D = C_W(O^p(K)) \) and \( U = W/D \). Then \( T \) centralizes \( U \). Since \( \overline{V} \cong VD/D \) and \( U = YV/D \), we can apply (iv) to conclude that \( W = DV \) and \( U \cong \overline{V} \). Since \( A \in \mathcal{A}(R) \), \( |W/W \cap A| \leq |A/C_A(W)| = |A/A \cap T| \). One the other hand by (iii), \( |A/A \cap T| \leq |V/C_V(A)| = |U/C_U(A)| \leq |W/C_W(A)D| \). Thus \( |V/C_V(A)| \leq |W/C_W(A)D| \) and \( D \leq C_W(A) \). Hence \( [D, A] = 1 \), \( D \leq Y \) and \( [D, K] = 1 \). Therefore \( [W, O_p(K)] \leq [D, K][V, T] = 1 \) and so \( O_p(K) \leq C_R(Y) = B(R) \).

Lemma 4.2 \([pu2(R) \text{ in } pu4(B(R))])\) Let \( R \) be a \( p \)-group. Then \( \mathcal{P} \mathcal{U}_2(R) \subseteq \mathcal{P} \mathcal{U}_4(B(R)) \).

Proof: Let \( L \in \mathcal{P} \mathcal{U}_3(R) \). Since \( N_L(R) \leq N_L(B(R)) \) we may assume that \( L \in \mathcal{P} \mathcal{U}_1(R) \). Set \( P = N_L(R) \). If \( P < H \leq L \), then clearly \( H \in \mathcal{P} \mathcal{U}_1(R) \). By 2.3(a) \( L \) is generated by the \( H \leq L \) such that \( P \) is contained in a unique maximal subgroup of \( H \). If \( H \in \mathcal{P} \mathcal{U}_4(B(R)) \) for all such \( H \), then by the definition of \( \mathcal{P} \mathcal{U}_4 \) also \( L \in \mathcal{P} \mathcal{U}_4(B(R)) \). Hence we may assume from now on that

1) \[ 1 \] \( P \) is contained in unique maximal subgroup \( H \) of \( L \).

Let \( D \) be the largest normal subgroup of \( L \) contained in \( P \). Then \( |D, R| \leq |P, R| \leq |R| \) and so \( |D, [R, D]| \leq O_p(D) \leq O_p(L) \).

Choose \( T \in Syl_p(L) \) with \( P_L(T) \leq P \). Then \( R \leq O_p(P_L(T) \leq O_p(C_L(\Omega_4 Z(T))) \) and \( [R, C_L(Z_L)] \leq O_p(C_L(Z_L)) = O_p(L) \leq R \). Thus \( C_L(Z_L) \leq N_L(R) \leq P \).

We proved:

2) \[ 2 \] \( |D, [R^L]| \leq O_p(L) \) and \( C_L(Z_L) \leq D \)

If \( J(R) \leq D \), then \( J(R) = J(O_p(D)) \) and so \( J(R) \leq H \). Thus \( |Z_L, J(R)| = 1 \) and so also \( |Z_L, B(R)| = 1 \). So by rr2, \( B(R) \leq D \) and \( B(R) = B(O_p(D)) \). Thus \( B(R) \leq H \) and so \( H \in \mathcal{P} \mathcal{U}_4(B(R)) \).
So we may assume that $J(R) \not\subset D$ and so by rr2 $[Z_L, J(R)] \neq 1$. Let $K = \langle J(R)^L \rangle$, $\overline{L} = L/C_L(Z_L)$ and $\overline{Z}_L = Z_L/C_{Z_L}(O_p(K))$. By ?? there exists a $L$-invariant set of normal subgroups $K_i$, $1 \leq i \leq l$, in $K$ such that

(3-i) $K_i = O_p^i(K_i)$,
(3-ii) $K = K_1 \times K_2 \times \ldots \times K_l,$
(3-iii) $\overline{Z}_L = [\overline{Z}_L, K_1] \times [\overline{Z}_L, K_2] \times [\overline{Z}_L, K_l],$
(3-iv) $\overline{K}_i \cong SL_n(q), Sp_{2n}(q), G_2(q)$ or $\text{Sym}(n)$, where $q$ is a power of $p$, $p = 2$ in the last two cases and $n \equiv 2, 3 \mod 4$ in the last case,
(3-v) $[\overline{Z}_L, K_i]$ is the natural module for $K_i$,
(3-vi) $\overline{J(R)} = (\overline{J(R)} \cap K_1) \times \ldots \times (\overline{J(R)} \cap K_l)

It is now easy to see that $\overline{L} = K N_{\overline{L}}(\overline{J(R)})$

By rr2 $O_p(C_L(Z_L)J(R)) = O_p(L)J(R)$ and so $J(R) = J(O_p(C_L(Z_L)J(R)))$. Thus $N_L(J(R)) = N_{\overline{L}}(\overline{J(R)})$ and so

3) $L = K N_L(J(R)).$

Suppose that $K \leq H$. Then by rr1 and rr4 $J(R)$ is normal in $L$ and $J(R) \leq O_p(L) \leq D$, a contradiction to the assumptions.

Thus $K \not\leq H$. Pick $j$ with $K_j \not\leq H$. Then by 1) $L = \langle K_j, P \rangle = \langle K_j^P \rangle P$.

Thus $\langle K_j^P \rangle J(R)$ is normal in $L$. So $P$ acts transitively on $\{K_i \mid 1 \leq i \leq l\}$, and $L = KP$. By 2) $[C_L(Z_L), J(R)] \leq O_p(L)$ and so $C_L(Z_L), K \leq O_p(L)$.

Hence $C_K(Z_L)$ is $p$-closed. Also $C_K(Z_L) = C_K(\overline{Z}_L)$.

Note also that $B(R) \leq KO_p(L)$ and so $\langle B(R)^L \rangle = K B(R)$.

Suppose that $B(R)O_p(L) = O_p(P \cap KO_p(L))$ or that $\overline{K}_j \cong G_2(q)$. Then it is easy to see that the assumptions of 4.1 are fulfilled. We conclude that $O_p(K B(R)) \leq B(R)$. Moreover, either $\overline{K}_j \cong G_2(q)$ or $B(R) = O_p(P \cap K B(R))$. By 2.2(a)

$$C_{K_1}(\Omega_1 Z(T \cap K_j B(R))) = C_{K_j}(\Omega_1 Z(T \cap K_i)) = C_{K_j}(\Omega_1 Z(T))$$

and we conclude that $P \cap K_j B(R)$ contains a point stabilizer of $K_i B(R)$.

Suppose in addition that $K_j \not\cong \text{Sym}(n), n \geq 7$. Then $K_i B(R) \in \mathcal{P} U_3(B(R))$.

Also $P \leq N_L(B(R))$ and $L = \langle P, K_i B(R) \mid 1 \leq i \leq l \rangle$ and so $L \in \mathcal{P} U_4(B(R))$.

Suppose now that $\overline{K}_j \cong G_2(q)$ and either $B(R)O_p(L) \neq O_p(P \cap KO_p(L))$ or $K_j \cong \text{Sym}(n), n \geq 7$. Put $q := 2$ in the second case. Then $\overline{K}_i \cong Sp_{2n}(q)$.
or $\text{Sym}(n)$ and $|B(R)/O_p(K_iB(R))| = q$. Hence there exists a subgroup $D_i$ of $K_iB(R)$ with $B(R) \leq D_i$, $D_i = (B(R)^D_i)$ and $D_i/O_p(D_i) \cong \text{SL}_2(q)$. By 4.1 $B(R) \in \text{Syl}_p(D_i)$. Thus $D_i \in \mathcal{P}U_3(B(R))$. Moreover, $K_i = (D_i, N_{K_i}(B(R)))$ and so $L = \langle D_i, N_L(B(R)) \rangle \mid 1 \leq i \leq n$. Thus again $L \in \mathcal{P}U_4(B(R))$. □

**Lemma 4.3 [P(T) in PU4(B(T))]** Let $P$ be a finite group of characteristic $p$. Let $T \in \text{Syl}_p(T)$ and suppose that $T$ lies in a unique maximal subgroup of $P$. Then either $Z_L = \Omega_1Z(L)$ or $P \in \mathcal{P}_4(B(T))$.

**Proof:** Suppose that $[J(T), Z_L] = 1$. Then also $[B(T), Z_L] = 1$ and so by the Frattini argument $L = C_L(Z_L)N_L(B(T))$. Since $L$ is minimal parabolic, $L = C_L(Z_L)S$ or $B(T)$ is normal in $L$. In the first case $Z_L = \Omega_1Z(L)$ and in the second case $L \in \mathcal{P}U_4(T)$.

So we may assume that $[B(T), Z_L] \neq 1$. Using 3.2 we can argue just as in 4.2. □

5 A solution to the principal amalgam problem

Let $R$ be a group and $\Sigma$ a set of groups containing $R$. Then

$$O_R(\Sigma) = \langle N \leq R \mid N \triangleleft \forall L \in \Sigma \rangle$$

So $O_R(\Sigma)$ is the largest subgroup of $R$ which is normal in all the $L \in \Sigma$.

**Theorem 5.1 [simultaneous pushing up]** Let $R$ be a finite $p$-group with $R = B(R)$ and $\Sigma$ a subset of $\mathcal{P}U_3(R)$. If $O_R(\Sigma) = 1$, then one of the following holds

(a) [a] who knows

The proof will be achieved in a long sequence of lemmas. Let $G^*$ be the free amalgamated product of the $\Sigma$ over $R$. We view $L \in \Sigma$ as a subgroup of $G^*$. Let $\Gamma$ be the graph with vertices $G^*$ and edges $(L_1g, L_2g)$, $g \in G^*$, $L_1 \neq L_2 \in \Sigma$. Note that $G^*$ acts on $\Gamma$ by right multiplication. For $\alpha \in \Gamma$ let $G_\alpha = \{g \in G^* \mid \alpha = a^g\}$, $Q_\alpha = O_p(G_\alpha)$ and $Z_\alpha = Z_{G_\alpha}$ and $U_\alpha = [Z_\alpha, G_\alpha]$. For an edge $(\alpha, \beta)$ let $Q_{\alpha\beta} = G_\alpha \cap G_\beta$ and $Z_{\alpha\beta} = \Omega_1Z(Q_{\alpha\beta})$. Let $\Delta(\alpha)$ be the set of neighbors of $\alpha$ and $G_{\alpha}^{(1)} = G_\alpha \cap \bigcap_{\beta \in \Delta(\alpha)} G_\beta$. Let $U_\alpha = [Z_\alpha, G_\alpha]$. Then by definition of $\Gamma$ and of $\mathcal{P}U_3(R)$.
Lemma 5.2 [basics of pushing up]

(a) \[ G_\alpha = L^9 \] for some \( L \in \Sigma \) and \( g \in G^* \), and \( G_\alpha \) is of characteristic \( p \).

(b) \[ \overline{G_\alpha} := G_\alpha/C_{G_\alpha}(Z_\alpha) \cong SL_{n_\alpha}(q_\alpha), Sp_{2n}(q_\alpha) \text{ or } G_2(q_\alpha), q_\alpha \text{ a power of } p. \]

(c) \[ \overline{Z_\alpha} := Z_\alpha/C_{Z_\alpha}(G_\alpha) \text{ is a natural module.} \]

(d) \[ Q_{\alpha\beta} = B(Q_{\alpha\beta}) \text{ and } G_\alpha = \langle Q_{\alpha\beta} \rangle. \]

(e) \[ P_{\alpha\beta} := N_{G_\alpha}(Q_{\alpha\beta}) \text{ contains a point stabilizer of } G_\alpha. \]

(f) \[ \text{If } \overline{G_\alpha} \neq G_2(q) \text{ then } Q_{\alpha\beta} = O_p(P_{\alpha\beta}). \]

Next we show

Lemma 5.3 [more basics of pushing up]

(a) \[ Z_{\alpha\beta} \leq Z_\alpha = \Omega_1 Z(Q_\alpha) \]

(b) \[ C_{G_\alpha}(Z_\alpha) = Q_\alpha. \]

(c) \[ Q_\alpha = G_\alpha^{(1)}. \]

(d) \[ \text{One of the following holds:} \]

1. \[ [1] \text{ } U_\alpha \cap \Omega_1 Z(G_\alpha) = 1, \text{ that is } U_\alpha \text{ is the natural module.} \]

2. \[ [2] \text{ } \overline{G_\alpha} \cong Sp_{2n}(q) \text{ or } G_2(q) \text{ and } U_\alpha \text{ is a quotient of the natural } O_{2n+1}(q)-\text{module for } \overline{G_\alpha}, \text{ (where } n = 3 \text{ in the } G_2(q)-\text{case).} \]

(e) \[ [e] \text{ For all } H \leq G_\alpha, \text{ } C_{Z_\alpha}(H) = \widetilde{C_{Z_\alpha}(H)}. \]

(f) \[ [f] \text{ Let } T \in \text{Syl}_p(P_{\alpha\beta}) \text{ and } x \in \Omega_1 Z(T) \text{ with } x \notin \Omega_1 Z(G_\alpha). \text{ Then } C_{G_\alpha}(x) = O^p(P_{\alpha\beta}). \]

(a) follows from 5.2(d),(e) and 3.1.

Let \( T \in \text{Syl}_p(P_{\alpha\beta}) \). Since \( C_{G_\alpha}(Z_\alpha) \leq C_{G_\alpha}(\Omega_1 Z(T)) \leq P_{\alpha\beta} = N_{G_\alpha}(Q_{\alpha\beta}) \) we get

\[ [C_{G_\alpha}(Z_\alpha), Q_{\alpha\beta}] \leq C_{G_\alpha}(Z_\alpha) \cap Q_{\alpha\beta} \leq O_p(C_{G_\alpha}(\Omega_1 Z(T))) \leq Q_\alpha. \]
Thus 5.2(d), $[C_{G_\alpha}(Z_{\alpha}), G_\alpha] \leq Q_\alpha$. we proved this before, should have been recorded

Thus (b) follows from 2.4 and 5.2 (d).

By 5.2(f) $Q_\alpha \leq Q_{\alpha\beta} = G_\alpha \cap G_\beta$. So (c) holds.

(d) follows from 3.4, and (e) follows from (d). Finally (f) follows from (b),(e), and 5.2 (c),(e).

We say that $\beta \in \Gamma$ is symplectic if $G_\beta \cong Sp_{2n}(q)$ with $n \geq 2$, $\beta$ is linear if $G_\beta \cong SL_n(q)$ and $\beta$ is a hex if $G_\beta \cong G_2(q)$. Let $\alpha \in \Delta(\beta)$.

$$X_{\alpha\beta} := \begin{cases} [Z_\alpha, Q_{\alpha\beta}] & \text{if } \alpha \text{ is symplectic.} \\ Z_\alpha & \text{otherwise.} \end{cases}$$

Put

$$A_{\alpha\beta} = [X_{\alpha\beta}, Q_{\alpha\beta}]$$

**Lemma 5.4** [agammadelta] Let $(\alpha, \beta)$ be an edge in $\Gamma$. Then $A_{\alpha\beta} \leq \Omega_1 Z(Q_{\alpha\beta}) \leq \Omega_1 Z(Q_\beta) \leq Z_\beta$ and $A_{\alpha\beta} \not\leq Z(G_\alpha)$.

**Proof:** Readily verified. \(\square\)

**Lemma 5.5** [offenders on xgammadelta] Let $(\alpha, \beta)$ be an edge in $\Gamma$, $D = X_{\alpha\beta}$ or $D = Z_\alpha$ and $B \leq Q_{\alpha\beta}$ be a non-trivial offender on $D$

(a) [a] $|D/C_D(B)| = |B/C_B(D)|$.

(b) [b] One of the following holds:

1. [1] $[D, Q_{\alpha\beta}] \leq [D, B]$.
2. [2] $\alpha$ is a symplectic, $D = Z_\alpha$ and $[D, C_{Q_{\alpha\beta}}(X_{\alpha\beta})] \leq [D, B]$.

(c) [c] One of the following holds

1. [1] $[D, B, Q_{\alpha\beta}] = 1$.
2. [2] $\alpha$ is symplectic, $D = Z_\alpha$, $[X_{\alpha\beta}, B] \neq 1$ and $[D, Q_{\alpha\beta}, Q_{\alpha\beta}] = A_{\alpha\beta}$.\(\square\)

**Proof:** This follows easily from the action of $Q_{\alpha\beta}$ on $D$

**Lemma 5.6** [agd in zgd] Let $(\alpha, \beta)$ be an edge in $\Gamma$ and suppose that $Z_\beta \leq Q_\alpha$. 

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(a) If $X_{\alpha\beta} \not\subseteq Z_\beta$ then $A_{\alpha\beta} \leq Z(G_\beta)$.

(b) Suppose $\alpha$ is symplectic and that $N$ is a normal $p$-subgroup of $G_\beta$ with $[X_{\alpha\beta}, N] = 1$. Then $[Z_{\alpha}, N] \leq Z(G_\beta)$.

Proof: For the proof of (b) we may assume (a) has been proved and that $[Z_{\alpha}, N] \neq 1$.

We prove (a) and (b) simultaneously. For the proof of (a) let $D_\alpha = X_{\alpha\beta}$ and $U = Q_\beta$. Note that $D_\alpha$ also depends on $\beta$ but $\beta$ will be fixed throughout the proof. For the proof of (b) let $D_\alpha = Z_\alpha$ and $U = N$. Let $A_\alpha = [D_\alpha, U]$.

From the definition of $A_\alpha$ we obtain:

1) $A_\alpha \leq Z_{\alpha\beta}$

Next we show:

2) Let $B \leq Q_{\alpha\beta}$ and suppose that $B$ is a non-trivial offender on $D_\alpha$. Then $A_\alpha \leq [D_\alpha, B] \cap Z_{\alpha\beta}$.

By 1) we only need to show that $A_\alpha \leq [D_\alpha, B]$. We apply 5.5(b) with $D_\alpha$. If 1 holds we have $A_\alpha = [D_\alpha, U] \leq [D_\alpha, Q_{\alpha\beta}] \leq [D_\alpha, B] = Z_{\alpha\beta}$ and we are done. Suppose that 2 holds. Then $D_\alpha \neq X_{\alpha\beta}$ and so we must be in the proof of (b). So $U = N \leq C_{Q_{\alpha\beta}}(X_{\alpha\beta})$ and again $A_\alpha \leq [D_\alpha, B]$.

3) Let $B \leq Q_\beta$ and suppose that $B$ is a non-trivial offender on $D_\alpha$. $[D_\alpha, B, Q_{\alpha\beta}] \leq \Omega_1 Z(G_\beta)$.

We apply 5.5(c). If 1 holds we are done. So suppose 2 holds. Then we are in the proof of (b), $[X_{\alpha\beta}, B] \neq 1$ and $[D_\alpha, B, Q_{\alpha\beta}] = A_{\alpha\beta}$. Since $B \leq Q_\beta$, we get $X_{\alpha\beta} \not\subseteq Z_\beta$ and so by (a) $A_{\alpha\beta} \leq \Omega_1 Z(G_\beta)$ and 3) is proved.

Thus there exists $A \in A(Q_{\alpha\beta})$ with $A \not\subseteq Q_\beta$. Let $a \in A$ with $a \notin Q_\beta$. If $\beta$ is a hex we choose $a$ such that in addition $C_{Z_\beta}(a) = Z_{\alpha\beta}$. Let $\gamma \in \alpha^{G_\beta}$ with $Z_{\alpha\beta} \cap Z_{\gamma\beta} = \Omega_1 Z(G_\beta)$ and $a \notin P_{\beta\gamma}$. The choice of $a$ implies

4) $Z_{\gamma\beta} \cap Z_{\gamma\beta}^a = \Omega_1 Z(G_\beta)$

Suppose first that

(*) $[D_\gamma, D_{\gamma}^a] \neq 1$. 

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Then by 5.5 $D^a_\gamma$ is an offender on $D_\gamma$ and vice versa. So by 2) applied to $(D^a_\gamma, \gamma)$ in place of $(B, \alpha)$

$$A_\gamma \leq [D_\gamma, D^a_\gamma] \cap Z_{\gamma^a}$$

By 3) applied to $(D_\gamma, \gamma^a)$ in place of $(B, \alpha)$ we have $[[D_{\gamma^a}, D_\gamma], Q^a_\gamma] \leq Z(G_\beta)$. Hence 5.3(f) implies $Z_\beta \cap [D_{\gamma^a}, D_\gamma] \leq Z^a_{\gamma^a}$ and thus

$$A_\gamma \leq [D_{\gamma^a}, D_\gamma] \cap Z_{\gamma^a} \leq Z_{\gamma^a} \cap Z^a_{\gamma^a} \leq \Omega_1 Z(G_\beta)$$

and we are also done in this case.

Suppose next that

$$\text{(**) } [D_\gamma, D^a_\gamma] = 1.$$ 

Set $B := A \cap Q_\beta$ and $C := C_B(D_\gamma)$. Then $Z_\beta B \in A(Q_\beta) \subseteq A(Q_\alpha^\beta)$. Since $Z_\beta$ centralizes $Z_{\gamma^a}$, $B$ is an offender on $D_\gamma$. Since $A$ is abelian and $C \leq B \leq A$ we have $B = B^a$ and $C = C^a$. Thus $C = C_B(D^a_\gamma)$ and $C$ centralizes $D^a_\gamma$. Since by assumption $Z_\beta \leq Q_\alpha$ we get $Z_\beta \leq Q_\gamma^a$. Thus by (**$)$ $Z_\beta D_\gamma C$ centralizes $D^a_\gamma$. By 1) $Z_\beta D_\alpha C \in A(Q_\beta)$ and we conclude that $D^a_\gamma \leq Z_\beta D_\gamma C$. By symmetry in $\gamma$ and $\gamma^a$ we conclude $Z_\beta D_\gamma C = Z_\beta D^a_\gamma C$.

Thus

$$[D_\gamma, B] = [D^a_\gamma, B].$$

Suppose that $B$ does not centralize $D_\gamma$. Then by 2) applied to $\gamma$ in place of $\alpha$, $A_\gamma \leq [D_\gamma, B] \cap Z_{\gamma^a}$. From $[D_\gamma, B] = [D^a_\gamma, B]$ and 3) applied to $\gamma^a$ in place of $\alpha$ we get $[D_\gamma, B, Q^a_\gamma] \leq Z(G_\beta)$ Now as in the (*) case $A_\gamma \leq Z(G_\beta)$ and we are done.

Suppose next that $B$ centralizes $D_\gamma$. Then also $Z_\beta B$ centralizes $D_\gamma$ and so $D_\gamma \leq Z_\beta B$. Since $a$ centralizes $B$ we conclude that $D_\gamma Z_\beta = D^a_\gamma Z_\beta$. Hence

$$A_\gamma = [D_\gamma, U] = [D_\gamma Z_\beta, U] = [D^a_\gamma, U] = A_{\gamma^a} \leq Z_{\gamma^a} \cap Z^a_{\gamma^a} \leq \Omega_1 Z(G_\beta)$$

and we are also done in this final case.

For adjacent vertices $\alpha, \beta$ let $V^B_\alpha = \langle Z^G_{\beta} \rangle$.

**Lemma 5.7 [qgamma cap qdelta normal]** Let $(\beta, \alpha)$ be an edge of $\Gamma$ and suppose that $V^B_{\alpha}$ and $V^B_{\beta}$ are abelian. Then $Q_{\alpha} \cap Q_{\beta}$ is normal in $G_{\alpha}$.

**Proof:** Choose $A$, $a$ and $\gamma$ as in the proof of 5.6. Assume that $Q_{\alpha} \cap Q_{\beta}$ is not normal in $G_{\alpha}$. By conjugation $Q_{\gamma} \cap Q_{\beta}$ is not normal in $G_{\gamma}$ and so $Q_{\gamma} \cap Q_{\beta} \neq Q_{\delta} \cap Q_{\gamma}$ for some $\delta \in \beta^{G_{\gamma}}$. Then $[Q_{\gamma} \cap Q_{\beta}, Z_{\delta}] \neq 1$. 

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If possible, choose $\delta$ such that $[Q_\gamma \cap Q_\delta, X_{\delta\gamma}] \neq 1$. In this case put $D_{\delta\gamma} = X_{\delta\gamma}$.

If not possible, put $N = \langle (Q_\alpha \cap Q_\beta)^{G_\gamma} \rangle$ and $D_{\delta\gamma} = Z_\delta$. Then $[X_{\delta\gamma}, N] = 1$.

Note that $Z_{\gamma} \leq V_\beta^\gamma$ and so $Z_{\gamma} \leq Q_\beta$. Thus we can apply 5.6 and to $(\beta, \gamma)$ in place of $(\alpha, \beta)$. We conclude that $A_\gamma := \langle D_{\delta\gamma}, Q_\alpha \cap Q_\beta \rangle \leq \Omega_1 Z(G_\gamma)$. Since $A_\gamma \nsubseteq \Omega_1 Z(G_\delta)$ and $\delta \in \beta^{G_\gamma}$ we get $A_\gamma \nsubseteq \Omega_1 Z(G_\beta)$. Since $Z_{\gamma\beta}^{-1} \cap Z_{\gamma\beta} \leq \Omega_1 Z(G_\beta)$ we have

1) [1] $A_\gamma \leq \Omega_1 Z(G_\gamma)$ and $Z_{\gamma\beta}^{-1} \nsubseteq A_\gamma \nsubseteq Z_{\gamma\beta}$.

From the definition of $D_{\delta\gamma}$ and 5.5(b) we deduce

2) [2] Let $F \leq Q_{\delta\gamma}$ be an offender on $D_{\delta\gamma}$, then $A_\gamma \leq [D_{\delta\gamma}, F]$.

Let $B = A \cap Q_\beta$ and $C = B \cap Q_\gamma$. Then $Z_\beta B$ and $Z_\beta Z_\gamma C$ are in $A(Q_\gamma)$. Next we show

3) [3] $D_{\delta\gamma} \leq Z_\beta Z_\gamma C$ for all $\delta \in \beta^{G_\gamma}$ with $[Q_\beta \cap Q_\gamma, D_{\delta\gamma}] \neq 1$.

Assume that $[C, D_{\delta\gamma}] = 1$. Since $V_\gamma^\beta$ is abelian, $Z_\gamma Z_\beta$ centralizes $Z_\delta$ and so also $D_{\delta\gamma}$. Since $Z_\beta Z_\gamma C \in A(Q_\gamma)$ we conclude that 3) holds in this case. So assume for a contradiction that $[C, D_{\delta\gamma}] \neq 1$ and put $D = CC(D_{\delta\gamma})$. Then by 2), $A_\gamma \leq [C, D_{\delta\gamma}]$ and by 5.5(a) $E := Z_\beta Z_\gamma D_{\delta\gamma} D \in A(Q_\gamma)$.

We will show that $[E, D_{\delta\gamma}] = 1$. Since $V_\gamma^\beta$ is abelian, $D_{\delta\gamma}^a$ centralizes $Z_\beta$.

Suppose that $[D_{\delta\gamma}^a, Z_\gamma] \neq 1$. Since $V_\beta^\gamma$ is abelian, $Z_\gamma \leq Q_\beta \cap Q_\gamma$. From 5.5(a) we conclude that $Z_\beta$ is an offender on $D_{\delta\gamma}$ and vice versa. By 2) $A_\gamma = [D_{\delta\gamma}^a, Z_\gamma] \leq Z_{\gamma\beta}$, a contradiction to 1).

Thus $[D_{\delta\gamma}^a, Z_\gamma] = 1$ and $D_{\delta\gamma}^a \leq Q_\beta \cap Q_\gamma$. By symmetry $D_{\delta\gamma} \leq Q_\beta \cap Q_\gamma$. Hence by 5.5(a) $D_{\delta\gamma}$ and $D_{\delta\gamma}^a$ are offenders on each other.

Suppose that $[D_{\delta\gamma}, D_{\delta\gamma}^a] \neq 1$. Then by 2) $A_\gamma \leq [D_{\delta\gamma}, D_{\delta\gamma}^a] \leq Z_{\gamma\beta}^a$, again a contradiction to 1).

Thus $[D_{\delta\gamma}, D_{\delta\gamma}^a] = 1$. Since $D$ centralizes $D_{\delta\gamma}$ and since $D = D^a$, $D$ centralizes $D_{\delta\gamma}^a$. Thus $E$ centralizes $D_{\delta\gamma}^a$ and so $D_{\delta\gamma}^a \leq E$. Note that $C$ is a non-trivial offender on $D_{\delta\gamma}$ and so by 2) $A_\gamma \leq [C, D_{\delta\gamma}]$. Since $a$ centralizes $C$ we get

$A_\gamma \leq [C, D_{\delta\gamma}] \leq [C, E] = [C, D_\beta] \leq Z_{\gamma\beta}$

contradicting 1). This completes the proof of 3).
Suppose that $B \neq C$, that is $B \not\leq Q_\gamma$. By 3) $[B, D_\delta] \leq [B, Z_\gamma] \leq Z_\gamma$ and so $B \leq N_{G_\gamma}(D_\delta Z_\gamma)$. In particular, $B$ normalizes $C_{Q_\gamma}(D_\delta)$. Let $\rho \in \beta^{G_\gamma}$ with $[Q_\beta \cap Q_\gamma, D_\rho] = 1$. Then

$$[Q_\gamma, B] \leq [Q_\gamma, Q_\beta] \leq Q_\beta \cap Q_\gamma \leq C_{Q_\gamma}(D_\rho)$$

So $B$ normalizes $C_{Q_\gamma}(D_\rho)$ . It follows that $B$ normalizes $C_{Q_\gamma}(D_\tau)$ for all $\tau \in \beta^{G_\gamma}$. Since $B \not\leq Q_\gamma$ we conclude that $C_{Q_\gamma}(D_\beta)$ is normal in $\langle B^{G_\gamma} \rangle Q_\beta = G_\gamma$. But then

$$Q_\beta \cap Q_\gamma \leq C_{Q_\gamma}(D_\beta) = C_{Q_\gamma}(D_\delta)$$

a contradiction.

Thus $B = C$. So $B$ centralizes $Z_\gamma$, $Z_\gamma \leq Z_\beta B$ and by 2) $D_\delta B \leq Z_\beta B$. Since $A$ centralizes $B$, we conclude that $A$ normalizes $Z_\gamma Z_\beta$ and $D_\delta Z_\beta$. But then $A$ also normalizes $Q_\gamma \cap Q_\beta$ and $[Q_\gamma \cap Q_\beta, D_\delta Z_\beta]$. Since this latter group is $A_\gamma$ we get a contradiction to 1). 

**Lemma 5.8** [zalpha offender] Let $(\alpha, \beta)$ and $(\gamma, \delta)$ be edges in $\Gamma$ such that $Z_\alpha Z_\delta \leq Q_{\alpha \beta} \cap Q_{\delta \gamma}$ and $[Z_\alpha, Z_\delta] \neq 1$. Then

(a) [a] $Z_\alpha$ is an offender on $Z_\delta$ and vice versa.

(b) [b] $|Z_\alpha Q_\delta / Q_\delta| = |Z_\delta Q_\alpha / Q_\alpha|$.

(c) [c] $G_\alpha = \langle Z_\delta^{G_\alpha} \rangle Q_\alpha$.

**Proof:** (a) and (b) follows from the fact that $Q_{\alpha \beta}$ contains no over-offender on $Z_\alpha$.

Note that $O^p(G_\alpha)Q_\alpha = G_\alpha$ unless $G_\alpha \cong SL_2(2), SL_2(3), Sp_4(2)$ or $G_2(2)$. In each of the four exceptionell case $O^p(G_\alpha)Q_\alpha$ has index $p$ in $G_\alpha$ and $Q_{\alpha \beta} \cap O^p(G_\alpha)Q_\alpha$ contains no non-trivial offender on $Z_\alpha$. Thus (c) follows from (a). 

**Lemma 5.9** [critical pairs] Let $(\alpha, \beta)$ and $(\gamma, \delta)$ be edges in $\Gamma$ such that $Z_\alpha Z_\delta \leq Q_{\alpha \beta} \cap Q_{\delta \gamma}$ and $[Z_\alpha, Z_\delta] \neq 1$.

Then $q := q_\alpha = q_\beta$ and one of the following holds.

1. [1] $G_\alpha \cong G_\delta \cong G_2(q)$.
2. [2]
(a) \[ a \] \( \overline{G}_\alpha \cong Sp_{2n_\alpha}(q) \) and \( \overline{G}_\delta \cong Sp_{2n_\delta}(q) \)
(b) \[ b \] \( |Z_\alpha Q_\delta/Q_\delta| = |Z_\delta Q_\alpha/Q_\alpha| = q \).
(c) \[ c \] \( [Z_\alpha, [Z_\delta, Q_\gamma]] = 1 \) and \( [Z_\delta, [Z_\alpha, Q_{\alpha\beta}]] = 1 \).

3. \[ 3 \]

(a) \[ a \] \( \overline{G}_\alpha \cong Sp_{2n_\alpha}(q), \overline{G}_\delta \cong Sp_{2n_\delta}(q), n_\alpha, n_\delta \geq 2 \),
(b) \[ b \] \( |Z_\alpha Q_\delta/Q_\delta| = |Z_\delta Q_\alpha/Q_\alpha| = q^2 \),
(c) \[ c \] \( [X_{\alpha\beta}, X_{\delta\gamma}] = 1 \).
(d) \[ d \] One of the following holds:
1. \[ \text{[1]} \] \( [X_{\alpha\beta}, Z_\delta] = [X_{\delta\gamma}, Z_\alpha], U_\alpha \) is the natural module for \( G_\alpha \) and \( U_\delta \) is the natural module for \( G_\delta \).
2. \[ \text{[2]} \] \( q = 2, [X_{\alpha\beta}, Z_\delta] \neq [X_{\delta\gamma}, Z_\alpha] \) and \( U_\alpha \cap Z(G_\alpha) = U_\delta \cap Z(G_\delta) \)

4. \[ 4 \]

(a) \[ a \] \( \overline{G}_\alpha \cong SL_{n_\alpha}(q) \) and \( \overline{G}_\delta \cong SL_{n_\delta}(q) \)
(b) \[ b \] \( ||Z_\alpha, Z_\delta|| = q \).

5. \[ 5 \] After interchanging \((\alpha, \beta)\) with \((\delta, \gamma)\) if necessary:

(a) \[ a \] \( \overline{G}_\alpha \cong SL_{n_\alpha}(q), n_\alpha > 2 \) and \( \overline{G}_\delta \cong Sp_{2n_\delta}(q), n_\beta > 1 \)
(b) \[ b \] \( |Z_\alpha Q_\delta/Q_\delta| = |Z_\delta Q_\alpha/Q_\alpha| = q \),
(c) \[ c \] \( [X_{\delta\gamma}, Z_\alpha] = 1 \)
(d) \[ d \] \( ||Z_\alpha, Z_\gamma|| = q \)

Proof:

Let \( I_{\alpha\delta} = \{ ||Z_\alpha, y|| \mid 1 \neq y \in Z_\delta Q_\alpha/Q_\alpha \} \) and \( J_{\alpha\delta} = \{ ||x, Z_\delta|| \mid x \in Z_\alpha \setminus C_{Z_\alpha}(Z_\delta) \}

By \( \text{??(??)} \) implies \( ||Z_\alpha, y|| = ||Z_\delta, y|| \) and \( ||x, Z_\delta|| \), for all \( y \in Z_\delta \) and \( x \in Z_\alpha \). Definition the positive integer \( k_{\alpha\delta} \) by \( |Z_\alpha/C_{Z_\alpha}(Z_\delta)| = q_{\alpha\delta}^k \) and note that

\[ q_{\alpha\delta}^k = |Z_\alpha Q_\delta/Q_\delta| = Z_\delta Q_\alpha/Q_\alpha| = q_{\delta\alpha}^{k_{\delta\alpha}} \]

Also \( Z_\delta \) is a quadratic offender on \( Z_\alpha \) and the action of \( \overline{G}_\alpha \) on \( \overline{Z}_\alpha \) implies:

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Note that the definitions of $I_{\alpha\beta}$ and $J_{\alpha\beta}$ imply $I_{\alpha\beta} = J_{\delta\alpha}$. This allows us to relate $\mathcal{G}_\alpha$ and $\mathcal{G}_\delta$. In particular we see that

$$q := q_\alpha = q_\delta \quad \text{and} \quad k := k_{\alpha\delta} = k_{\delta\alpha}.$$ 

Furthermore, $\mathcal{G}_\alpha \cong G_2(q_\alpha)$ we conclude that also $\mathcal{G}_\delta \cong G_2(q_\delta)$ So (a) holds in this case.

If $\mathcal{G}_\alpha \cong SL_n(q_\alpha)$ and $n_\alpha > 2$, we get $\mathcal{G}_\alpha \cong SL_n(q_\delta)$ or $Sp_n(q_\delta)$. In the latter case we get $k = 1$. In any case since $n_\alpha > 2$, $|[Z_\alpha, Z_\gamma]| = q$ and so (4) or (5) holds.

If $\mathcal{G}_\alpha \cong Sp_{2n_\alpha}(q)$ and $\mathcal{G}_\delta \cong Sp_{2n_\delta}(q)$ we get $k \in \{1, 2\}$. If $k = 1$, (2) holds.

So suppose that $k = 2$. Then clearly $n_\alpha, n_\delta > 2$. We will show that (3) holds. We already proved (3)(a) and (b). Also both $[X_{\alpha\beta}, Z_\delta]$ and $[X_{\delta\gamma}, Z_\alpha]$ have order $q$. It follows that $X_{\alpha\beta}Q_\delta/Q_\delta$ is the unique full transvection group in $Q_{\gamma\delta}/Q_\delta$ and thus (3)(c) holds.

If $q > 2$, then $|[X_{\delta\gamma}, Z_\alpha]| = q$ implies that $U_{\alpha}$ is a natural module and so also $[X_{\alpha\beta}, Z_\delta] = [Z_\alpha, X_{\delta\gamma}] = U_\alpha \cap Z_{\alpha\beta}$. Thus (3) holds in this case.

So suppose that $q = 2$. Note that $U_\alpha \cap Z_{\alpha\beta} = [X_{\alpha\beta}, Z_\delta][Z_\alpha, X_{\delta\gamma}]$. If $[X_{\alpha\beta}, Z_\delta] = [Z_\alpha, X_{\delta\gamma}]$ we conclude that $U_\alpha$ is a natural module and (3) holds. If $[X_{\alpha\beta}, Z_\delta] \neq [Z_\alpha, X_{\delta\gamma}]$ we get that $U_\alpha \cap Z(G_\alpha)$ is the unique subgroup of order two in $[X_{\alpha\beta}, Z_\delta][Z_\alpha, X_{\delta\gamma}]$ distinct from $[X_{\alpha\beta}, Z_\delta]$ and $[Z_\alpha, X_{\delta\gamma}]$. The same is true for $U_\delta \cap Z(G_\delta)$ and again (3) holds.

**Lemma 5.10 [q=2 for g2(q)]** Let $(\alpha, \beta, \gamma, \delta)$ be as in Case 1. of 5.9. Then $q = 2$ and $U_\alpha \cap Z(G_\alpha) = U_\delta \cap Z(G_\delta)$.

**Proof:** The following argument is taken from [MS].

Let $R = [Z_\alpha, Z_\delta]$ and $X = R \setminus \{[x, y] \neq 1 \mid x \in Z_\alpha, y \in Z_\delta\}$. Then it is not too difficult to see that $X = C_{U_\alpha}(G_\alpha) = C_{U_\delta}(G_\delta)$. We will compare the actions of $U_\alpha/X$ on $U_\delta/X$ as seen in $G_\delta$ with the action of $U_\delta/X$ on $U_\alpha/X$.
as seen in $G_\alpha$. Let $F_\alpha = \text{End}_{G_\alpha}(U_\alpha/X)$. Then $F_\alpha$ is a field isomorphic to $GF(q)$.

Let

$$K_{\delta\alpha} = \{ C_{U_\alpha}(y) \mid y \in Z_\alpha, U_\delta \cap Q_\alpha < C_{U_\delta}(y) < U_\delta \}.$$ 

and similarly define $K_{\alpha\delta}$. If $A \in K_{\delta\alpha}$ then $C_{U_\alpha}(A) \neq U_\alpha \cap Q_\delta$ and $C_{U_\alpha}(A)/R$ is a 1-dim. $F_\alpha$-subspace of $U_\alpha/R$. Also $C_{U_\alpha}(A) = C_{U_\alpha}(a)$ for all $a \in A \setminus Q_\alpha$. So $C_{U_\alpha}(A) \in K_{\alpha\delta}$ and we obtained a bijection between $K_{\alpha\delta}$ and $K_{\delta\alpha}$. Moreover, $A$ is a long root subgroup of $G_\alpha$. Let $t \in Z_\alpha$ with $[t, A] \neq 1$.

We show next that

(*) $[t, A]X/X$ is a 1-dim. $F_\alpha$ and $F_\delta$ subspace of $R/X$ and a

Clearly it is a 1-dim $F_\delta$-subspace. Let $P = C_{G_\alpha}(A)$. Then $W := U_\alpha/C_{U_\alpha}(A)$ is a natural module for $P/O_\rho(P) \cong SL_2(q)$. Let $t^*$ be the image of $t$ in $W$. Then $S := C_P(t^*)$ is a Sylow $p$-subgroup of $P$ and so of $G_\alpha$. Since $S$ centralizes $[t, A]$ we conclude that $[t, A]X/X = C_{U_\alpha/X}(S)$, which is a 1-dim. $F_\alpha$-space.

The preceding argument also shows that every 1-dim. $F_\alpha$ subspace of $[U_\alpha, A]X/X$ is of the form $[t, A]$ for some $t \in Z_\alpha$. Moreover each 1-dim. $F_\alpha$ subspace of $R/X$ is contained in $[U_\alpha, A]X/X$ for some $A \in K_{\delta\alpha}$. Thus (*) implies

(**) The $F_\alpha$ and $F_\delta$ subspaces in $R/X$ coincide.

Let $W_{\alpha\beta} = [U_\alpha, O_p(P_{\alpha\beta})]X$ and $U_{\alpha\beta} = C_{U_\alpha}(O_p(P_{\alpha\beta}))$. Then $U_{\alpha\beta}/X$ is a 1-dim. $F_\alpha$ subspace of $R/X$. Moreover, $U_{\alpha\delta} \leq [U_\alpha, A]X$ for all $A \in K_{\delta\alpha}$. Considering the action of $U_\alpha Q_\delta/Q_\delta$ on $U_\delta/X$ we conclude that $U_{\alpha\beta} = U_{\gamma\delta}$.

Fix $z \in U_\alpha \setminus W_{\alpha\beta}$ and define $Y/U_{\alpha\gamma} := C_{U_\delta/U_{\delta\gamma}}(z)$. Then $Y/R$ is a 1-dimensional $F_\delta$ subspace of $U_\delta/R$. Since $[Y, z] \leq U_{\alpha\delta} = U_{\alpha\delta}$ we also have $[Y, F_{\alpha}zX/X] \leq U_{\alpha\delta}$. Since $[z, Q_{\alpha\beta}]R = W_{\alpha\beta}$, the Frattini-argument shows that $L := C_{P_{\alpha\beta}}(zR/R)$ has a quotient $SL_2(q)$. Since $L$ normalizes $Y$, we conclude that $YQ_{\alpha}/Q_{\alpha}$ is a short root subgroup of $G_\alpha$.

Hence there exists a subgroup $M$ of $G_\alpha$ with $YQ_{\alpha}/Q_{\alpha} \leq M$ and $M \cong SL_2(q)$. Note that for all $t \in Y_\alpha$, $[t, Y]X/X$ is an $F_\alpha$-submodule of $R/X$. Hence $[t, Y]X/X$ is also an $F_\alpha$-submodule of $U_\alpha/X$. But this implies that $U_\alpha/X$ is as an $F_\alpha M$-module the direct sum three isomorphic natural module. But this implies $q = 2$. (For example let $P$ be a minimal parabolic of $G_\alpha/Q_{\alpha}$ with $M$ as a Levi complement, $V_1 = C_{U_\alpha/X}(O_p(P))$ and $V_2 = [U_\alpha/X, O_p(P)]/V_1$. Then $O_p(P)/\Phi(O_p(P))$ is isomorphic to a $F_p$-submodule
of $\text{Hom}_{F_\alpha}(V_2, V_1)$. Since $V_2$ and $V_1$ are isomorphic $F_\alpha M$ modules, we conclude that every composition factor for $M$ in $O_p(P)$ is either natural or trivial. Thus $q = 2$.

Comment: a quote from [BBSM] would be more appropriate

\begin{lemma} [b=1 sigma=2] \label{lemma511} Suppose that $|\Sigma| = 2$, $\Sigma = \{\alpha, \beta\}$ and $[Z_\alpha, Z_\beta] \neq 1$. Then for $\gamma \in \Sigma$ there exists $K_\gamma \leq \Omega_1 Z(G_\gamma)$ and $L_\gamma \leq G_\gamma$ such that $G_\gamma = K_\gamma \times L_\gamma$ and one of the following holds.

1. [1] $L_\alpha \sim L_\beta \sim q^n SL_n(q)$ and $|K_\alpha| = |K_\beta| \leq q$.

2. [2] $p = 2$ and (after interchanging $\alpha$ and $\beta$ if necessary), $G_\alpha = L_\alpha \sim q^{1+2n}Sp_{2n}(q)$, $G_\beta = L_\beta \sim q^{1+2+2(2n-2)}SL_2(q)$.

3. [3] $p = 2$, $L_\alpha \sim L_\beta \sim 2^6 G_2(2)$ and $|K_\alpha| = |K_\beta| \leq 2^3$.

4. [4] $p = 2$ and $G_\alpha = L_\alpha \sim G_\beta = L_\beta \sim q^{1+6+8}Sp_6(q)$.

5. [5] $p \neq 2$, $L_\alpha \sim L_\beta \sim q^{2n}Sp_{2n}(q)$, $n \geq 2$ and $|K_\alpha| = |K_\beta| \leq q$.

6. [6] $q = 2$, $G_\alpha \sim 2^{1+2n}Sp_{2n}(2)$ and $G_\beta \sim 2^{1+2+1-m+1-m+2k}SL_2(2)$ for some $m, k$ with $m + k = n - 2$ and $k$ even.

7. [7] who knows

\end{lemma}

Proof:

By assumption, $[Z_\alpha, Z_\beta] \neq 1$. Clearly $Z_\alpha Z_\beta \leq Q_{\alpha\beta}$ and we can apply \ref{59} with $(\delta, \gamma) = (\beta, \alpha)$.

For $\langle \gamma, \delta \rangle = \{\alpha, \beta\}$ define $H_\gamma = (Z_\delta^G_{\gamma})$. Let $R = [Z_\alpha, Z_\beta]$, $I = \{1 \neq [x, y] \mid x \in Z_\alpha, y \in Z_\beta\}$ and $D_\gamma = C_{Q_\gamma}(O_p(G_\gamma))$.

We divide the proof in a series of Steps.

Step 1 [da cap db] $D_\alpha \cap D_\beta = 1$.

Proof: This holds since $D_\alpha \cap D_\beta$ is normalized by $G_\alpha = O^p(G_\alpha)Q_{\alpha\beta}$ and $G_\beta = O^p(G_\beta)Q_{\alpha\beta}$.

We call $\alpha$ non-abelian if $\alpha$ is symplectic, $p \neq 2$ and $n_\alpha \geq 2$. Otherwise $\alpha$ is called abelian.

Step 2 [abelian]
(a) \(A\) \(\alpha\) is abelian if only if \(Q_\alpha/\alpha\) is elementary abelian.

(b) \(B\) If \(\alpha\) is abelian, then \(\Phi(\beta) \leq D_\beta\).

(c) \(C\) If \(\alpha\) and \(\beta\) are abelian, then \(Q_\alpha \cap Q_\beta\) is elementary abelian.

**Proof:** (a) is obvious. If \(\Phi(\beta) \leq \alpha\), then \(Z_\alpha\) centralizes \(\Phi(\beta)\) and so \(\Phi(\beta) \leq D_\alpha\). Thus (b) holds.

Since \(\Phi(Q_\alpha \cap Q_\beta) \leq \Phi(Q_\alpha) \cap \Phi(Q_\beta)\), Step 1 and (b) imply (c).

\(\square\)

**Step 3** \(B=1\) case 1] Suppose that 5.9(1) holds. Then 5.11(3) holds.

**Proof:** Note first that \(Q_\alpha \leq Q_\alpha \beta = Z_\alpha \beta\). Thus \(Q_\alpha = Z_\alpha (Q_\alpha \cap Q_\beta)\) and Step 2(c) implies that \(Q_\alpha\) is elementary abelian. Thus by 5.3(a), \(Q_\alpha = Z_\alpha\).

By 5.10, \(q = 2\) and

\[U_\alpha \cap Z(G_\alpha) = U_\beta \cap Z(G_\beta) \leq D_\alpha \cap D_\beta = 1.\]

Thus \(|U_\alpha| = 2^6\).

By [Schur, Schur Multiplier] we get \(O^2(G_\alpha)/U_\alpha \cong G_2(2)'\). Since \(G_\alpha = Q_\alpha Z_\alpha O^2(G_\alpha)\) and \([Q_\alpha, Z_\beta] \leq [U_\alpha, Z_\beta] \leq U_\alpha \leq O^2 * G_\alpha\) we get that \(G_\alpha/O^2(G_\alpha)\) is elementary abelian. Hence there exists \(L_\alpha \leq G_\alpha\) with \(G_\alpha = D_\alpha \times L_\alpha\) and \(L_\alpha \sim 2^6 G_2(2)\). Since \(D_\alpha \leq Z_\alpha \beta\) and \(D_\alpha \cap D_\beta = 1\) we have \(|D_\alpha| \leq |Z_\alpha \beta/D_\beta| = 2^d\), a proof of Step 3 is complete.

\(\square\)

**Step 4** \(B=1\) case 2] Suppose that 5.9(2) holds. Then

**Proof:**

Let \(D_{\beta \alpha} = [Z_\alpha, Q_{\alpha \beta}]\) and \(A_{\beta \alpha} = [D_{\beta \alpha}, Q_{\alpha \beta}] \leq Z_\alpha \beta\).

We will show first

1) \(6\) \([D_{\beta \alpha}, Q_{\alpha}] \leq \Omega_1 Z(G_\alpha)\). In particular, either \(D_{\beta \alpha} \leq Z_\alpha\) or \(A_{\beta \alpha} \leq \Omega_1 Z(G_\alpha)\).

Choose \(\delta \in \beta G_\alpha\) with \([Z_{\delta \alpha}, Z_\beta] \neq 1\). If \([D_{\delta \alpha}, D_{\beta \alpha}] \neq 1\), then

\([D_{\delta \alpha}, Q_{\alpha}] \leq A_{\beta \alpha} = [D_{\delta \alpha}, D_{\delta \alpha}] \leq Z_{\alpha \beta} \cap Z_{\alpha \delta} \leq \Omega_1 Z(G_\alpha)\)

So suppose that \([D_{\delta \alpha}, D_{\beta \alpha}] = 1\). Then \([D_{\delta \alpha}, Z_{\beta}] \leq Z_{\alpha \beta} \leq Z_\alpha\) and so \(D_{\beta \alpha} Z_\alpha\) is normal in \(G_\alpha = (Q_{\alpha \delta}, Z_\beta)\). Hence also \([D_{\beta \alpha}, Q_{\alpha}]\) is normal in \(G_\alpha\).

Since \(Q_{\alpha \beta}\) centralizes \(D_{\beta \alpha}\) and \(G_\alpha = (Q_{\alpha \beta})\), the first statement in 1) hold.

If \([D_{\beta \alpha}, Q_{\alpha}] = 1\) then since \(\Omega_1 Z(Q_{\alpha}) = 1\) we get \(D_{\beta \alpha} \leq Z_\alpha\). If \(D_{\beta \alpha}, Q_{\alpha} \neq 1\), then \(A_{\beta \alpha} = [D_{\beta \alpha}, Q_{\alpha}] \leq \Omega_1 Z(G_\alpha)\), completing the proof of 1).

Next we prove:
2) [7] If \([D_{\beta \alpha}, Q_{\alpha}] = 1\), then \(D_{\beta \alpha} \leq Z_{\alpha} \cap Q_{\beta} = D_{\alpha \beta}Z_{\alpha \beta}\). 3.4 implies.

By 5.3, \(D_{\beta \alpha} \leq Z_{\alpha}\). Also \(D_{\beta \alpha} \leq Z_{\beta} \leq Q_{\beta}\) and so 2) holds.

3) [8] If \(p\) is odd, then 1. or 5 of 5.11 holds.

If \([D_{\beta \alpha}, Q_{\alpha}] \neq 1\), then by 1), \(R = A_{\beta \alpha} = [D_{\beta \alpha}, Q_{\alpha}] \leq Z(G_{\alpha})\) a contradiction. Thus \([D_{\beta \alpha}, Q_{\alpha}] = 1\) and by 2) \(D_{\beta \alpha} \leq D_{\alpha \beta}Z_{\alpha \beta}\). By symmetry \(D_{\alpha \beta} \leq D_{\beta \alpha}Z_{\alpha \beta}\). Hence \(Z_{\alpha} \cap Z_{\beta} = Z_{\alpha} \cap Q_{\beta} = Z_{\beta} \cap Q_{\alpha}\). Thus \(Z_{\alpha} \cap Z_{\beta}/Z_{\alpha \beta} = q^{2n_{\alpha} - 2}\) and \(n_{\alpha} = n_{\beta}\). Since \(Q_{\alpha} \leq Z_{\alpha}Q_{\beta}\) we get that \(Q_{\alpha} \cap Q_{\beta}\) is elementary abelian, \(Q_{\alpha} = Z_{\alpha}\) and \(Q_{\beta} = Z_{\beta}\). Also \(D_{\alpha} \leq Z(G_{\alpha})\), \(D_{\alpha} \leq Z_{\alpha \beta}\) and \(D_{\alpha} \cap D_{\beta} = 1\). Thus \(|D_{\alpha}| \leq q\). Hence 5. holds and 3) is proved.

We may assume from now on that \(p = 2\). Set \(D = D_{\alpha \beta}D_{\beta \alpha}\) and \(T = C_{Q_{\alpha \beta}}(D)\). By ?? \(Q_{\alpha} \cap Q_{\beta}\) is elementary abelian. Since \(C_{Q_{\alpha \beta}}(D_{\alpha \beta} = Z_{\beta}Q_{\alpha}\) we have \(T = Z_{\alpha}Z_{\beta}(Q_{\alpha} \cap Q_{\beta})\). Since \(p = 2\) we conclude that

4) [10] \(A(T) = \{Z_{\alpha}(Q_{\alpha} \cap Q_{\beta}), Z_{\beta}(Q_{\alpha} \cap Q_{\beta})\}\)

Let \(A \in A(Q_{\alpha \beta})\). Then \(C_{A}(D_{\alpha \beta})D_{\alpha \beta}\) is in \(A(Q_{\alpha \beta})\). Then \(C_{A}(D) \in A(T)\) and so \(C_{A}(D)D = Z_{\gamma}(Q_{\alpha} \cap Q_{\beta})\) for some \(\gamma \in \{\alpha, \beta\}\). In particular, \(C_{A}(D)D \leq Q_{\gamma}\). Let \(\{\alpha, \beta\} = \{\gamma, \delta\}\). Since \(E := C_{A}(D_{\gamma})D_{\delta} \in A(Q_{\alpha \beta})\), \(E\) is an offender on \(Z_{\gamma}\). Moreover, \(C_{E}(D) \leq C_{A}(D)D \leq Q_{\gamma}\), the action of \(Q_{\gamma \delta}\) on \(Z_{\gamma}\) implies \(E \leq Q_{\gamma}\). Since \(E \in A(Q_{\alpha \beta})\) we conclude, \(Z_{\gamma} \leq E\).

Thus \([Z_{\gamma}, A] \leq [E, A] \leq [D_{\delta \gamma}, A]\). Suppose that \([Z_{\gamma}, A] \neq 1\), then also \([Z_{\gamma}, A] \leq Z(G_{\gamma})\) and 1) implies \([D_{\delta \gamma}, Q_{\gamma}] = 1\). By 2), we get \(D_{\delta \gamma} \leq D_{\gamma}Z_{\delta \gamma}\), so \(Z_{\gamma} \leq AD_{\gamma}Z_{\delta \gamma}\) and thus \(Z_{\gamma} = C_{Z_{\gamma}}(A)D_{\gamma}\). This implies \([Z_{\gamma}, A] = 1\). So \([Z_{\gamma}, A] = 1\) and \(A \leq Q_{\gamma}\). Hence

5) [11] \(A(Q_{\alpha \beta}) = A(Q_{\alpha}) \cup A(Q_{\beta})\).

Since \(Q_{\alpha \beta} = J(Q_{\alpha \beta})\) we conclude \(Q_{\alpha \beta} = J(Q_{\alpha})J(Q_{\beta})\). In particular \(Q_{\alpha} \leq J(Q_{\alpha})Q_{\beta}\) and so \(Q_{\alpha} = J(Q_{\alpha})(Q_{\alpha} \cap Q_{\beta})\). Since \(Z_{\alpha}(Q_{\alpha} \cap Q_{\beta}) \in A(Q_{\alpha \beta})\) we get \(Q_{\alpha} = J(Q_{\alpha})\). Thus

6) [12] \(Q_{\alpha} = J(Q_{\alpha}), Q_{\beta} = J(Q_{\beta})\) and \(Q_{\alpha \beta} = Q_{\alpha}Q_{\beta}\).

Let \(A \in A(Q_{\alpha})\). Then \(Z_{\alpha} \leq A\) and \(C_{A}(D_{\beta \alpha})D_{\beta \alpha} = Z_{\alpha}(Q_{\alpha} \cap Q_{\beta})\). Thus \(Q_{\alpha} \cap Q_{\beta} = (A \cap Q_{\beta})D_{\beta \alpha}\) and \([Q_{\alpha} \cap Q_{\beta}, A] = [D_{\beta \alpha}, A] \leq A_{\beta \alpha} \leq Z_{\beta}\).

So

7) [13] \([Q_{\alpha} \cap Q_{\beta}, Q_{\beta}] \leq A_{\alpha \beta}\) and \([Q_{\alpha} \cap Q_{\beta}, Q_{\alpha \beta}] \leq A_{\alpha \beta}A_{\beta \alpha} \leq Z_{\alpha \beta}\)
Let \( \widehat{Q}_\beta = Q_\beta/Z_\beta \). We conclude that

8) \([14]\) \([Q_\alpha \cap \widehat{Q}_\beta]Z_\beta, Q_\alpha = 1 \) and \([\widehat{Q}_\beta, Q_\alpha] \leq Q_\alpha \cap Q_\beta \)

We will now prove

9) \([9]\) Suppose \( p = 2, \) and \( D_{\beta_\alpha}Z_\alpha \) is normal in \( G_\alpha, \) then \( 1. \) or \( 2. \) of 5.11 holds.

Since \([Q_\alpha, Z_\beta] \leq D_{\beta_\alpha} \) and \([D_{\beta_\alpha}, Z_\beta] = 1 \) we get \([Q_\alpha, O^p(G_\alpha)] \leq Z_\alpha \). Let \( \overline{Q}_\alpha = Q_\alpha/D_\alpha \). Then \( Q_\alpha \) centralizes \( \overline{Q}_\alpha, C_{\overline{Q}_\alpha}(O^p(G_\alpha)) = 1 \) and \([\overline{Q}_\alpha, O^p(G_\alpha)] = U_\alpha \) is a natural module. Thus the structure of \( \overline{Q}_\alpha \) is determined by 3.4. From \([Q_\alpha \cap Q_\beta, Z_\beta] = 1, Q_\alpha Q_\beta = Q_\alpha Z_\beta \) and \((^*\)\) we get \( Q_\alpha \cap Q_\beta = D_{\beta_\alpha} \). Hence \( Q_\alpha \cap Q_\beta \leq D_\alpha D_{\beta_\alpha} \) and so

\[
Q_\alpha \cap \beta = (D_\alpha \cap Q_\beta)D_{\beta_\alpha}
\]

Since \([D_\alpha \cap Q_\beta, Z_\beta] \leq D_\alpha \cap D_\beta = 1 \) we have \( D_\alpha \cap Q_\beta \leq Z_\beta \). As \( Z_\alpha \) centralizes \( D_\alpha, D_\alpha \cap Q_\beta \subseteq Z_\beta \cap Q_\alpha = D_{\beta_\alpha}Z_\alpha \beta \). We conclude

\[
Q_\alpha \cap Q_\beta = D_{\beta_\alpha}D_{\beta_\alpha}Z_\alpha \beta \text{ and } T = Z_\alpha Z_\beta = U_\alpha Z_\beta
\]

Since \( Q_\beta \) centralizes \( D_{\beta_\alpha}, 3.4 \) implies \( D_{\beta_\alpha} \leq D_\alpha Z_\alpha \beta \) and so

\[
D_{\beta_\alpha}Z_\alpha \beta = (D_\alpha \cap (D_{\beta_\alpha}Z_\alpha \beta))Z_\alpha \beta.
\]

Note that \( r := |Q_\alpha/D_\alpha U_\alpha| \leq q \). Let \( F = O^p(G_\alpha) \cap Q_\alpha \beta \). Then \( U_\alpha \leq F \) and \(|Q_\alpha \beta/F| = e \), where \( e = 2 \) if \((n_\alpha, q) = (2, 2) \) or \((1, 2) \) and \( e = 1 \) otherwise. Since \( D_{\beta_\alpha} \leq D_\alpha Z_\alpha \) and so \( F \leq U_\alpha Q_\beta \) and \( F = U_\alpha(F \cap Q_\beta) \). Let \( F_1 = C_F(D_{\beta_\alpha}) \). Since \( F \) centralizes \( D_{\beta_\alpha}, F_1 \leq T = U_\alpha Z_\beta \). Since \( U_\alpha \leq F_1, F_1 = U_\alpha(F_1 \cap Z_\beta) \).

Suppose that \( G_\alpha/Q_\alpha \cong Sp_2(2) \). Then \( Q_\alpha = D_\alpha \times U_\alpha \). Moreover \( Q_\beta \leq Z_\beta Q_\alpha \) and \( Q_\beta = Z_\beta(Q_\alpha \cap Q_\beta) = Z_\beta D_{\alpha \beta} = Z_\beta \). Since \([D_\alpha, Z_\beta] \leq R \cap D_\alpha = 1, D_\alpha \leq Z_\beta \). Thus \( D_\alpha \) is abelian and \( D_\alpha \) is centralized by \( D_\alpha U_\alpha Z_\beta = Q_\alpha \beta \). Thus \( D_\alpha \leq Z_\alpha \beta \) and \( Q_\alpha = Z_\alpha \). Hence \( Z_\beta \cap Q_\alpha = Z_\alpha \beta \) and so \( G_\beta/Q_\beta \cong \text{SL}_2(2) \). Thus 1. or 2. of ?? holds in this case.

Suppose that \( G_\alpha/Q_\alpha \notin \{Sp_2(2), Sp_4(2)\} \). Then \( F_1 \cap Z_\beta \notin Q_\alpha \). Since \( D_\alpha \) centralizes \( F_1 \cap Z_\beta \) we conclude that \( D_\alpha \leq Q_\beta \). Since \(|Q_\alpha \beta/D_\alpha(F \cap Q_\beta)Z_\beta \leq r q \leq q^2 \) we get \(|Q_\alpha \beta/Q_\beta| \leq q^2 \) and so \( q_\beta = 1 \). Thus \( D_{\beta_\alpha} \leq Z_\alpha \beta \) and so \( Q_\alpha \cap Q_\beta = D_{\beta_\alpha}Z_\alpha \beta = Z_\alpha \cap Q_\beta \). Moreover, \( Q_\alpha \leq U_\alpha Q_\beta \) and so \( Q_\alpha = U_\alpha(Q_\alpha \cap Q_\beta) = Z_\alpha \). Assume that \((Z_\alpha \cap Q_\beta)Z_\beta \) is normal in
We claim that \( G_\beta \cong SL(2) \), the preceding paragraph gives a contradiction. If \( G_\beta/Q_\beta \cong Sp_4(2) \) ???. And if \( G_\beta/Q_\beta \not\in \{Sp_2(2), Sp_4(2)\} \), the first half of this paragraph applied with the roles of \( \alpha \) and \( \beta \) reversed, gives \( n_\alpha = 1 \). But then case (1) or (2) holds. Assume now that \((Z_\alpha \cap Q_\beta)\) is not normal in \( G_\beta \). Let \( W = (Z_\alpha \cap Q_\beta)Z_\beta, V = \langle W^{G_\beta} \rangle \) and \( U = \bigcap_{g \in G_\beta} W^g \). Since \( [W, Q_\beta] \leq Z_\beta \leq U \) and \( [V, Q_\alpha] \leq Q_\alpha \cap Q_\beta \leq W \) we have \( [V, Q_\alpha \beta] \leq W \) and \( [W, Q_\alpha \beta] \leq U \). Thus we can apply 3.3 to \( V/U \) and conclude that \( W = [Z_\alpha, V]U \). Hence

\[
Z_\alpha \cap Q_\beta = [Z_\alpha, V](Z_\alpha \cap U)
\]

We claim that \( Z_\alpha \cap U = C_{Z_\alpha}(V) \). Indeed, \( U \leq Z(V) \) and so \( Z_\alpha \cap V \leq C_{Z_\alpha}(V) \). For the converse let \( g \in G_\beta \). Then \( [C_{Z_\alpha}(V), Z_\alpha^3] \leq R^g \leq Z_\alpha \) and so \( C_{Z_\alpha}(V)Z_\beta \) is normal in \( G_\beta \). Thus \( C_{Z_\alpha}(V) \leq U \). This proves the claim and so

\[
Z_\alpha \cap Q_\beta = [Z_\alpha, V]C_{Z_\alpha}(V).
\]

The action of \( Q_\alpha \beta \) on \( Z_\alpha \) implies \( [Z_\alpha, V] \cap C_{Z_\alpha}(V) \leq Z_\alpha \). Let \( V^* = [V, H_\beta] \). Since \( H_\beta \) is generated by two conjugates of \( Z_\alpha \) we derive

\[
V/Z_\beta = V^*/Z_\beta \times U/Z_\beta
\]

\( U \leq X \leq Z(V) \) with \( [X, Q_\alpha \beta] \leq U \). Then \( X \leq W \) and so \( X = Z_\beta(X \cap Z_\alpha) \). Since \( Z(V) \cap Z_\alpha \leq U \) we conclude that \( X \leq Z(V) \). Since \( Q_\alpha \beta \) normalizes \( Z(V)/U \) we get \( U = Z(V) \). Since \( [W, Q_\beta] = A_{\alpha \beta} \) and \( \Phi(Q_\beta) \leq D_\beta \) we get that \( A_\beta := A_{\alpha \beta} \leq Z(G_\beta) \) and \( A_\beta = [V, Q_\beta] \). Hence also \( [V^*, Q_\beta] = A_\alpha \). Put \( D^* = C_{Q_\beta}(V^*) \). Then \( Q_\beta/D^* \) is dual to \( V^*/Z_\beta \) as \( G_\beta \) module. Hence \( Q_\beta = V^*D^* \). Note that \( [D^*, O^p(G_\beta)] \leq Z_\beta \). Suppose that \( q \neq 2 \). Then

\[
[Z_\alpha, Q_\beta] \leq ([D^*V^*O^p(G_\beta), D^*] \cap Z_\alpha)[Z_\alpha, V] \leq (D_\beta \cap Z_\alpha)[Z_\alpha, V]
\]

But \( D_\beta \cap Z_\alpha \) is \( \square \) \( \square \)

For \( \alpha \in \Sigma \) let

\[
\Sigma_1(\alpha) = \{ \beta \in \Sigma \mid [Z_\alpha, Z_\beta] \neq 1 \}
\]

and

\[
\Sigma_2(\alpha) = \{ \beta \in \Sigma \mid [Z_\alpha, Z_\beta] = 1 \neq [Z_\alpha, V^*_\beta] \}
\]

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Lemma 5.12 Let \( \alpha \in \Sigma \) and \( \beta \in \Sigma_1(\alpha) \). Define the conjugacy classes \( \langle G_\alpha, G_\beta \rangle \),\( L^* := \langle \Omega_1 Z(R^L) \rangle \), \( K := O_R(\{G_\alpha, G_\beta\}) \) and \( \widetilde{L} := L/K \). For \( \{\alpha, \beta\} = \{\gamma, \delta\} \), put \( K_\gamma = C_{Q_\gamma}(\langle Z_{G_\delta}^{G_{\gamma}} \rangle) \). Then for \( \gamma \in \{\alpha, \beta\} \) there exists a normal subgroup \( L_\gamma \) of \( G_\gamma \) such that

(a) [a] \[ K, L^* \] = 1.

(b) [b] \( K = K_\alpha \cap K_\beta \) and \( \Phi(K_\alpha K_\beta) \leq K \).

(c) [c] \( G_\alpha = K_\alpha L_\alpha \) and \( G_\beta = K_\beta L_\beta \).

(d) [d] Interchanging \( \alpha \) and \( \beta \) if necessary one of the following holds (where \( q \) is a power of \( p \)).

1. [1] \( \bar{L}_\alpha \sim \bar{L}_b \sim q^n SL_n(q) \).
2. [2] \( p = 2, \bar{L}_\alpha \sim q^{1+2n} SL_2(q), \) and \( \bar{L}_\beta \sim q^{1+2+2(2n-2)} SL_2(q) \).
3. [3] \( p = 2 \) and \( \bar{L}_\alpha \sim \bar{L}_\beta \sim 2^6 G_2(2) \).
4. [4] \( p = 2 \) and \( \bar{L}_\alpha \sim \bar{L}_\beta \sim q^{1+6+8} Sp_6(q) \).

Proof: Note that \( K \) is normal in \( L \) and \( K \leq R \), indeed \( K \) is the largest normal subgroup of \( L \) contained in \( R \). Let \( g \in K \) then

\[ [\Omega_1 Z(R)^9, K] = [\Omega_1 Z(R)^9, K^9 = [\Omega_1 Z(R), K]^9 = 1. \]

Thus (a) holds.

Let \( H_\alpha = \langle Z_{\bar{G}_\alpha}^{G_{\alpha}} \rangle, R = [Z_\alpha, Z_\beta] \) and \( D_{\alpha\beta} = [Z_\beta, Q_{\alpha\beta}] \).

Note that by (a), \( K \leq K_\alpha \cap K_\beta \) also \( K_\alpha \cap K_\beta \) is normalized by

\[ \langle O^2(G_\alpha), O^2(G_\beta), Q_{\alpha\beta} \rangle = L. \]

Thus \( K = K_\alpha \cap K_\beta \). So the first part of (b) holds. By definition \([K_\alpha, Z_\beta] = 1\) and so \( K_\alpha \leq Q_\beta \). Thus \( \Phi(K_\alpha) \leq \Phi(Q_\beta) \cap K_\alpha \). Note that \( \Phi(Q_\beta) \leq \Phi(Q_{\alpha\beta}) \).

Since \( Q_{\alpha\beta}/Q_\alpha \) is elementary abelian, unless \( \alpha \) is symplectic, \( n_\alpha > 1 \) and \( p \neq 2 \), we get

\[ (*) \quad \Phi(K_\alpha) \leq K \) and \( [\Phi(Q_\beta), H_\beta] = 1 \), unless \( \alpha \) is symplectic, \( n_\alpha > 1 \) and \( p \neq 2 \).

Note that by definition of \( \Sigma_1(\alpha) \), \([Z_\alpha, Z_\beta] \neq 1\). Clearly \( Z_\alpha Z_\beta \leq Q_{\alpha\beta} \) and we can apply 5.9 with \( (\delta, \gamma) = (\beta, \alpha) \).
Suppose that Case c.1 of 5.9 holds. Then $Q_\alpha \leq Q_{\alpha \beta} = Z_\alpha Q_\beta$. Since $Q_\alpha$ normalizes $Z_\beta$, $H_\alpha$ is generated by two conjugates of $Z_\beta$. Thus $|Q_\alpha/K_\alpha| \leq q^6$ and so $Q_\alpha = K_\alpha U_\alpha$. By 5.10, $q = 2$ and $U_\alpha \cap Z(G_\alpha) = U_\beta \cap Z(G_\beta)$. Thus $U_\alpha \cap Z(G_\alpha) \leq K$ and $|\tilde{U}_\alpha| = 2^6$. Using [Schur, Schur Multiplier] we get $O^2(G_\alpha)/U_\alpha \cong G_2(2)'$ also by (*) $G_\alpha/O^2(G_\alpha)K$ is elementary abelian. Hence there exists $L_\alpha \leq G_\alpha$ with $O^2(G_\alpha)K \leq L$, $G_\alpha = K_\alpha L_\alpha$ and $L_\alpha \cap K_\alpha = K$. Thus d.3 holds in this case.

Suppose next that Case c.2 of 5.9 holds.

Suppose that $n_\beta = 1$. Then $[Q_\alpha, Z_\beta] \leq [Z_\alpha, Z_\beta] \leq U_\alpha$ and so $[Q_\alpha, H_\alpha] \leq U_\alpha$. Also $\Phi(Q_\alpha) \leq Q_\beta$ and so $[\Phi(Q_\alpha), H_\alpha] = 1$. Suppose that also $n_\alpha = 1$. Then $H_\alpha$ is generated by two conjugates of $Z_\beta$ and we conclude that $|Q_\alpha/K_\alpha| = q^2$ and $Q_\alpha = K_\alpha U_\alpha$. Let $I = \{1 \neq [x, y] \mid x \in Z_\alpha, y \in Z_\beta\}$. If $q \leq |[Z_\alpha, Z_\beta]| = q^2$ then $U_\alpha \cap Z(G_\alpha) = [Z_\alpha, Z_\beta] \cap I = U_\beta \cap Z(G_\beta)$ and thus d.1 holds. If $|[Z_\alpha, Z_\beta]| = q^2$, then $[Z_\alpha, Z_\beta] \cap I$ contains exactly two subgroups of order $q$ and these two subgroups have trivial intersection. Hence either $U_\alpha \cap Z(G_\alpha) = U_\beta \cap Z(G_\beta)$ and d.1 holds; or $U_\alpha \cap Z(G_\alpha) \cap U_\beta \cap Z(G_\beta) = 1$ and d.2 holds.

Suppose next that $n_\beta > 1$ and that $D_{\beta \alpha} Z_\alpha$ is normal in $G_\alpha$. Then $A_\alpha := [D_{\beta \alpha}, Q_{\alpha \beta}] = [D_{\beta \alpha}, Z_\alpha, Q_\alpha]$ is normal in $G_\alpha$. Since $Q_{\alpha \beta}$ centralizes $A_\alpha$ we get $A_\alpha \leq Z(G_\alpha)$. Let $D_\alpha := C_{Q_\alpha}(O^p(G_\alpha))$. We conclude that $D_{\alpha \beta} \leq U_\alpha D_\alpha$ and $D_{\alpha \beta} \leq D_\alpha Z_\beta$. Note that $[Q_\alpha, Z_\beta] \leq D_{\beta \alpha}$ and so $[Q_\alpha, H_\alpha] \leq U_\alpha D_\alpha$.

Note that $|Q_\alpha/RA_\alpha| \geq q$ and so $p = 2$ and $|U_\beta \cap Z(G_\beta)| = q$. By (*) $[\Phi(Q_\alpha), H_\alpha] = 1$. Thus $|Q_\alpha/UA_\alpha D_\alpha| \leq q$. Note that $O^2(G_\alpha) \cap Q_{\alpha \beta}$ centralizes $D_\alpha Z_\alpha$ and so we have $O^2(G_\alpha) \cap Q_{\alpha \beta} \leq C_{Q_\alpha}(D_{\beta \alpha}) = Z_\alpha Q_\beta$. Note also that $Z_\beta \leq Q_\beta$, $G_\alpha = O^2(G_\alpha)Z_\beta$ and $Z_\alpha \leq Q_\alpha$. Thus $Q_{\alpha \beta} = Q_\alpha Q_\beta$.

If $q > 2$, then $A_\alpha \leq R$ and we conclude that $A_\alpha = U_\alpha \cap Z(G_\alpha)$.

Let $\gamma \in \beta G_\alpha$ with $[Z_{\gamma \alpha}, Z_\beta] \neq 1$.

**Lemma 5.13** [sigma symmetric] Let $\alpha, \beta \in \Sigma$ and $i \in \{1, 2\}$. Then $\alpha \in \Sigma_i(\beta)$ if and only if $\beta \in \Sigma_i(\alpha)$.

**Proof:** For $i = 1$ this is obvious. Suppose now that $\beta \in \Sigma_2(\alpha)$ but $\alpha \notin \Sigma_2(\beta)$. The $Z_\alpha Z_\beta \leq Q_\alpha \cap Q_\beta$, $V_\alpha^\beta \leq Q_\alpha$ and $V_\beta^\alpha \leq Q_\beta$.

**Lemma 5.14** [vdelta non abelian] There exists an edge $(\gamma, \delta)$ in $\Gamma$ such that $\langle Z_\delta^G \rangle$ is not abelian.

**Proof:** Suppose not. Let $V = \langle Z_L \rangle L \in \Sigma$ and $Q = \bigcap O_p(L)$ $L \in \Sigma$. Then $V \leq Q$ and so $Q \neq 1$. Let $L \in \Sigma$. Then $Q = \bigcap (O_p(L) \cap O_p(H)) \mid L \neq H \in \Sigma$ and so by 5.7 $Q$ is normal in $L$. Hence $Q$ is a non-trivial subgroup of $R$ which is normal in all the $L\Sigma$, a contradiction. □
Some ideas on the rest of the proof. Define a relation $\approx$ on $\Sigma$ by $L \approx H$ if and only if $Z_L = Z_H$. This should be an equivalence relation and $L \approx H$ if and only if $O_p(L) \cap O_p(H)$ is not normal in $L$. If $L \not\cong H$ we should have $[(R \cap O^p(L), O^p(H)] = 1$. $b = 2$ (that is $L \approx H$ and $Z_L \leq O_p(H)$) seems to occur only for the $G_2(3^k)$ situation, and $2^{1+4+6}L_4(2)$.

What still needs to be discussed in this section is the consequences of 5.1 for the sets $PU_i$, $i = 1, 2, 4$. There are some interesting cases: for example an amalgam if $Z_L$ is the 6-dimensional module for $L/O_2(L) \cong 3\text{Alt}(6)$, then $L \in PU_4(R)$. Same for $\text{Alt}(6)$ or $\text{Alt}(7)$ on the four dimensional module.

Also it seems possible to enlarge the set $PU_3$ without having to change the $b < 3$ part of the proof of 5.1. Namely can drop the assumption on $N_L(R)$ containing a point stabilizer one can allow $[Z_L, L]$ to be the four dimensional module for $SL_3(2)$. This would be useful for the $-E!$ case. Other exceptional $FF$-modules could be included.

For example $\text{Alt}(n)$ on the natural module should be o.k. This also would be o.k for $D_{10}(q)$ on the 16-dimensional spinmodule and $L_n(q)$, $n \geq 5$ on the exterior square. But the choice of $a \in A$ will cause some problems. Might not be so important though, maybe we only need $\bigcap_{a \in A} Z_a \leq \Omega_1 Z(G_3)$.

6 The C(G,T)-Theorem

Suppose that $G$ fullfills $CGT$. Then $S$ is contained in unique maximal subgroup $M$ of $G$, but there exists $L \in \mathcal{L}(S)$ such that $L \not\cong M$ and $|L \cap M|_p \neq 1$. Choose such an $L$ such that $|H \cap L|_p$ is maximal. Let $T$ be a Sylow $p$-subgroup of $H \cap T$. Without loss $T \leq S$. If $T = S$ we get that $L \in \mathcal{L}(S)$ contradicting our assumption $M$ is the unique maximal $p$-local subgroup of $M$. Thus $T \neq S$. Let $C$ be a non-trivial characteristic subgroup of $S$. Then $N_S(T) \leq N_G(C)$ and so $|M \cap N_C(C)|_p > |M \cap L|$. Hence the maximal choice of $|M \cap L|_p$ implies $N_G(C) \leq M$. In particular, $N_L(C) \leq M \cap L$. For $C = S$ we conclude that $T \in \text{Syl}_p(T)$. Then we can apply the

Theorem 6.1 (Local C(G,T)-Theorem) [local CGT] Let $L$ be a finite $K_p$ group of characteristic, $T$ a Sylow $p$-subgroup of $L$, and suppose that

$$C(L, T) := \langle N_L(C) \mid 1 \neq C \text{ a characteristic subgroup of } S \rangle$$

is a proper subgroup of $L$. Then there exists a $L$-invariant set $D$ of subnormal subgroup of $L$ such that

(a) $L = \langle D \rangle C(L, T)$
(b) \[ [D_1, D_2] = 1 \] for all \( D_1 \neq D_2 \in \mathcal{D} \).

(c) \( [c] \) Let \( D \in \mathcal{D} \), then \( D \not\in C(L, T) \) and one of the following holds:

1. \( [1] \) \( D/Z(D) \) is the semidirect product of \( SL_2(p^k) \) with a natural module for \( SD_2(p^k) \). Moreover \( O_p(D) = [O_p(D), D] \) is elementary abelian.

2. \( [2] \) \( p = 2 \) and \( D \) is the the semidirect product of \( Sym(2^k + 1) \) with a natural module for \( Sym(2^k + 1) \).

3. \( [3] \) \( p = 3 \), \( D \) is the semidirect product of \( O_3(D) \) and \( SD_3(3^k) \), \( Z(D) = O_p(D) \) has order \( 3^k \) and both \( [Z(O_3(D)), D] \) and \( O_3(D)/Z(O_3(D)) \) are natural \( SL_2(3^k) \) modules for \( D \).

For \( p = 2 \) the local \( C(G, T) \)-theorem was proved by Aschbacher in [Asch].

For general \( p \) by GLS? For us it will be consequence of the ??.

Back to \( G \). Case 3 can be rules out using that \( N_5(T)/T \) is odd. Let \( m = |\mathcal{D}| \) and suppose that \( m > 1 \). Let \( g \in N_5(T) \setminus T \). Then there exists \( X, Y \in \mathcal{D} \) such that \( R := [[V, X], [V, Y]] \neq 1 \). Let \( H = N_G(R) \). Then for all \( Z \in \mathcal{D} \) with \( D \neq D, D \leq N_G(R) \) and since \( [[V, D], V] \neq 1, [V, D] \not\subseteq O_p(N_{L_5}(R)) \). Thus \( [V, D] \not\subseteq O_p(H) \). Let \( U = O_p(H) \). We conclude that \( [Q \cap T, D] = 1 \).

Since \( H \) is of characteristic \( p \), \( D \) acts non-trivially on \( Q/Q \cap T \).

Let \( T^* \in Syl_p(H) \) with \( N_T(R) \leq T^* \). The maximal choice of \( |T| \) implies \( |T'/N_T(R)| \leq |T/N_T(R)| = T/N_T(X) \). In particular \( |U/\cap T| \leq |T/N_T(X) | \). Thus \( T \) does not normalize \( X \). Let \( e := |T/N_T(X)| \). Then there are at least \( e - 1 \) choices for \( D \), each two of which commute and each acting non-trivially on \( U/\cap T \) which has order at most \( e \). This is impossible.

Hence there exists a unique \( D \in \mathcal{D} \).

Suppose that case 2. holds and \( n \geq 3 \). Then \( O_2(M \cap L) = O_2(L) \). Let \( Q = O_2(M) \). Then \( T \cap Q \leq O_2(M \cap L) \leq O_2(L) \). On the other hand the maximality of \( |T| \) implies \( N_Q(O_2(L)) \leq T \). Thus \( N_Q(O_2(L)) \leq O_2(L) \) and so \( Q \leq O_2(L) \).

If \( Q \) is not elementary abelian that \( [\Phi(Q), D] = 1 \) implies \( D \leq M \), a contradiction. Hence \( Q \) is elementary abelian.

Since \( |Q, O_2(D)| = 1 \) and \( M \) is of characteristic \( p \) we conclude \( O_2(D) \leq Q \). Thus \( |Q, D| \leq |O_2(L), D| \leq O_2(D) \leq Q \) and so \( D \leq N_G(Q) \leq M \). Thus also \( L = D(M \cap L) \leq M \), a contradiction.

Suppose that case 2 holds and \( n = 2 \). Then we can choose \( x \in [V, D] \) so that \( R := [V^g, x] \) has order two. Also \( C_D(x) \) is divisible by 3 and \( [V, O^2(C_D(x))] \), \( C_D(x) \) is not a 2-group. Argue as above we get \( C_D(x) \) acts non-trivially on \( Q/Q \cap T \). But \( |Q/Q \cap T \) has order 2 a contradiction.
Thus Case 1. holds. We have proved:

References


[Schur] Some Schur multipliers