Groups of characteristic $p$–type, The structure theorem

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1 Introduction

This paper is part of the classification of the finite simple groups of characteristic $p$–type. To state the main result we just need some definitions.

In what follows we fix a prime $p$, a group $G$ and a Sylow $p$–subgroup $S$. With $\mathcal{L}(S)$ we will denote the set of all subgroups $H$ containing $S$, with $F^*(H) = O_p(H)$. We will assume that

1) Let $U$ be any $p$–local of $G$ then $F^*(U) = O_p(U)$

2) $O_p(\langle H | H \in \mathcal{L}(S) \rangle) = 1$

Because of 2) we may assume without loss that $G = \langle H | H \in \mathcal{L}(S) \rangle$.

Let $L \in \mathcal{L}(S)$ then we denote by $Y_L$ the maximal elementary abelian normal $p$–subgroup of $L$ with $O_p(L/C_L(Y_L)) = 1$. Such a group exists see 2.2.

Now we denote by $C = C_G(\Omega_1(Z(S)))$. Let $\tilde{C} \in \mathcal{L}(S)$ with $C \leq \tilde{C}$ and $\tilde{C}$ be maximal. Set $E = O_p(F^*(C_G(Y_{\tilde{C}})))$, where $F^*_p(X)$ is the inverse image of $F^*(X/O_p(X))$.

In a separate paper the case that there is some $p$–local $X$ containing $E$ with $X \not\leq \tilde{C}$ has been treated. So we assume

$E$–uniqueness If $X$ is some $p$–local of $G$ with $E \leq X$, then $X \leq \tilde{C}$.

In particular $\tilde{C}$ above was uniquely determined.

Now the purpose of this paper is to get hand on $p$–locals $M$ which are not contained in $\tilde{C}$. In a successor we will determine the structure of $\tilde{C}$.

For this we make the following definitions. Let $Q = O_p(\tilde{C})$. Then for $M \in \mathcal{L}(S)$ we set $M_0 = \langle Q^M \rangle$. As we will see in 2.5 $M = M_0(M \cap \tilde{C})$, we have that $M_0$ dominates the structure of $M$. Hence if we know $\tilde{C}$ lateron we will know all $p$–locals containing $S$. 

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In this paper we will make one further assumption. A subgroup $P \in \mathcal{L}(S)$ is called a minimal parabolic if $S$ is contained in a unique maximal subgroup of $P$ and $S \not\leq P$. So we will assume that if $P_1, P_2$ are minimal parabolics in $\mathcal{L}(S)$ then $O_p(\langle P_1, P_2 \rangle) \neq 1$. What we really use from this is that $M_0S$ is not a minimal parabolic for the $M$ in question (see 2.7).

In particular going through the proof we can see what kind of minimal parabolics we have:

- $F^*(M_0S/O_p(M_0S)) = K_1 \times K_2 \times \cdots \times K_r$, where the $K_i \cong SL_2(p^n)$ and $Y_M$ is a direct sum of natural modules for the $K_i$, and $Q$ acts transitively on the $K_i$.
- $M_0$ is a solvable $\{2, 3\}$–group $M_0S/C_{M_0S}(Y_M)$ has a normal subgroup $K_1 \times K_2 \times \cdots \times K_r$, where the $K_i \cong \Sigma_3$, or $SL(2, 3)$ and $Y_M$ is a direct sum of natural modules for the $K_i$ and $Q$ acts transitively on the $K_i$.
- Here for $p = 2$ $C_{M_0S}(Y_M)/O_2(M_0S)$ might be a cyclic 3–group.
- $F^*(M_0S/O_2(M_0S)) \cong A_9$.
- $M_0/O_2(M_0) \cong \Sigma_3$ and $Y_M$ is a direct sum of natural modules for $M_0/O_2(M_0)$.

Before we can start to state the main theorem of this paper we need a few definitions concerning modules. Let $G$ be a group and $V$ be some nontrivial $GF(p)G$–module for $G$. Then

- We call $V$ an $F$–module, if there is some elementary abelian $p$–subgroup of $G$ with $|V : C_V(A)| \leq |A|$. The group $A$ then will be called an offender.
- We call $V$ an $2F$–module, if there is some elementary abelian $p$–subgroup of $G$ with $|V : C_V(A)| \leq |A|^2$. The group $A$ then will be called an offender.
- We call $V$ a strong $F$–module, if there is some elementary abelian $p$–subgroup of $G$ with $|V : C_V(A)| \leq |A|$ and $C_V(A) = C_V(a)$ for all $a \in A^2$.
- We call $V$ a strong dual $F$–module, if there is some elementary abelian $p$–subgroup of $G$ with $|[V, A]| \leq |A|$ and $[A, v] = [V, A]$ for all $v \in V$, with $[v, A] \neq 1$. For $p > 2$ we assume $[V, A, A] = 1$.  

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We call $V$ a weak strong dual $F$–module, if there is some elementary abelian $p$–subgroup of $G$ with $|V : C_V(A)| \leq |A|$ and $[A, W] = [A, V]$ for all $W \leq V$, $|W| = p^2$ and $C_W(A) = 1$. For $p > 2$ we assume $[V, A, A] = 1$ (In fact these module just show up for $p = 2$ in the proof)


The purpose of this paper is to prove the following theorem

\textbf{Theorem 1.1} Let $p = 2$ and $M \in \mathcal{L}(S)$, $M \not\leq \tilde{C}$, with $M_0$ maximal. Then one of the following holds

i) $Y_M \not\leq Q$ and $F^*(M_0/O_p(M_0))$ is quasisimple and isomorph to $3A_6$, $Sp(2n, q)$, $SL(n, q)$ or $\Omega^\pm(2n, q)$, $U_n(q)$, $G_2(q)$, $E_6(q)$, $M_{22}$ or $M_{24}$ and $Y_M$ is a $2F$–module with quadratic or cubic offender. Further never $M_0S$ induces diagram automorphisms.

ii) $Y_M \leq Q$ and one of the following holds and $F^*(M_0/O_p(M_0))$ is quasisimple and isomorph to $3A_6$, $Sp(2n, q)$, $SL(n, q)$ or $\Omega^\pm(2n, q)$. Further $[Y_M, F^*_p(M_0)]$ is the natural module.

iii) $Y_M \leq Q$ and $F^*(M_0/O_p(M_0)) \cong SL(n, q)$ and $[Y_M, F^*(M_0/O_p(M_0))]$ is a direct sum of two natural $SL(n, q)$–modules.

In all cases we also allow the solvable variant, i.e $L_2(q)$ might be $\Sigma_3$. In particular in the whole paper, if we speak about a component of a group $H$, then it might be a nonsolvable component or it might be solvable which then usually will be $F(H)$.

The proof will follow from 4.1 and 5.15.

As one can see the first assumtion of the theorem is that $p = 2$. There should be a similar theorem for arbitrary $p$. In fact we state most of our results in this paper for arbitrary $p$. But there is a part of the proof which cannot be done so far. To describe this we need some further notation.
Let $M \in \mathcal{L}(S)$, $M \not\leq \tilde{C}$. We say $b(M) = 1$ if $Y_M \not\leq Q$. Then we will attach to $M$ a parameter $b(M)$. Here we say that $b(M) = 2$ if $Y_M \leq Q$ but there is some $g \in G$ with $1 \neq [Y_M, Y_M^g] \leq Y_M \cap Y_M^g$, we say $b(M) > 2$ if $Y_M \leq Q$ but $b(M) \neq 2$.

Now the first step in the proof is to determine the structure of $M_0$ in case of $b(M) = 1$ (4.1). This is dependent on the structure of $2F$–modules with cubic offender. At the moment I do not have any classification at hand for this modules in $p > 2$, neither for simple groups nor for solvable groups. For $p = 2$ things are a little bit easier as then we get fairly large quadratic subgroups in our offender and the we can use the well known classification from [GBSM]. The case $b(M) \geq 2$ is independent. It just goes as follows. We find some minimal parabolic $P$ in $\tilde{C}$ such that $(M, P)$ is an amalgam. As $Y_M \leq Q$ the usual parameter $b$ of this amalgam greater than one (that is the reason for calling $b(M) > 1$). Now there are two possibilities for $b$ it might be even, which yields the case of $b(M) = 2$, or odd, which is the case for $b(M) > 2$. Hence the structure theorem for $b(M) \geq 2$ just follows from the amalgam method applied to this coset graph.

So the first result one has to prove for general $p$ would be 2.16. After this one has to prove 3.7 but for all simple groups and solvable groups, i.e. $2F$–modules with cubic offender in general.

As this paper is part in the revision of the classification of the finite simple groups some remarks on quotations might be in order. Most of the quotations are results about representations of the finite simple groups. The quotation of [Mei] seams to be of different nature. This has been used just once. In 5.3 we determine the structure of $M$ if $Y_M \leq Q$ but there is some $1 \neq x \in Y_M$ with $Y_M \not\leq O_p(C_G(x))$ (this in fact for all $p$). Just to get rid of two cases we need the quotation above. The cases are $M_0/O_2(M_0) \cong L_n(2)$, or $L_n(2) \times \Sigma_3$ and $Y_M$ is a direct sum of two natural modules. If we allow to add this possibility to the main theorem, we will not have to quote [Mei]. If we go over the theorem we will see that this possibility shows up anyway. But the only problem is that we would not know in applying the theorem in that case whether $Y_M \leq Q$ or not. As long as the final part of the classification not has been written up, it is not clear how important this information will be. So we decided to use the result of Meierfrankenfeld, but having in mind that we may drop it.
2 Preliminaries

Definition 2.1 Let $M \in \mathcal{L}(S)$. Then a $p$–reduced normal subgroup of $M$ is an elementary abelian normal $p$–subgroup $Y$ of $M$ such that $O_p(M/C_M(Y)) = 1$.

Lemma 2.2 a) Let $M \in \mathcal{L}(S)$ then there exists a unique maximal $p$–reduced normal subgroup $Y_M$ of $M$

b) Let $S \leq L \leq M$, $M \in \mathcal{L}(S)$ and $X$ a $p$–reduced normal subgroup of $L$, then $\langle X^M \rangle$ is a $p$–reduced normal subgroup of $M$.

c) Let $M, L \in \mathcal{L}(S)$, $L \leq M$, then $Y_L \leq Y_M$.

d) Let $M \in \mathcal{L}(S)$, $C_M = C_M(Y_M)$ and $M^* = N_M(S \cap C_M)$. Then $S \cap C_M = O_p(M^*)$ and $Y_M = \Omega_1(Z(S \cap C_M))$.

e) Let $M^*$ be as in d). Then $M = M^*C_M$ and $Y_M = Y_{M^*}$.

Proof: a) Let $Y_M$ be the subgroup generated by all $p$–reduced normal subgroups. If $O_p(M/C_M(Y_M))$ is nontrivial, this also holds for all the generators of $Y_M$, a contradiction.

b) Let $Y = \langle X^M \rangle$ and $D = C_M(Y)$. Set $N/D = O_p(M/D)$. Then $N = (N \cap S)D = (N \cap L)D$. As $X$ is $p$–reduced for $R$, we have $[X, N \cap R] = 1$. Further $[D, X] = 1$, so $[N, X] = 1$. As $N$ is normal in $M$, we have $[N, Y] = 1$, hence $Y$ is $p$–reduced.

c) Follows from b) with $X = Y_L$.

d) As $O_p(M/C_M) = 1$, we have $O_p(M^*) \leq C_M$. So we get $O_p(M^*) \leq C_M \cap S$ and so $O_p(M^*) = C_M \cap S$. Set $X = \Omega_1(Z(S \cap C_M))$. Then $Y_M \leq X$. Set $Y = \langle X^M \rangle = \langle X^{C_M} \rangle$ as $M = M^*C_M$. Now $X$ is $p$–reduced for $S$ and so by b) $Y$ is $p$–reduced for $C_M$. Set $D = C_M(Y)$ and $N/D = O_p(M/D)$. Since $Y_M \leq X \leq Y$ and $Y_M$ is $p$–reduced for $M$, we get $N \leq C_M$. As $Y$ is $p$–reduced for $C_M$, we get $[N, Y] = 1$. Hence $Y$ is $p$–reduced for $M$ and so $Y \leq Y_M \leq X$. This shows $X = Y_M$, the assertion.

e) The first assertion is just the Frattini argument. Hence now $Y_M \leq Y_{M^*}$. By c) we have $Y_{M^*} \leq Y_M$.

Lemma 2.3 We have $[E, \Omega_1(Z(Q))] = 1$.

Proof: Let $L$ be a $p$–component of $E$ with $[\Omega_1(Z(O_p(L))), L] = 1$. We have $O_p(L) \leq Q$, so $[\Omega_1(Z(Q)), L] \leq \Omega_1(Z(O_p(L)))$ and then $[\Omega_1(Z(Q)), L, L] = 1$.

The 3-subgroup lemma now implies $[\Omega_1(Z(Q)), L] = 1$. 

Let now \([\Omega_1(Z(O_p(L))), L] \neq 1\). Set \(K = \langle L, S \rangle\). We have \(\Omega_1(Z(O_p(K))) \leq Y_K \leq Y_\mathcal{C}\) by 2.2. Now \([\Omega_1(Z(O_p(L))), L] \leq \Omega_1(Z(O_p(K)))\). But then \([\Omega_1(Z(O_p(L))), L, L] = 1\), which gives a contradiction with the 3–subgroup lemma.

So we just have to handle \(F_p(E)\). We have \(F_p(E) \leq C(Z(S))\) and so \([\Omega_1(Z(Q)), F_p(E)] \cap C(S) = 1\). Hence \([\Omega_1(Z(Q)), F_p(E)] = 1\).

**Lemma 2.4** Let \(1 \not= U \leq Z(Q)\), then \(N_G(U) \leq \mathcal{C}\).

**Proof:** As \([E, \Omega_1(Z(Q))] = 1\) by 2.3 we get \(E \leq N_G(U)\) and so by \(E\)–uniqueness we have the assertion.

**Lemma 2.5** Let \(M \in \mathcal{L}(S)\), then

\[M = M_0(M \cap \mathcal{C}).\]

**Proof:** Let \(S_1 \in \text{Syl}_p(M_0)\). Then we have \(M = N_M(S_1)M_0\). As \(Q \leq S_1\) and \(Z(S_1) \leq Z(Q)\), we see mit 2.4 that \(N_M(S_1) \leq \mathcal{C}\), the assertion.

**Lemma 2.6** Let \(M \not\leq \mathcal{C}\), then \(M_0 \leq M^*\).

**Proof:** We have \(C_M \cap M_0 = O_p(M_0)\), as this group has to normalize \(Q\). Now \([M_0, C_S(Y_M)] \leq O_p(M_0) \leq C_S(Y_M)\), so \(M_0 \leq M^*\).

**Lemma 2.7** Let \(M \in \mathcal{L}(S)\) with \(M_0\) maximal. Then \(M_0S\) is not a minimal parabolic.

**Proof:** Set \(R = N_G(M_0)\). Then by choice of \(M\) we have \(R_0 = M_0\). Let \(P_1 \in \mathcal{P}(S)\) be a minimal parabolic in \(ES\), which is not in \(R\). Then for \(H = \langle M_0S, P_1 \rangle\) we have \(O_p(H) \neq 1\). By maximality of \(M_0\) we have \(M_0 = H_0\). But then we would have \(H \leq R\), a contradiction.

**Lemma 2.8** Let \(H\) be a finite group, \(p\) be a prime, \(S \in \text{Syl}_p(H)\) and \(S\) be contained in a unique maximal subgroup \(M\) of \(H\). Let \(P \leq S\) with \(P \not\leq O_p(H)\). Then there are \(L \leq H\) and \(h \in H\) such that

a) \(P \leq L\), \(P \not\leq O_p(L)\)

b) \(O_p(L)P \leq M^h \cap L\), which is the unique maximal subgroup in \(L\) containing \(P\)

c) \(S^h \cap L \in \text{Syl}_p(L)\).
Moreover for any such $L$, we have $L = \langle P^L \rangle$.

Proof: If $M$ is the unique maximal subgroup containing $P$ we may set $L = H$. So assume there is a maximal subgroup $K \neq M$, $P \leq K$. Among all such $K$ we choose $K$ with $|K \cap S|$ maximal and then $|K|$ minimal. Set $T = K \cap S$. By the minimal choice of $K$ we know that $M \cap K$ is the unique maximal subgroup of $K$ containing $T$. Set $R = \langle P^g \mid P^g \leq T, g \in H \rangle$. As $K \not\leq M$, we have $T \neq S$. So $T < N_S(T) \leq N_H(R)$. By the choice of $K$ we now have $N_H(R) \leq M$ and so $N_K(R) \leq K \cap M$. In particular $T \in \text{Syl}_p(K)$.

Now $O_p(K) \leq T \leq M$. If $R \leq O_p(K)$, then $R \leq K$ and so $K \leq M$. Hence there is $P^g \leq T$ with $P^g \not\leq O_p(K)$. Now we may replace $H$ by $K$, $P$ by $P^g$ and $M$ by $M \cap K$. By induction we get $L_1 \leq K$ with $P^g \leq L_1$, $P^g \not\leq O_p(L_1)$ and $h_1 \in K$ with $P^g \leq (M \cap K)^{h_1} \cap L_1$ and this is the unique maximal subgroup of $L_1$ containing $P^g$. Further $T^{h_1} \cap L_1 \in \text{Syl}_p(L_1)$. Set $h = h_1g^{-1}$ and $L = L_1^{g^{-1}}$, then (a) - (c) hold.

Now let $D = \langle P^L \rangle \neq L$. As $P \leq D$ we have $D \leq M^h \cap L$. The Frattini argument shows $L = DN_L(S^h \cap D)$, so $L = N_L(S^h \cap D)$, otherwise by $P \leq N_L(S^h \cap D)$, we get $N_L(S^h \cap D) \leq M^h \cap L$ and then $L = DN_L(S^h \cap D) \leq M^h \cap L$, a contradiction. But now $P \leq O_p(L)$, a contradiction.

Lemma 2.9 Let $\langle b, c \rangle$ be an elementary abelian group of order $p^2$ acting quadratically on a $GF(p)$--module $V$. Let $1 \neq v \in V$, then $\langle v^b \rangle \leq \langle v \rangle \langle v^b \rangle \langle v^c \rangle$

Proof: We have $(v^bv^{-1})^c = v^bv^{-1}$. So $v^{bc}v^{-c} = v^bv^{-1}$. Hence $v = v^{-bc}v^c v^b$, the assertion.

Lemma 2.10 Let $H$ be a group and $A$ be a $p$--subgroup, $A \not\leq O_p(H)$ but $A$ contained in a unique maximal subgroup $M$ of $H$. Let $V$ be a faithful $GF(p)H$--module with $[V, A, A] = 1$ such that for some $Z \leq V$ with $[Z, A] = 1$, we have $V = \langle Z^H \rangle$ or $V = C_V(A)[V, H]$. Then the following hold

a) $C_V(t) = C_V(A)$ for all $t \in A \setminus O_p(H)$.

b) $|V : C_V(A)| \geq |A/A \cap O_p(H)|^c$, where $c$ is the number of non trivial chief factors in $V$.

c) $[V, t] \cap C_V(H) = 1$ and $|[V, t]|^2 = |V : C_V(H)|$ for all $t \in A \setminus O_p(H)$.

d) $[V, H] \cap C_V(H) \leq [V, A]$

e) For all chief factors $W$ we have $W = C_W(H)[W, H]$.  


f) If \([Z, O_p(H)] \leq Z\), then we have one of the following

(\alpha) \ [V, A \cap O_p(H)] \leq C_V(H)

(\beta) p = 2, H/O_2(H) is dihedral of order \(2r^k\), where \(r\) is some odd prime \(k > 1\) and \(A \not\leq O_2(C_H([V, A \cap O_2(H)])) \not\leq O_2(H)\).

**Proof:** First notice that if \(V = \langle Z^H \rangle\) then also \(V = C_V(A)[V, H]\). So we may assume the latter. Set \(N = \bigcap_{\eta \in H} M^\eta\). By the Frattini argument we have \(O_p(H) \in \text{Syl}_p(N)\). So the element \(t\) in question is not contained in \(N\). Now choose \(h \in H\) with \(t \not\in M^h\) and set \(B = A^h\). Then as \(M^h\) is the unique maximal subgroup containing \(B\), we see \(H = \langle t, B \rangle\).

This now shows

\[ [V, H] = [V, t][V, B] \]

By quadratic action \(V = C_V(A)[V, B] = C_V(t)[V, B]\). So

\( C_V(B) = C_V(H)[V, B] \)

In the same way we see \(V = C_V(B)[V, A]\). Now

\[ C_V(t) = C_V(B) \cap C_V(t)[V, A] = C_V(H)[V, A] = C_V(A), \]

which is (a).

Let \(W\) be an irreducible nontrivial chief factor. Then

\( W = [W, A] \oplus [W, B] \)

Hence we get \([W, A] = C_W(t)\) and so \(|[W, A]| \geq |A/A \cap O_p(H)|\), this is (b).

We have \([V, A] = [V, t][\langle V, A \rangle \cap [V, B]\). Set \(Y = [V, B] \cap C_V(t) \geq [V, A] \cap [V, B]\). We have

\[ ||[V, t]| = ||[V, B, t]| = ||[V, B]/C_{[V, B]}(t)| = ||[V, A]/Y| \]

So we see \(|[V, t]| | Y = ||[V, A]|\). This shows that \(Y = [V, A] \cap [V, B]\) and so \([V, A] = [V, t] \oplus Y\). So we see \([V, t] \cap C_V(H) \leq [V, t] \cap Y = 1\) and then

\[ ||[V, H]| = ||[V, t]|^2|Y|\), so \(|[V, t]|^2 = |V : C_V(H)|\), which is (c) and \(C_{[V, H]}(A) = [V, A]Y\) so \(C_{[V, H]}(H) = [V, A]Y \cap [V, B]Y = Y\), which is (d).

To prove (e) let \(W\) be a submodule of \(V\). We may assume \(C_V(H) = 1\).

By induction the assertion holds for \(V/[V, H]\). In particular \(W/[W, H] \leq [V, A][W, H]/[W, H]\). Hence \(W = [W, H]/(\langle V, A \rangle \cap W)\), this is (e).

To prove (f) let \(h \in H \setminus M\). We have

\[ [Z^h, A \cap O_p(H)] \leq Z^h \cap C(A) \leq C((A, A^h)) \leq C_V(H). \]
Set \( Y = \langle Z^h \mid h \in H \setminus M \rangle \). Then \([Y, A \cap O_p(H)] \leq C_V(H)\).

Assume now that \(|AO_p(H)/O_p(H)| \geq 3\). We then show that \( B \) normalizes \( Y \). If \( h \in H \setminus M \) and \( b \in B \) such that \( hb \notin M \), then \( Z^{hb} \leq Y \). So let \( hb \in M \). As \(|AO_p(H)/O_p(H)| \geq 3\), there is some \( c \in B, c \notin O_p(H) \) such that \( c \notin O_p(H)b \). If \( hc \in M \), then \( c^{-1}b \in M \cap B \). But let \( 1 \neq a \in A \cap M^h \setminus O_p(H) \). Then there is some \( k \in M^h \) with \( \langle a, B^k \rangle \) is a \( p \)-group. But then \( C_V(a) \cap C_V(B^k) \neq 1 \), a contradiction as we may assume \( C_V(H) = 1 \). So we have \( A \cap M^h = O_p(H) \) and so \( c^{-1}b \in O_p(H) \), a contradiction. Hence we have \( hc \notin M \). Similar \( hbc \notin M \). But \( \langle b, c \rangle \) acts quadratically. So by 2.9 we have \( Z^{hb} \leq Z^h Z^{hbc} Z^{hc} \). Hence \( Y \) is \( B \)-invariant. As \( H = \langle t, B \rangle \) we have \( Y = V \).

So we are left with \(|A/O_p(H) \cap A| = 2\). Then \( p = 2 \) and \( H/O_2(H) \) is dihedral of order \( 2^k \), \( r \) an odd prime. If \( r = 1 \), then \( M = AO_2(H) \) normalizes \( Z \) and so \( V = ZY \). Now \([V, A \cap O_2(H)] = [Y, A \cap O_2(H)] \leq C_V(H) \), which is \( (α) \).

Let \( k > 1 \) and \( L = C_H([V, A \cap O_2(H)]) \). Choose \( H^* \) minimal with \( A \leq H^* \) and \( H^*O_p(H) = M \). Set \( V^* = \langle Z^{H^*} \rangle = \langle Z^M \rangle \). Then \( V = V^*Y \). Now by induction we have \( A \not\leq O_2(C_{H^*}([V^*, A \cap O_2(H^*)])) \not\leq O_2(H^*) \). As \([V, A \cap O_2(H)] = [V^*, A \cap O_2(H)][Y, A \cap O_2(H)] \) we get \([V, A \cap O_2(H), O_2(C_{H^*}([V^*, A \cap O_2(H^*)]))) = 1 \). As \( O_2(C_{H^*}([V^*, A \cap O_2(H^*)])) \) does not normalize \( O_2(H) \) we have \( (β) \).

**Lemma 2.11** Let \((G_α, G_β)\) be an amalgam with \( S \in \text{Syl}_p(G_α \cap G_β) \) and \( S \leq M_αβ \), where \( M_αβ \) is the unique maximal subgroup of \( G_β \) which contains \( G_α \cap G_β \). Let further \( b \) be odd, \( b \geq 3 \). Then \( (Y^β_α) = V_β \not\leq O_p(G_α') \). Set \( β = β^+ \) and \( α' = β^- \). Then one of the following holds.

1. There is \( L^β \leq G_β^- \) and \( µ^\epsilon \in Δ(β^\epsilon) \), \( ε = \pm \), such that for both values of \( ε \) and \( V^ε = \langle Y^β^ε \rangle \) we have
   a) \( V^ε \not\leq O_p(L^β) \)
   b) \( V^ε \leq G_µ^ε \) and \( G_β^- \cap G_µ^ε \) contains a Sylow \( p \)-subgroup of \( L^β \).
   c) \( L^ε \cap M_β^ε \) is the unique maximal subgroup of \( L^β \), which contains \( V^ε \)
   d) \([V^ε, Y_µ^ε] \neq 1\).
2. There is \( ε \in \{+, -\} \) and \( L^ε \), \( µ^- \in Δ(β^-) \), \( µ^+ \in Δ(β^+) \) with
   a) \( V^ε \leq G_µ^−ε \), \( Y_µ^−ε \leq L^ε \), \( Y_µ^−ε \leq O_p(L^ε) \)
   b) \( Y_µ^−ε \leq G_µ^ε \) and \( G_µ^ε \cap G_β^- \) contains a Sylow \( p \)-subgroup of \( L^ε \).
c) $L^e \cap M_{\beta^e\mu^e}$ is the unique maximal subgroup in $L^e$ which contains $Y_{\mu^e}$.

d) $[Y_{\mu^+}, Y_{\mu^-}] = 1$.

3) There is $\mu^e \in \delta(\beta^e)$, $\epsilon = \pm$, such that $Y_{\mu^e} \leq G_{\mu^{-\epsilon}}$ and $[Y_{\mu^+}, Y_{\mu^-}] \neq 1$.

**Proof:** Assume that 3) does not hold. Then choose $L^e \leq G_{\beta^e}, \mu^e \in \Delta(\beta^e)$ with $|L^+||L^-|$ minimal such that

1) for all $\epsilon$ we have $V^{-\epsilon} \leq L^e \cap G_{\beta^e\mu^e}$

2) for all $\epsilon$ the group $L^e \cap G_{\beta^e\mu^e}$ contains a Sylow $p$–subgroup of $L^e$ and is contained in a unique maximal subgroup $M_{\beta^e\mu^e} \cap L^e$.

3) For at least one $\epsilon$ we have $V^e \not\leq O_p(L^e)$

Such a set up exists. Choose for example $\mu^e = \alpha + 2, \mu^- = \alpha' - 1$, and $L^e = G_{\beta^e}$.

Let us first assume

(*) There is some $\epsilon = \pm$ and some $\mu \in (\mu^e)^{L^e}$ with

$[V^{-\epsilon}, Y_{\mu}] \neq 1$ and $V^{-\epsilon} \leq G_{\mu}$.

We may choose notation such that $\epsilon = -$. If $Y_{\mu} \leq O_p(L^+)$, then as $O_p(L^+) \leq G_\rho$ for all $\rho \in (\mu^+)^{L^+}$, we may choose $\rho$ such that $[Y_{\mu}, Y_{\rho}] \neq 1$, which contradicts the assumption that we do not have 3).

So $Y_{\mu} \not\leq O_p(L^+)$. As $V^- \leq G_{\mu^+}$ and we do not have 3), we see again $[Y_{\mu^+}, Y_{\mu^-}] = 1$. Now set $\mu^- = \mu$. By 2.8 there is $L \leq L^+, h \in L^+$, and $Y_{\mu} \leq L, Y_{\mu} \not\leq O_p(L)$ such that $(G_{\beta^e\mu^e} \cap L^-)^h \cap L$ contains a Sylow $p$–subgroup of $L$ and $(M_{\beta^e\mu^e} \cap L^+)^h \cap L$ is the unique maximal subgroup containing $Y_{\mu}$. Now consider $L^+, L, \mu^+$ and $(\mu^e)^h$. By the minimal choice of $|L^+||L^-|$, we see $L = L^+$ and we have 2).

Assume now that (*) does not hold. Suppose $[V^-, V^+] = 1$. As $Y^{-\epsilon} \not\leq O_p(L^e)$, we would get $Y_{\mu^e} \leq L^e$, so $Y_{\mu^e} \leq (G_{\mu^e}, L^e)$. But then $Y_{\mu^e} \leq (G_{\beta^e\mu^e}, L^e) = G_{\beta^e}$ and then $Y_{\mu^e} \leq (G_{\beta^e\mu^e}, L^e)$, a contradiction. So we have shown $[V^-, V^+] \neq 1$. Assume now $V^{-\epsilon} \leq O_p(L^e)$. Then there is some $\mu \in (\mu^e)^{L^e}$ with $[Y_{\mu}, V^{-\epsilon}] \neq 1$, which is (*). So we have $V^{-\epsilon} \not\leq O_p(L^e)$ for both $\epsilon$. If $[V^{-\epsilon}, Y_{\mu^e}] \neq 1$ we get 1) with $\mu = \mu^e$. So we may also assume that for both $\epsilon$ we have $[V^{-\epsilon}, Y_{\mu^e}] = 1$. Then application of 2.8 again yields a contradiction.
Lemma 2.12 Let the notation be as in 2.11. Assume $b(G_\alpha) > 1$. Assume further that there is exactly one nontrivial chief factor of $L^\epsilon$ in $V_\epsilon$. Then $Y_\alpha$ is a dual $F$-module with offender $X$ such that $[Y_\alpha, X] = [y, X]$ for all $y \in Y_\alpha \setminus C_{Y_\alpha}(X)$, or $[V^\epsilon, O_p(G_{\beta^p})] = 1$.

Proof: Set $\alpha = \mu^\epsilon$ and $V = V^\epsilon$, $R = O_p(G_{\beta^p})$ and $L = L^\epsilon$ in 2.11. Let $[V, R, L] \neq 1$. Then by 2.10(e) we have $V = C_V(L)[V, L]$. As there is just one chief factor we get $Y_\alpha[V, R] \leq L$. This implies $V = Y_\alpha[V, R]$. But $R$ is a $p$-group so we get $[V, R] \leq Y_\alpha$ and then $V = Y_\alpha \leq \langle G_\alpha, L \rangle$, a contradiction.

So we have $[V, R, L] = 1$. By 2.10 again $V/C_V(L)$ is an irreducible module. Now choose $x \in Y_\alpha \setminus C_V(L)$. Then $V = \langle x \rangle C_V(L)$ and so

$$[Y_\alpha, R] \geq [x, R] = \langle [x, R] \rangle = [V, R] \geq [Y_\alpha, R]$$

so

$$|R/C_R(Y_\alpha)| \geq |R/C_R(x)| = |[R, x]| = |[Y_\alpha, R]|.$$

If $[Y_\alpha, R] \neq 1$ we have the assertion. So assume $[Y_\alpha, R] = 1$, then $[V, R] = 1$ and $V \leq \Omega_1(Z(R))$.

Remark. In the application of 2.12 the second alternative will never show up. We usually choose $G_\beta$ as a subgroup of $\tilde{C}$. Then $Q = O_p(\tilde{C}) \leq R$, so $[Y_\alpha, Q] = 1$. By 2.4 we then have $G_\alpha \leq \tilde{C}$ too, a contradiction.

Lemma 2.13 Suppose that we have the situation of 2.11(1) with more than one nontrivial chief factor in $V^-$ and $V^+$. Assume further $b(G_\alpha) > 1$ and $Y_\alpha \leq O_p(C_G(x))$ for all $1 \neq x \in Y_\alpha$. Then $Y_\alpha$ is a strong $F$-module with $V^- \cap O_p(L^+)$ as offender.

Proof: Choose $\alpha \in (\mu^+)L^+$ with $Y_\alpha \not\leq O_p(L^-)$. Then $V^- \cap O_p(L^+) \leq G_\alpha$. By 2.10(a) applied to $L^-$ with $V^+$ acting on $V^-$, we get $C_{V^-}(Y_\alpha) = C_{V^-}(V^+) \leq V^- \cap O_p(L^+)$. Let $1 \neq x \in [Y_\alpha \cap O_p(L^-), V^- \cap O_p(L^+)]$. Then we have $Y_\alpha \leq C_{L^-}(x)$. By 2.10(f) we have $Y_\alpha \not\leq O_p(C_{L^-}(x))$, a contradiction. So we have

$$[Y_\alpha \cap O_p(L^-), V^- \cap O_p(L^+)] = 1$$
Suppose now that $Y_\alpha$ is not an $F$–module. Then

$$\left| V^- / V^- \cap O_p(L^+) \right| \left| V^- \cap O_p(L^+) / C_{V^-} (V^+) \right| = \left| V^- / C_{V^-} (V^+) \right| \overset{2.10(a)}{=}$$

$$\left| V^- / C_{V^-} (Y_\alpha) \right| \overset{2.10(b)}{\geq} \left| Y_\alpha / Y_\alpha \cap O_p(L^-) \right|^2 \geq$$

$$\left| Y_\alpha / C_{Y_\alpha} (V^- \cap O_p(L^+)) \right|^2 \overset{2.10(b)}{\geq} \left| V^- \cap O_p(L^+) / C_{V^-} (Y_\alpha) \right|^2$$

The last inequality is because $Y_\alpha$ is not an $F$–module. Further this inequality is strict besides $V^- \cap O_p(L^+) = C_{V^-} (Y_\alpha)$. By 2.10(a) we have

$$\left| V^- \cap O_p(L^+) / C_{V^-} (Y_\alpha) \right| = \left| V^- \cap O_p(L^+) / C_{V^-} (V^+) \right|$$

so

$$\left| V^- / V^- \cap O_p(L^+) \right| \overset{2.10(b)}{\geq} \left| V^- \cap O_p(L^+) / C_{V^-} (V^+) \right|.$$ 

By 2.10(b) we have

$$\left| V^+ / C_{V^+} (V^-) \right| = \left| V^- / V^- \cap O_p(L^+) \right|^2.$$ 

Hence

$$\left| V^+ / C_{V^+} (V^-) \right| \overset{2.10(b)}{\geq} \left| V^- / V^- \cap O_p(L^+) \right| \left| V^- \cap O_p(L^+) / C_{V^-} (V^+) \right| = \left| V^- / C_{V^-} (V^+) \right|.$$ 

By symmetry we also have

$$\left| V^- / C_{V^-} (V^+) \right| \overset{2.10(b)}{\geq} \left| V^+ / C_{V^+} (V^-) \right|.$$ 

Hence we have equality everywhere. But this implies $V^- \cap O_p(L^+) = C_{V^-} (Y_\alpha)$ and then also $V^- = C_{V^-} (V^+)$, a contradiction. Hence we have that $Y_\alpha$ is an $F$–module and even more that $V^- \cap O_p(L^+)$ is an offender. By 2.10(a) we get that it is a strong $F$–module.

**Lemma 2.14** Assume that we have 2.11(2) with more that one nontrivial chief factor in $V^\epsilon$. Further assume $b(G_\alpha) > 1$ and $Y_\alpha \leq O_p(C_{G_\alpha} (x))$ for all $1 \neq x \in Y_\alpha$. Then $Y_\alpha$ is a strong $F$–module. Let without loss $\epsilon = -$. Then we have $[V^-, a] = [V^-, Y_\alpha]$ for all $a \in Y_\alpha \setminus C_{Y_\alpha} (V^-)$.

**Proof:** Without loss we may assume $\epsilon = -$. As in 2.13 we see with 2.10(f) that $[Y_\alpha \cap O_p(L^-), V^-] = 1$, recall that $V^- \leq G_\alpha$ and $Y_\alpha \not\leq O_p(L^-)$. Again we get

$$\left| V^- / C_{V^-} (V^+) \right| = \left| V^- / C_{V^-} (Y_\alpha) \right| \overset{2.10(a)}{=} \left| V^- / C_{V^-} (Y_\alpha) \right| \overset{2.10(b)}{\geq}$$

$$\left| Y_\alpha / Y_\alpha \cap O_p(L^-) \right|^2 \overset{2.10(b)}{\geq} \left| Y_\alpha / C_{Y_\alpha} (V^-) \right|^2 \overset{\text{not } F\text{–mod.}}{\geq}$$

$$\left| V^- / C_{V^-} (Y_\alpha) \right|^2 = \left| V^- / C_{V^-} (V^-) \right|^2.$$
Again this is only possible if $V^- = C_{V^-}(V^+)$, a contradiction. So we again have an $F$–module. If $1 \neq x \in [Y_\alpha, V^-] \cap Z(L^-)$, then $Y_\alpha \not\subseteq O_p(C_G(x))$, a contradiction. So we have $[Y_\alpha, V^-] \cap C_{V^-}(L^-) = 1$. By 2.10(d) we have $[V-, L^-] \cap C_{V^-}(L^-) \subseteq [V-, Y_\alpha]$, so $V^- = C_{V^-}(L^-) \oplus [V-, L^-]$. Now let $a \in Y_\alpha \setminus O_p(L^-)$. Then $[V-, a] = C_{[V-, L^-]}(a) = C_{[V-, L^-]}(Y_\alpha)$.

In particular $Y_\alpha$ is a strong $F$–module. Further $[V-, a] = [V-, Y_\alpha]$ and $C_{V^-}(a) = C_{V^-}(Y_\alpha)$.

**Remark.** From the classification of $F$–modules we see that an irreducible dual $F$–module is an $F$–module and an irreducible strong dual $F$–module is a strong $F$–module.

**Lemma 2.15.** Let $O^2(L/O_2(L)) \cong Sp(2n, q)$ and $S$ a Sylow 2–subgroup of $L$. Suppose that no nontrivial characteristic subgroup of $S$ is normal in $L$. Then one of the following holds

1. $O_2(O^2(L))$ is the $O(2n + 1, 4)$–module and $O^2(L)$ does not split over $O_2(O^2(L))$.
2. $|O_2(O^2(L))| = q^3$ and $O^2(L/O_2(L)) \cong L_2(q)$
3. $O^2(L/O_2(L)) \cong Sp(6, q)$ and $O_2(O^2(L))$ is as in $F_4(q)$.

**Proof:** [Mei]

The next lemma I’m not sure whether it is also valid in odd characteristic.

**Lemma 2.16.** Let $G$ be a group with $F^*(G) = O_2(G) \neq 1$ and $A \leq G$ be elementary abelian with $A \not\subseteq O_2(G)$ but $A \leq S$ for some Sylow 2–subgroup $S$ of $G$. Then there is some $g \in G$ such that for $X = \langle A, A^g \rangle$ the following hold

1. $X/O_2(X) \cong L_2(q), S_2(q)$ or $X/O_2(X)$ is a dihedral group of order $2u$, $u$ odd.
2. $S \cap X$ is a Sylow 2–subgroup of $X$
3. $Y = (A \cap O_2(X))(A^g \cap O_2(X)) \leq X$
(4) \( Y \neq A \cap O_2(X) \)

(5) \(|A : C_A(Y)| \leq |Y : C_Y(A)|q \leq |Y : C_Y(A)|^2\), where \( q = 2 \) if \( X/O_2(X) \) is dihedral.

(6) If \( X/O_2(X) \) is not dihedral, then \( Y/(A \cap A^g) \) is a direct sum of natural modules for \( X/O_2(X) \).

or there is some \( g \in G \) such that \( B = A^g \leq S, A^g^2 = A, [B, A] \neq 1 \) and \(|A : C_A(B)| = |B : C_B(A)|\).

**Proof:** We start the proof with some general remarks. Let \( X \) be as in (1). Then obviously (3) follows. If (4) would be false, then as \([O_2(G), A] \leq O_2(G) \cap A \leq O_2(X) \cap A\), we get that \([O_2(G), X, X] = 1\), which contradicts \( C_G(O_2(G)) \leq O_2(G)\). Hence also (4) holds. Next we see that \( C_Y(A) = A \cap Y \) and so we see that \( C_{Y/(A \cap A^g)} = (A \cap Y)/(A \cap A^g) \) and \( Y/(A \cap A^g) = (Y \cap A)/(A \cap A^g) \oplus (Y \cap A^g)/(A \cap A^g)\). So (5) follows. Further we see that elements of odd order in \( X \) act fixed point freely on \( Y/(A \cap A^g)\). Hence [Hig] and [Mar] yield (6). So we see that in case (1) we just have to prove (2) which will be clear by the specific construction.

Set \( \bar{G} = G/O_2(G) \). Let \( r \) be some odd prime with \([O_r(\bar{G}), \bar{A}] \neq 1\). Then we get some \( g \in O_r(\bar{G}) \) with \( X/O_2(X) \cong D_{2r} \), where \( X = \langle A, A^g \rangle \). If there is some component \( L \) with \( 1 \neq [L, \bar{A}] \) and \(|\bar{A} : C_A(L)| = 2\), then again we get \( X \) with \( X/O_2(X) \cong D_{2u} \), \( u \) odd. In both case of course \( S \cap X \) is a Sylow 2-subgroup of \( X \).

So we may assume that \( F^*(\bar{G}) = E(G) \). Further for any component \( L \) we may assume that \(|\bar{A} : C_A(L)| \geq 4\). We have that \( A \) acts quadratically on \( O_2(G) \). So assume first that \( L \) is of Lie type in odd characteristic, which is not also of Lie type in even characteristic. Then by [MeiStr1] we have that \( L/Z(L) \cong U_4(3) \). As \( A \leq S \), there is some 2-central involution \( s \) in \( \bar{A} \cap L \). Let \( B \) be the projection of \( \bar{A} \) onto \( L \). If \( B \not\leq O_2(C_L(s)) \), then there is a conjugate \( B^g \) such that \( W = \langle B, B^g \rangle \cong D_6 \) and \( S \cap W \) is a Sylow 2-subgroup of \( W \). Hence we may set \( X = \langle A, A^g \rangle \). So we may assume that \( B \leq O_2(C_L(s)) \). As we may generate \( C_L(s) \) by elements \( g \) with \( g^2 \in S \), the action of \( C_L(s) \) on \( O_2(C_L(s)) \) gives us some \( A^g \leq O_2(C_L(s)) \) with \([\bar{A}, A^g] \neq 1\), and so we have the second alternative.

Let next \( L \cong G(r) \) be a group of Lie type in even characteristic. Let first \([\bar{A}, L] \not\leq L \). Then application of [Cher] shows \( L \cong L_{2r}(r) \). Let \( L_1 = N_{(L, A)}(\bar{S}) \) then \( L_1/O_2(L_1) \cong Z_{r-1} \times Z_{r-1} \) and \( \bar{A} \) acts nontrivially on this group and
so we get \( X \) with \( X/O_2(X) \) dihedral and \( S \cap X \) a Sylow 2–subgroup of \( X \). Hence we may assume that \( A \) normalizes \( L \).

Let \( R \) be a root subgroup in \( Z(S \cap L) \). Let \( B \) be the projection of \( A \) onto \( \text{Aut}_G(L) \). Suppose \( B \not\leq O_2(N_L(R)) \). Then we have an induction and the lemma holds. So we may assume that \( B \leq O_2(N_L(R)) \). If we may generate \( C_L(R) \) by elements \( g \) with \( g^2 \leq O_2(N_L(R)) \), then we get the second alternative, or \( \langle B^N_L(R) \rangle \) is abelian. If \( B \leq R \), then \( B \leq \tilde{L} \leq L \), with \( \tilde{L} \cong L_2(2) \) or \( Sz(2) \) and \( S \cap \tilde{L} \) is a Sylow 2–subgroup of \( \tilde{L} \).

Hence we just have to handle rank 1 groups or \( L \cong L_n(r), Sp(2n,r), F_4(r), 2F_4(r) \).

If we have a rank 1 group then as \(|B| \geq 4\), we either get \( X \) such that \( X/O_2(X) \) is dihedral or we get \( X \) with \( X/O_2(X) \cong L_2(q) \) or \( Sz(q) \) and a Sylow 2-subgroup of \( X \) is contained in \( S \). So we may assume that \( B \not\leq R \).

Assume next \( L \cong L_n(r), n \geq 3 \). Assume that \( B \) acts trivially on the Dynkin diagram. Let \( P_1, P_{n-1} \) be the two parabolics containing \( S \cap L \) which involve \( L_n-1(r) \). If \( B \not\leq O_2(P_i) \) for one \( i \), then we have induction. So we have \( B \leq O_2(P_1) \cap O_2(P_{n-1}) = R \), a contradiction. So let \( b \in B \) acting nontrivially on the Dynkin diagram.

Let first \( n = 3 \). Then we see that \([b,\bar{S}]\) is not abelian, which contradicts \( B \leq \bar{S} \). In any other case we get a parabolic \( P_3 \) such that \( P_3/O_2(P_3) \cong L_2(r) \times L_2(r) \) such that \( B \) acts nontrivially on \( P_3/O_2(P_3) \). If \( r > 2 \), we have induction. If \( r = 2 \) this is solvable and we get a dihedral group \( X/O_2(X) \).

Let next \( L \cong Sp(2n,r), n \geq 2 \). We may assume that \( B \leq Z(O_2(N_L(R))) \). So we may embed \( B \) into some \( \bar{L} \cong Sp(4,r) \) with \( S \cap \bar{L} \) a Sylow 2–subgroup of \( \bar{L} \). So we may assume \( L \cong Sp(4,r) \). Now we have two parabolics \( P_1, P_2 \), containing \( S \cap L \). By induction we may assume that \( B \leq O_2(P_1) \cap O_2(P_2) \).

As \( B \) is not contained in a root subgroup we see that \( \langle B^P_i \rangle = O_2(P_i) \) for \( i = 1,2 \). Let \( H_i \) be the preimage of \( P_i \), i.e. \( H_i/O_2(G) = P_i \). Now suppose that \( \langle A^{H_i} \rangle \) is abelian. Then we see that \( O_2(H_i) \leq C_{SL}(A)O_2(G) \). If this is true for both \( i \), we get \( S \cap L = C_{SL}(A)O_2(G) \). As \( B \) acts quadratically on \( O_2(G) \) there is a chief factor \( V \) in \( O_2(G) \) which is the natural module \([\text{Str}], \) or \( L \cong A_6 \) and some 6-dimensional module is involved. In the former we have \(|V,B| = r^2\), while \(|C_V(S \cap L)| = r \). As \([V,B]\) is covered by \( A \) this is a contradiction. So we have the latter. Now by quadratic action \( B \leq A_6 \) and then \(|B| = 2 \), which gives that we have a dihedral group \( X/O_2(X) \).

Let next \( L \cong E_6(r) \). By induction we may assume that \( B \) acts trivially on the Dynkin diagram. We have two root groups \( R_1 \) and \( R_2 \) and we may assume that \( B \leq Z(O_2(N_L(R_1))) \cap Z(O_2(N_L(R_2))) \). But this group is contained in some \( Sp(4,r) \) and we get the assertion by induction.
Let next \( L \cong 2F_4(r) \). As \( B \) acts quadratically we get with [Str] \( B \leq R \), a contradiction.

Let now \( L \cong A_n, n \geq 5 \). So we may assume \( n = 7 \), or \( n \geq 9 \). If \( n = 7 \) there is some \( \bar{L} \leq L \) such that \( \bar{L} \cong A_6 \). Hence we have induction. So we have \( n \geq 9 \). Furthermore by induction we may assume \( n \) to be even. Let \( n = 2^m \). Then there is a subgroup \( \bar{L} \leq L \) with \( S \cap L \leq \bar{L} \leq \Sigma^2 \). As \( n \geq 16 \) we may apply induction. Let \( m_1, \ldots, m_r \) be the dyadic decomposition of \( n \). Let \( \bar{L} \) be the subgroup of \( L \) with \( S \cap L \leq \bar{L} \leq \Sigma_{m_1} \times \cdots \times \Sigma_{m_r} \). Further we may assume that \( A \) acts nontrivially on all \( \Sigma_{m_i} \). Hence we have all \( m_i < 8 \), which has been handled before.

Let finally \( L \) be sporadic. By [MeiStr2] we get that \( L/Z(L) \cong M_{12}, M_{22}, M_{24}, J_2, Co_1, Co_2, \) or \( Suz \). Now we choose \( s \in Z(S \cap L \cap B) \). If \( B \not\leq O_2(C_L(s)) \), then by induction we get the assertion again. If there is some involution \( g \) in \( C_L(s) \) with \( [B, B^g] \neq 1 \), we have the second alternative. So we may assume that \( \langle B^{C_L(s)} \rangle \) is abelian. This gives \( L/Z(L) \cong M_{12} \). If \( L \cong M_{24} \) there is a subgroup \( \bar{L} \leq L \) with \( S \cap L \leq \bar{L} \) and \( L \cong E_{16}A_8 \). Now by induction we may assume \( B \leq O_2(\bar{L}) \). But there is no quadratic foursgroup in \( O_2(\bar{L}) \).

Let next \( L/Z(L) \cong M_{22} \). Then we may embed \( B \) into a subgroup \( SL(3, 4) \) and again we get the assertion by induction.

So we are left with \( L \cong M_{12} \). If \( B \not\leq L \), then with [MeiStr2] we see that \( B \not\leq S \cap L \), so we have \( B \leq L \). Now in \( L \) there are two parabolics \( P_1, P_2 \) such that \( P_i/O_2(P_i) \cong \Sigma_3 \). So if \( B \not\leq O_2(P_i) \) for some \( i \) we have induction again. Hence we may assume that \( B \) is contained in \( O_2(P_1) \cap O_2(P_2) \) and \( \langle B^{C_L(s)} \rangle \) is elementary abelian of order 8. Then this group contains an involution \( i \) which is fixed point freely on the 12 points moved by \( L \). So \( C_L(i) \cong Z_2 \times \Sigma_5 \). Further \( S \) contains a Sylow 2–subgroup of \( C_L(i) \). As \( B \leq C_L(i) \), we get the assertion by induction.
3 Modules

Lemma 3.1 Let $V$ be a module over $GF(2)$ for a solvable group $G$. Let $A \leq G$ be an elementary abelian subgroup such that $[V,A,A] = 1$. Then $V = \langle v \mid |A : C_A(v)| \leq 2 \rangle$.

Proof: \[GBSM\]

Lemma 3.2 Let $G$ be a group, $A$ a $p$–group, with $G = \langle A,a^g \rangle$ for some $a \in A^2$ and $g \in G$. Let $V$ be some $GF(p)$–module for $G$ with $[V,G] = V$. Let $A \leq B$ with $[V,B,B] = 1$, then $[V,A] = [V,B] = C_V(A)$.

Proof: We have $V = [V,G] = [V,A][V,a^g]$ and $C_V(A) = [V,A](C_V(A) \cap [V,a^g])$. As $C_V(A) \cap [V,a^g] \leq C_V(G)$ we get $C_V(A) \cap [V,a^g] \leq C_V(A) \cap [V,a] \leq [V,A]$. Hence $C_V(A) = [V,A]$. Now we have $[V,B] \leq C_V(A) = [V,A]$.

Lemma 3.3 Let $L \cong L_2(q)$ or $Sz(q)$, $q$ even, $V$ an irreducible $GF(2)$–module for $L$. Let $A \leq L$, $|A| \geq 4$, with $[V,A,A] = 1$. If $A \leq L_1$ with $L_1 \cong L_2(q_1)$ or $Sz(q_1)$ always implies $L_1 = L$, then $V$ is the natural module.

Proof: Let $A \leq S$, $S$ a Sylow 2–subgroup of $L$. By assumption we have $L = \langle A,a^g \rangle$ for any $a \in A^2$ and $g \in L \setminus N_L(S)$. By 3.2 we have $[V,A] = C_V(A)$ and $[V,a^g] \cap [V,A] = 1$. Hence $|[V,a]| = |[V,a^g]| = |V : [V,A]| = |V : C_V(a)|$. So we have $C_V(A) = C_V(a) = [V,a]$. This implies that all elements of odd order in $L$ which are inverted by $a$ act fixed point freely on $V$. Hence $V$ is the natural module by [Hig] or [Mar].

Lemma 3.4 Let $F^*(H)$ be quasisimple and $V$ be an irreducible $F^*(H)$–module over $GF(p)$ which is an $F$–module. Then $F^*(H)$ is classical, $G_2(q)$, $A_n$, or $3A_6$, and one of the following holds

1) $F^*(H)$ is classical or $A_n$ and $V$ is the natural module (in case of $A_n$ we have $p = 2$)

2) $F^*(H) \cong L_n(q)$ and $V$ is the exterior square of the natural module or its dual

3) $F^*(H) \cong Sp(6,q)$ or $\Omega^+(10,q)$ and $V$ is the spin module or half spin module, respectively. If $F^*(H) \cong \Omega^+(10,q)$, then this is sharp, i.e. there is no offender $A$ with $|V : C_V(A)| < |A|$.
4) $F^*(H) \cong G_2(q)$ and $V$ the natural module or $3A_6$, $p = 2$, and $V$ is the 6-dimensional module

5) $H \cong A_7$ and $V$ is the 4-dimensional module over $GF(2)$.

**Proof:** [GBSM]

**Lemma 3.5** Let $F^*(G)$ be quasisimple, $V$ be a faithful $GF(p)G$-module and $A$ an elementary abelian $p$-subgroup of $G$ which offends on $V$ as an $F$-module. Assume further $[V, A, A] = 1$ and $A \leq O_p(C_G(v))$ for all $1 \neq v \in [V, A]$. Then $A$ is a transvection group. In particular for $V$ irreducible we have one of the following

(i) $F^*(G) \cong SL(n, q)$, $Sp(2n, q)$, $V$ is the natural module and $A$ is a group of transvections to a point

(ii) $F^*(G) \cong \Omega^\pm(2n, 2)$ or $A_n$ and $|A| = 2 = p$, $V$ is the natural module.

**Proof:** By 3.4 we know the possible candidates for $F^*(G)$. Let first $F^*(G) \cong SL(n, q)$ and $V$ be the natural module. Now we have that $A \leq O_p(C_G(v))$ for some $v \in V$ and so it is a transvection group to the point $v$. Let next $V = V(\lambda_2)$. We may assume $n \geq 5$. Then by [GBSM] we get that $|A| = q^{n-1}$ and $A$ is a transvection group in the natural representation. Hence $V_1 = [V, A] \cong V(\lambda_2)$ for the group $GL(n-1, q)$. Further $V_1 \cap C_V(S) \neq 1$ for $A \leq S \in \text{Syl}_p(G)$. But now for $1 \neq v \in V_1 \cap C_V(S)$ we have $A \not\leq O_p(C_G(v))$.

Let next $F^*(G) \cong Sp(2n, q)$ and $V$ be the natural module. Now we choose $v \in V$ with $A \leq O_p(C_G(v))$. Suppose $[A, V] \not\leq (v)$. For $w \in [A, V] \setminus \langle v \rangle$ we have that $O_p(C_G(v)) \cap O_p(C_G(w))$ centralizes a subspace of codimension 2 in $V$. In particular $|O_p(C_G(v)) \cap O_p(C_G(w))| \leq q^2$. As $A$ offends and there is just one transvection group in $O_p(C_G(v))$, we get a contradiction. So $|A| = q$ and $A$ is a transvection group.

Let now $F^*(G) \cong Sp(6, q)$ and $V$ be the spin module. Then we get for some $v \in [V, A]$ that $C_G(v) = q^6GL(3, q)$. But there is no offender in $O_p(C_G(v))$.

Let $F^*(G) \cong \Omega^\pm(2n, q)$. Let $V$ be the natural module. Let $v$ be an isotropic vector, then $O_p(C_G(v))$ does not contain an offender. So we see that $|[V, A]| = q$ and $A$ is a transvection group. Hence $|A| = 2 = q$.

Let $F^*(G) \cong \Omega^+(10, q)$ and $V$ be the halfspin module. We have that $|V, A|$ contains some $1 \neq v$ such that $v$ is centralized by some Sylow $p$-subgroup which contains $A$. Hence $C_G(v) \cong q^{10}L_5(q)$. But then $O_p(C_G(v))$ does not contain any offender.
Let $F^*(G) \cong U_n(q)$ and $V$ be the natural module. Then there is some $1 \neq v \in [V, A]$ with $C_G(v) \cong q^{1+2(n-2)}U_{n-2}(q)$. But again $O_p(C_G(v))$ does not contain any offender.

Let next $F^*(G) \cong G_2(q)$ and $V$ be the natural module. Now for $1 \neq v \in [V, A]$ we have $C_G(v) \cong q^{2+1+2}GL_2(q)$. For $t \in A^2$ we have $| [V, t] | = q^2$ and so $| \cap_{v \in [V, t]} O_p(C_G(v)) | \leq q^2$, but an offender has order $q^3$.

Let finally $F^*(G) \cong A_n$. Then $p = 2$. If $V$ is the permutation module, we have some $1 \neq v \in [V, A]$ such that $C_{\Sigma_n}(v) \cong Z_2 \times \Sigma_{n-2}$ or $\Sigma_4 \times \Sigma_{n-4}$. In the former $A$ is a transvection group and so $|A| = 2$. In the latter $A$ is conjugated to a subgroup of $\langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$. But this group does not act quadratically.

Let next $F^*(G) \cong A_7$ and $V$ be the 4-dimensional module. Then $C_G(v) \cong L_3(2)$ and so $O_2(C_G(v)) = 1$ for any $1 \neq v \in V$.

Let $F^*(G) \cong 3A_6$ and $V$ be the 6-dimensional module over $GF(2)$. Then $| [V, A] | = 2^4$ and $\cap_{v \in [V, A]} O_2(C_G(v)) = 1$.

**Lemma 3.6** Let $F^*(H)$ be quasisimple and $V$ be an irreducible $F^*(H)$-module over $GF(p)$ which is a strong (dual) $F$-module. Then $H$ is classical or $\Sigma_n$ and $V$ is the natural module, or $F^*(H) \cong 3A_6$ and $V$ is one of the two 6-dimensional modules over $GF(2)$ or $H \cong A_7$ and $V$ the 4-dimensional module over $GF(2)$.

**Proof:** [GBSM]

**Lemma 3.7** Let $F^*(H)$ be a group of Lie type in characteristic two and $V$ be an irreducible $2F$-module in characteristic 2. Then $V$ is an $F$-module, or one of the following holds

1. $F^*(H) = L_m(r^2)$ and $V = V(\lambda_i) \otimes V(\lambda_i)^\sigma$, where $i = 1$ or $n - 1$ and $\sigma$ is a field automorphism of order two.
2. $F^*(H) = Sp(2n, r)$ and $V = V(\lambda_2)$
3. $F^*(H) = L_6(r)$ and $V = V(\lambda_3)$
4. $F^*(H) = Sp(2n, r)$, $n = 4, 5$, and $V = V(\lambda_n)$.
5. $F^*(H) = Sp(4, r^2)$ and $V = V(\lambda_1) \otimes V(\lambda_1)^\sigma$ or $V(\lambda_2) \otimes V(\lambda_2)^\sigma$, where $\sigma$ is a field automorphism of order two.
6. $F^*(H) = \Omega^-(8, r)$, $\Omega^-(10, r)$, $\Omega^+(12, r)$ and $V$ is the half spin module.
(7) \( F^*(H) = U_6(r) \) and \( V = V(\lambda_3) \)

(8) \( F^*(H) = U_3(r) \) or \( Sz(r) \) and \( V = V(\lambda_1) \)

(9) \( F^*(H) = G_2(q) \) and \( V = V(\lambda_2) \)

(10) \( F^*(H) = E_6(r) \) and \( V = V(\lambda_1) \) or \( V(\lambda_6) \)

(11) \( F^*(H) = E_7(r) \) and \( V = V(\lambda_7) \)

**Proof:** \([GBSM]\)

**Lemma 3.8** Let \( F^*(H) \cong A_n \), \( n = 7 \) or \( n \geq 9 \), and \( V \) be an irreducible 2F–module in characteristic 2. Suppose there is an offender \( A \) such that \( |V : C_V(A)| \leq |A|q \) for some 2-power \( q \). Assume further that \( A \) contains a quadratically acting subgroup \( B \) of order \( q \). Then \( V \) is an \( F \)–module, or \( n = 9 \) and \( V \) is an eight dimensional module.

**Proof:** If \( q = 2 \), we have that \( V \) is an \( F + 1 \)–module and the assertion follows with \([GBSM]\). So assume \( q > 2 \). Now we have a quadratic fours group. Now by \([MeiStr2]\) we have that \( V \) is either an \( F \)–module or the spin module and \( q = 4 \). Now in the latter we see that \( n = 9 \).

**Lemma 3.9** Let \( F^*(H) \), be a group of Lie type in odd characteristic and \( V \) be an irreducible 2F–module in characteristic 2. Suppose there is an offender \( A \) such that \( |V : C_V(A)| \leq |A|q \) for some 2-power \( q \), with \( |A| = q^s \), for some \( s \). Assume further that \( A \) contains a quadratically acting subgroup \( B \) of order \( q \) and \( |V : [V,A]C_V(A)| \leq q \). Then \( F^*(H) \) is a group of Lie type in characteristic two, too.

**Proof:** Suppose false. Suppose first \( q = 2 \). Then \( V \) is an \( F + 1 \)–module and \([GBSM]\) yields \( F^*(H) = 3U_4(3) \) and \( |V| = 2^{12} \). As there are no transvections we have \( |A| > 2 \). Hence we see that \( |A| = 2^5 \). But then \( A \) is uniquely determined and so \( A \) acts quadratically on \( V \). Then \( A \) would induce transvections, a contradiction.

So we have \( q > 2 \) and then we have a quadratic fours group. By \([MeiStr1]\) we get again \( F^*(H) = 3U_4(3) \) and \( |V| = 2^{12} \). Now we see that \( |A| \geq 16 \). As \( |A| \leq 2^5 \), we get \( q \leq 8 \). Now for \( q = 8 \), we get \( |A| = 64 \). Again \( A \) is uniquely determine and \( |C_V(A)| < 8 \). So we have \( q = 4 \). Hence again \( |A| \neq 64 \). So \( |A| = 16 \). In particular \( |C_V(A)| = 64 \). This shows again that \( A \) is uniquely determined and then that \( A \) acts quadratically. But then \( |V : C_V(A)| = 4 \), a contradiction.
Lemma 3.10 Let $F^*(H)$, be a sporadic simple group and $V$ be an irreducible $2F$–module in characteristic 2. Suppose there is an offender $A$ such that $|V : C_V(A)| \leq |A|q$ for some 2–power $q$, with $|A| = q^8$, for some $s$. Assume further that $A$ contains a quadratically acting subgroup $B$ of order $q$ and $|V : [V,A]C_V(A)| \leq q$. Then $F^*(H) = M_{22}$, $M_{23}$, or $M_{24}$ and $|V| = 2^{10}$, $2^{11}$, $2^{11}$, respectively.

Proof: If $V$ is an $F + 1$–module the assertion follows with [GBSM]. So we may assume $q > 2$. Then we have a quadratic fours group and the groups and modules are given by [MeiStr2]. If there is no quadratically acting group of order 8, then we get $q = 4$. Hence $|A| \geq 16$.

Suppose $|F^*(H)| = M_{12}$. Then $|V| = 2^{10}$. Further $A \not\leq F^*(H)$. Now there is $a \in A$ with $C(a)$ involves $A_5$ and $|[V,a]| = 2^5$. But then $A_5$ would induce transvections on $C_V(a)$, a contradiction.

Suppose next $F^*(H) = J_2$. Now $A$ contains a non 2–central involution $a$. Then $|[V,a]| = 2^6$, which contradicts $|[V,x]| \leq 2^5$ for any $x \in A$.

Suppose next $F^*(H) = Co_1$. Suppose there is a 2–central involution $z$ in $A$. Then as $V$ is the Leech lattice module, we see $|[V,a]| = 2^8$. Hence $2^8 \leq |A| \geq 2^{10}$, remember $|A|$ is a power of 4. We have $|[V,a] : C_{[V,a]}(A)| \leq 4$. As $C_{F^*(H)}(a)/O_2(C_{F^*(H)}(a)) = \Omega^+(8,2)$, we get $|A : A \cap O_2(C_{F^*(H)}(a))| \leq 2$. But as $O_2(C_{F^*(H)}(a))$ is extraspecial of order $2^9$ this is impossible. Hence we have that there is no 2–central involution in $A$. Let $a$ be as before, we get $A \cap O_2(C_{F^*(H)}(a)) = 1$. In particular $|A| \leq 2^6$, and so $|[V,z]| \leq 2^7$ for all $z \in A$. But $|[V,x]| \geq 2^8$ for all involutions in $H$.

Let next $F^*(H) = Co_2$. Now any quadratic fours group contains a 2–central involution, in particular $A$ contains a 2–central involution $a$. Now we see that $|A| \geq 2^8$, while $|A : A \cap O_2(C_{F^*(H)}(a))| \leq 2$, and so we have a contradiction as before.

Let $F^*(H) = 3Sz$. As $3Sz \leq Co_1$ and the modules are the same, we have a contradiction again.

So we are left with $F^*(H) = 3M_{22}$ and $|V| = 2^{12}$. Now $q > 2$. As $|A| \leq 2^6$, we get $q = 4$ and $|A| = 16$. Now $|V : C_V(a)| \leq 2^5$ for all $a \in A$. Hence we get $A \leq F^*(H)$ and there are exactly two possibilities. If $N(A)$ involves $A_6$, we get $|C_V(A)| = 64$ and $A$ acts quadratically, a contradiction. So we have that $N_A$ involves $\Sigma_5$ and then $|C_V(A)| = 4$. But we know $|V : C_V(A)| \leq 2^6$, a contradiction again.
Lemma 3.11 Let $p$ be an odd prime and $K$ be an automorphism group of a simple group and $V$ be an irreducible $GF(p)G$-module. Let $A \leq K$, $|A| = p^2$. Assume that $A$ acts quadratically on $V$ and $\langle Y^p \rangle$ is abelian for any proper parabolic $P$ of $K$. Then $K$ is a group of Lie type in $GF(p)$.

Proof: [GBSM]

Lemma 3.12 Let $F^*(H) \cong A_n$, $n \neq 5, 6, 8$, and $V$ be the natural module. Then there is no $1 \neq Q$ in $H$, $Q$ a 2-group such that $Q \leq O_2(C_H(x))$ for all $1 \neq x \in C_V(Q)$.

Proof: Let $S$ be a Sylow 2-subgroup with $Q \leq S$. Then $Q \leq O_2(C_H(x))$ for all $1 \neq x \in C_V(S)$. In particular $O_2(C_H(x)) \neq 1$. Hence we have that $Q$ is conjugate to a subgroup of $\Sigma_4$ acting on $\{1, 2, 3, 4\}$, say. But then we see that $H = \langle C_H(x) \mid 1 \neq x \in C_V(Q) \rangle$.

Lemma 3.13 Let $X \cong A_n, n \geq 5, V$ be a $GF(2)X$-module with $V/C_V(X)$ the natural irreducible permutation module. Assume $[V, X] = V$. Then $|C_V(X)| \leq 2$, and $|C_V(X)| = 1$ if $n$ is odd. Furthermore $V$ is a submodule of the permutation module.

Proof: For simplicity we prove the dual statement. Let $C_V(X) = 1$ and $[V, X]$ be the natural module. Then $|V : [V, X]| \leq 2$, $|V : [V, X]| = 1$ for $n$ odd, and $V$ is a factor of the permutation module.

This will be proved by induction on $n$. For $n = 5$ this is well known. So let $n > 5, K \cong A_{n-1}, K \leq X$. If $n - 1$ is odd, then we have that $[V, X] = T \oplus T_1, T_1$ the irreducible permutation module for $K, [T, K] = 1$. By induction $V = [V, K] \oplus \hat{T}, [V, K]$ the irreducible permutation module. Hence there is $v \in V \setminus [V, X], [v, K] = 1$, i.e. $\langle v^X \rangle = V$ is a factor of the permutation module.

Let $n - 1$ be even. Then we have a $K$-chain. $1 < T < T_1 < [V, X] < V$, with $|T| = 2, T_1/T$ the irreducible $K$-module and $|[V, X]/T_1| = 2$. Now by induction $C_{V/T}(K) \neq 1$. As $C_{V/T}(K) \nleq [V, X]/T$, we again get some $v \in V \setminus [V, X], [v, K] = 1$, and so $V$ is a factor of the permutation module.

Lemma 3.14 (a) Let $L$ be one of the following groups $(S)L_n(q), SU_n(q), Sp_{2n}(q), G_2(q), \Omega^+(2n, q), n \geq 2, q$ even, and $V$ be a module over $GF(2)$ with $[V, L]$ the natural module and $C_V(L) = 1$. Then $V = [V, L]$, or one of the following holds:
(i) \( L \cong L_2(q) \), and \([V : [V, L]] \leq q\)

(ii) \( L \cong L_3(2) \), and \(|V| = 16\)

(iii) \( L \cong U_4(2) \), and \(|V| \leq 2^{10}\)

(iv) \( L \cong Sp_{2n}(q) \), and \([V : [V, L]] \leq q\)

(v) \( L \cong G_2(q) \), and \([V : [V, L]] \leq q\).

(vi) \( L \cong \Omega^+(6, 2) \) and \(|V| = 2^7\).

(b) Let \( L \cong Sp_6(q), q \) even, \( V \) be a module over \( GF(2) \) with \( C_V(L) = 1 \) and \([V, L] \cong V(\lambda_3)\). Then \( V \cong V(\lambda_3)\).

(c) If in (a) or (b) we have \( V = [V, L] \) and \( V/C_V(L) \) is the natural module or \( V(\lambda_3) \), respectively, then we either get \( C_V(L) = 1 \) or we have the exceptions as in (a) and the bound for \(|C_V(L)|\) is the same as the bound for \(|[V, L]|\) in (a).

Proof: (c) follows by duality. Hence we just have to prove (a) and (b). Let us start with (a).

(1) Let first \( L \cong L_n(q)\):

If \( L \cong L_2(q) \), then for \( x \in L, o(x) = 2 \), we have \(|V : C_V(x)| = q\).
We have that \( L \) is generated by three conjugates of \( x \). Hence \(|V| \leq q^3\).

Let now \( L \cong SL_3(q) \). If \( q > 4 \), then there are three elements \( x_1, x_2, x_3 \) in \( L \) acting fixed point freely

\[
\begin{align*}
x_1 &= \begin{pmatrix} \omega^{-2} & \omega \\ \omega & \omega \end{pmatrix}, &
x_2 &= \begin{pmatrix} \omega & \omega^{-2} \\ \omega & \omega \end{pmatrix}, &
x_3 &= \begin{pmatrix} \omega & \omega \\ \omega & \omega^{-2} \end{pmatrix},
\end{align*}
\]

\( o(\omega) = q - 1 \).

We have \([x_i, V] = [V, L], i = 1, 2, 3 \), and so as \([x_i, x_j] = 1 \) for all \( i, j \),
we get \( C_V(x_1) = C_V(x_2) = C_V(x_3) \). But \( L = \langle C_L(x_i) | i = 1, 2, 3 \rangle \), the assertion.
If \( q = 4 \), we have \( V = [V, Z(L)] = [V, L] \). So let \( q = 2 \). Then there is \( x \in V \setminus [V, L], |x^L| = 8 \). Hence \( V \) is a factor module of the permutation module, which shows \( |V| \leq 16 \).

Let next \( L \cong L_4(2) \). There is \( \langle \rho \rangle \times A_5 \leq L, o(\rho) = 3, [V, L] = [V, \rho] \) and \([V, L] = [V, \gamma], \gamma \in A_5, o(\gamma) = 3 \). Hence \( C_V(\rho) = C_V(\gamma) \). Now as \( \langle C_L(\gamma), C_L(\rho) \rangle = L, \) we get \( V = [V, L] \).

Let now \( n \geq 4, q > 2 \) for \( n = 4 \). Let \( P \) be the parabolic in \( L \) with \( |O_2(P)| = q^{n-1} \) and \( P'/O_2(P) \cong SL_{n-1}(q) \). Assume \( |C_{[V, L]}(O_2(P))| = q^{n-1} \). Then \( [P', V] = C_{[V, L]}(O_2(P)) \), as \( P' = P'' \). This now shows that \( [C_V(O_2(P)), P'] \) is the natural module. By induction on \( n \) we have that \( C_V(O_2(P)) = C_V(P') \oplus C_{[V, L]}(O_2(P)) \). Now application of Gaschütz [Hu] shows \( C_V(P') = 1 \). Hence as \( |V : C_V(O_2(P))| = q, \) we get \( V = [V, L] \).

(2) Let next \( L \cong SU_n(q) \):

If \( L \cong SU_3(q) \). Then there are \( x_1, x_2 \in L, x_2 \notin \langle x_1 \rangle, o(x_1) = q + 1 = o(x_2), [x_1, x_2] = 1 \) and \( [V, L] = [V, x_i], i = 1, 2 \). Hence \( C_V(x_1) = C_V(x_2) \). As \( C_V(x_1) \cong Z_{q+1} \times L_2(q) \), is a maximal subgroup of \( L \), we get that \( L = \langle C_L(x_1), C_L(x_2) \rangle \), the assertion.

Let now \( L = U_4(q), q > 2 \). There are \( x_1, x_2, x_3 \in L, (o(\omega) = q + 1) \)

\[
\begin{align*}
x_1 &= \begin{pmatrix} \omega^{-3} & \omega & \omega \\
\omega & \omega & \omega \\
\omega & \omega & \omega^{-3} \end{pmatrix}, \\
x_2 &= \begin{pmatrix} \omega & \omega^{-3} & \omega \\
\omega & \omega & \omega \\
\omega & \omega & \omega \end{pmatrix}, \\
x_3 &= \begin{pmatrix} \omega & \omega & \omega \\
\omega & \omega & \omega \\
\omega^{-3} & \omega & \omega \end{pmatrix}.
\end{align*}
\]

As \( q > 2 \), we have \( |V, L| = [V, x_i], i = 1, 2, 3 \). Now \( C_V(x_1) = C_V(x_2) = C_V(x_3) \). As \( C_L(x_1) = \langle x_1 \rangle SU_3(q) \), we get \( L = \langle C_L(x_1), C_L(x_2), C_L(x_3) \rangle \). Hence we have the assertion. Let now \( q = 2 \). Let \( P \) be the parabolic with \( P'/O_2(P) \cong L_2(4) \). Now \( |C_{[V, L]}(O_2(P))| = 16 \). As \( C_{[V, L]}(O_2(P)) \not\cong O_2(P) \) as \( P'/O_2(P) \)-modules, we get \( V = [V, L] \oplus C_V(O_2(P)) \). We have \( |C_V(O_2(P)) : [C_V(O_2(P)), P]| \leq 4 \). Hence \( |V : [V, L]| \leq 4 \) by Gaschütz.
Let now $L \cong U_5(2)$. In this case we have $x_1, x_2, x_3, x_4$,

\[
x_1 = \begin{pmatrix} \omega^{-1} & \omega \\ \omega & \omega \end{pmatrix}, \quad x_2 = \begin{pmatrix} \omega^{-1} & \omega \\ \omega & \omega \end{pmatrix}, \ldots
\]

and so on, $o(\omega) = 3$.

Further $[x_i, x_j] = 1$. Now $[V, L] = [V, x_j], i = 1, \ldots, 4$ and so $C_V(x_1) = C_V(x_2) = C_V(x_3) = C_V(x_4)$. We have $C_L(x_i) \cong \langle x_i \rangle \times U_4(2)$. Hence $L = \langle C_L(x_i) | i = 1, 2, 3, 4 \rangle$ and then $V = [V, L]$.

If $L \cong SU_6(2)$, then $[Z(L), V] = [V, L]$ and we get $V = [V, L]$, as $|Z(L)| = 3$. Let now $L \cong SU_n(q), q > 2$ for $n = 5$ or $6$.

Let $P$ be the normalizer of a root group $R$ in $L$. We have $|[V, L], R]| = q^2$. We have $P'/O_2(P) \cong SU_{n-2}(q)$ and $C_{[V,L]}/[[V, L], R] \cong O_2(P)/R$. Now $[V, R] = [[V, L], R]$. Furthermore as $[P', V] \leq C_{[V,L]}(R)$, we see that $V/[V, L], R] = [V, L]/[[V, L], R] \cdot C_V/[[V, L], R](O_2(P))$. Let $V_1$ be the preimage of $C_V/[[V, L], R](O_2(P))$. Then $[V_1/[[V, L], R], P']$ is the natural $SU_{n-2}(q)$-module. By induction $V_1/[[V, L], R] = (V_1/[[V, L], R])C_{V_1/[[V, L], R]}(P')$.

Now as $P' = P''$, we get $[V_2, P'] = 1$ for a preimage $V_2$ of $C_{V_1/[[V, L], R]}(P')$. Hence $V_2 = 1$ and so $V = [V, L]$, the assertion.

(3) Let next $L \cong Sp_{2n}(q)$:

If $L \cong Sp_{4}(2)'$, the assertion is well known. Let now $L \cong Sp_{2n}(q), q > 2$, for $n = 2$. Let $P$ be the parabolic with $P'/O_2(P) \cong Sp_{2n-2}(q), A_6$ for $L \cong Sp_{6}(2)$. Now we see $V = [V, L]C_V(Z(O_2(P)))$. Set $V_1 = C_V(Z(O_2(P)))$.

Now $[V_1, P'] = [[V, L], P']$. Set $V_2 = [V, Z(O_2(P))]$. Then $|V_2| = q$. We see that $[V_1/V_2 : C_{V_1/\langle V_2 \rangle}(O_2(P))] = q$. We have

\[
V_1/V_2 = ([V, L], P'/V_2)C_{V_1}(O_2(P))/V_2.
\]

Hence as $P' = P''$ we have $[C_{V_1}(O_2(P)), P'] = 1$. This shows $C_{V_1}(O_2(P)) \leq [V, L]$ and so $|V : [V, L]| \leq q$ by Gaschütz [Hu], or we have that $L \cong Sp_{6}(2)$ and $[P, C_{V_1}(O_2(P))] \neq 1$. Hence there is some $t \in P \setminus P', o(t) = 2$, with $[C_{V_1}(O_2(P)), t] = V_2$. We may assume that $t$ induces a transvection on the natural module. As $[t, [V, L]/V_2] \neq 1$ we now see $|[V, t]| \geq 4$. But $\langle t \rangle$ is
conjugate to $Z(O_2(P))$ and $[V, Z(O_2(P))] = V_2$ is of order 2.

(4) Let $L \cong G_2(q)$:

Let $R$ be root group and $P = C_L(R)$. Assume first $q \geq 4$. If $q > 4$, then $O_2(P/R)$ is an irreducible $P$–module. For $q = 4$, this is true for $N_L(R)$, while for $P$ it is a diirct some of two orthogonal modules. Now let

$$1 < V_1 < V_2 < [V, L] = V_3$$

where $V_i/V_{i-1}$ is the natural $L_2(q)$–module. Let $x \in R^\delta$. Then we see $V = C_V(x)[V, L]$. We have that $P$ acts on $C_V(x)$ and $|C_V(x) \cap [V, L]| = q^4$.

As $C_V(x) \cap [V, L]$ is not isomorphic to $O_2(P)/R$ as $P$–module we get $V = C_V(O_2(P))[V, L]$ and $|C_V(O_2(P)) \cap [V, L]| = q$. By (1) applied to $P/O_2(P)$ we get $|C_V(O_2(P)) : C_V(P)| \leq q$. As $C_V(P) = 1$, the assertion follows.

Let now $q = 2$. Then the normalizer $N$ of a Sylow 3-subgroup in $G_2(2)'$ is $3^{1+2}Z_8$. Further the element of order three in the center of a Sylow 3-subgroup acts fixed point freely on $[V, L]$. Hence $V = [V, L] \oplus C_V(N)$. Let $S$ be a Sylow 2–subgroup of $N$. Then $|C_{[V,L]}(S)| = 2$. Let $x \in N_L(S)$, $x \notin S$, $x^2 \in S$. Then $C_V(N) \cap C_V(x) \leq \langle N, x \rangle = l'$. So we have $C_V(N) \cap C_V(x) = 1$. In particular $|C_V(x) \cap C_V(S)| = 2$, which shows $|C_V(S)| \leq 4$ and so $|V : [V, L]| \leq 2$.

53) Let next $L \cong \Omega^+(2n, q)$:

Let $L \cong \Omega^+(6, q), q > 2$. Let $P$ be the parabolic with $P'/O_2(P) \cong L_2(q) \times L_2(q)$. Then let $V_1 = C_V(O_2(P))$. We have $|V_1| = q, [V, O_2(P)]/V_1 \cong O_2(P)$ and $[P', V] = [V, O_2(P)]$, as $P' = P''$. This shows $|V/V_1 : C_{V/V_1}(O_2(P))| = q$. Now $|C_{V/V_1}(O_2(P)) : C_V(O_2(P))/V_1| = q^4$. Hence $|C_V(O_2(P)), P'| = 0$. This gives with [Hu, (1.17.4)] that $C_V(O_2(P)) \leq [V, L]$ and then $V = [V, L]$.

Let $L \cong \Omega^+(6, 2)$. Now $A_8 \cong \Omega^+(6, 2)$ and $[V, L]$ is the permutation module. Hence the assertion follows with 3.13.

Let $L \cong \Omega^+(2n, q), n \geq 4$ for $L \cong \Omega^+(2n, q)$. Let $P$ be the parabolic with $P'/O_2(P) \cong \Omega^+(2n-2, q)$. Set $V_1 = C_{[V/L]}(O_2(P)), |V_1| = q$. We have $|P', V| = [V, O_2(P)]$, as $P' = P''$. Hence we see $|V/V_1 : C_{V/V_1}(O_2(P))| = q$. Furthermore $C_{V/V_1}(O_2(P)) = [C_{V/V_1}(O_2(P)), P']C_{V(O_2(P))/V_1}$. Now we get $|C_V(O_2(P)), P'| = 0$. By [Hu, (1.17.4)] we have $C_V(O_2(P)) \leq [V, L]$ and so $[V, L] = V$. 26
(6) Let finally \( L \cong \Omega^+(4,q) \):

We write \( L \) as \( L_1 \times L_2, \ L_i \cong L_2(q) \). We may assume \( q > 2 \), as the
assertion is obvious for \( q = 2 \). Now there is \( \omega_i \in L_i \), with \( o(\omega_i) = q + 1 \) and \( \omega_i \)
acting fixed point freely on \([V,L], i = 1,2\). Now choose \( v \in V \setminus [V,L] \) with
\([v_1,\omega_1] = 1\). Then \( v_1 \) is uniquely determined in \([V,L]v_1\). So \( K_2 \) centralizes \( v_1 \) too. Hence \([\omega_2,v_1] = 1\). As there is a unique fixed point of \( \omega_2 \) in \([V,L]v_1\),
we see \([K_1,v_1] = 1\), so \([v_1,L] = 1\), a contradiction. This proves (a).

To prove (b) let \( L_1 \leq L, \ L_1 \cong \Omega^+_6(q) \cong L_4(q) \). Then \([V,L] \) is an
extension of the natural \( L_4(q) \)-module by the natural module. Hence \( V = \[V,L] \oplus C_V(L_1) \).
Let \( P_1 \leq L_1 \) be the parabolic which is the stabilizer of
a 2-space in the natural representation of \( L_4(q) \). Then \( C_V(P_1) = C_V(L_1) \).
We have \( P_1 \leq P, P \) the stabilizer of a 1-space in the natural representation
of \( Sp_6(q) \). Now \( Z(P') \) centralizes \( P_1 \) and so \([Z(P'),C_V(P_1)] = 1\). Hence
\( C_V(P_1) \) is centralized by \( \langle L_1, Z(P') \rangle = L \). This shows \( C_V(L_1) = 1 \) and then
\( V = [V,L] \). This finishes (b).

Lemma 3.15 Let \( G = L_3(2) \) and \( V \) some \( GF(2) \)-module such that \([V,G] \)
is the natural module. If \([V/[V,G]] = 2 \), and some element in \( G \) induces a
transvection on \( V \), then \( C_V(G) \neq 1 \). By duality if \( V/C_V(G) \) is the natural
module and \([C_V(G)] = 2 \) and some element induces a transvection, then
\([V,G] \) is the natural module.

Proof: Let \( x \in V \setminus [V,G] \) with \( |C_G(x)| = 21 \), then \(|x^G| = 8 \) and so
\( x[V,G] = x^G \). But if there are transvections in \( G \) they have to centralize
some elements in \( V \setminus [V,G] \), a contradiction.

Lemma 3.16 Let \( G = SL_n(q), q = p^f \), and \( V \) an \( GF(q) \)-module with a
submodule \( V_1 \) such that \( V_1 \) and \( V/V_1 \) are natural modules. Assume further
that \( n \geq 4 \) or \( q = 2 \), then \( V \cong V_1 \oplus V_1 \) as \( G \)-module.

Proof: Let \( H \leq G \) be the point stabilizer on \( V_1 \). Let \( W \leq V \) with \( V_1 \leq W \) and \([W,H] \leq V_1 \) and \(|W : V_1| = q|\). Now \( O_2(H) \) induces transvections
on \( W \) and so we get \(|C_W(O_2(H))| = q^2|\). Hence \( W = V_1 \oplus \langle R \rangle \) as \( H \)-module.
Let \( T \) be the conjugacy class of \( r \in R^2 \). Then we see that \( T \)
is a transversal of \( V_1 \) in \( V \). Hence for \( xO_2(H) \) we get \(|[V,x]| = q^2|\). Let
\( \tilde{V} \) such that \( \tilde{V}/\langle v \rangle = C_{V/\langle v \rangle}(O_2(H)) \). Then \( \tilde{V}/C_{\tilde{V}}(O_2(H)) \) is the natural
module and \( V/C_V(O_2(H)) = \tilde{V} \oplus V_1/C_V(O_2(H)) \). If \( \tilde{V}/\langle v \rangle \) is a direct sum
as \( SL_{n-1}(q) \)-module, then we get an \( S \)-invariant complement to \( V_1 \), where
$S$ is a Sylow $p$-subgroup of $G$ and application of Gaschütz [Hu] yields the assertion. Hence we may assume that not, then 3.14 implies $n - 1 = 3$ and $q = 2$. (The proof of 3.14(1) also works for $p$ odd. Now by 3.15 $x$ does not induce a transvection on $V/\langle v \rangle$ and so $|V/\langle v \rangle| \geq 8$, a contradiction.

**Lemma 3.17** Let $V$ be a $K_1$-module and $W$ be a $K_2$-module, both over $GF(q)$. Let further $X = V \otimes W$ and $B = A_1 \times A_2$, with $A_i \leq K_i$, $i = 1, 2$, where $A_i$ act quadratically on $X$. Then for at least one $i$ we have $|A_i| \neq |X : C_X(A_{3-i})|.$

**Proof:** Assume $|A_1| = |X : C_X(A_2)|$ and $|A_2| = |X : C_X(A_1)|$. Let $|V| = q^n$ and $|W| = q^m$. Further let $|V : C_V(A_1)| = q^a$ and $|W : C_W(A_2)| = q^b$. Then $|X : C_X(A_1)| = q^{am}$ and $|X : C_X(A_2)| = q^{bm}$. Now the maximal order of a quadratic subgroup in $GL(V)$ is $q^{a(n-a)}$ and in $GL(W)$ is $q^{b(m-b)}$. So we get $am \leq b(n-b)$ and $bn \leq a(m-a)$. But the former yields $a < b$ while the latter inequality yields $b < a$, a contradiction.

**Lemma 3.18** Let $V$ be an $F$-module for $K_1 \times K_2$ with $C_V(K_1 \times K_2) = 1$ and quadratic offender $A$, where $K_i$ are automorphism groups of quasisimple groups. Assume $[K_1, A] \leq K_i$ and $C_A(K_i) = 1$, for $i = 1, 2$. Then for some $1 \neq v \in [V, A]$, we get $A \not\leq O_p(C_{K_1 \times K_2}(v)).$

**Proof:** Set $K_1 A = K_1 \times T$. Let $V_1$ be an irreducible submodule for $K_1$. As $C_A(K_2) = 1$, we get that there is $V_1$ with $[V_1, K_2] \neq 1$. Set $W = V_1 \oplus \cdots \oplus V_s = V_1^T$. We have $s > 1$. Now we have that $V_1$ is better than $F$-module for $K_1$ and so with 3.5 we get $E(K_1) \cong L_n(q)$, $Sp(2n, q)$, $\Omega^\pm(2n, 2)$ or $A_n$. As $|V_1 : C_{V_1}(A)| < |A|$, this is not possible.

**Lemma 3.19** Let $M \in \mathcal{L}(S)$, $M \not\leq \bar{C}$, with $M_0$ be maximal and $K$ be a $p$-component of $F_p^*(M_0)$. Assume that $\langle K, S \rangle$ induces a strong $F$-module on $Y_M$, then $\langle K, Q \rangle = KQ$.

**Proof:** If $K$ is solvable, then $K = F_p(M_0)$ and of course $K$ is normalized by $Q$. So we may assume that $K$ is nonsolvable. Let $K_1 K_2 \ldots K_r = \langle K^Q \rangle$ with $r > 1$. By [GBSM] we have $[Y_M, K_1 K_2 \ldots K_r] = V_1 \oplus \cdots \oplus V_r$, where $[V_i, K_i] = V_i$ is a strong $F$-module for $K_i$ and $[V_i, K_j] = 1$ for $i \neq j$. Now let $1 \neq x_i \in C_{V_i}(S \cap K_i)$. Hence we may assume that $y = x_1 x_2 \cdot \cdots x_r \in C_{Y_M}(Q)$. Now we have with 2.4 that $[C_{K_1}(y), Q] \leq Q$. Hence we have $C_{K_1}(y)$ is a two group. So application of 3.6 shows that $K_1/O_p(K_1) = SL(2, q)$ and so $\langle K, S \rangle = M_0 S$ is a minimal parabolic. But this contradicts 2.7.
Lemma 3.20 Let $M \in \mathcal{L}(S)$, $M \not\leq \mathcal{C}$, with $M_0$ maximal. Assume $b(M) > 1$ and there is no $g \in G$ such that $1 \neq [Y_M, Y_M^g] \leq Y_M \cap Y_M^g$. Let $P \leq ES$, $P \in \mathcal{P}(S)$, be a minimal parabolic with $O_p((M, P)) = 1$, then $Y_M$ is a strong $F$-module or the a strong dual $F$-module.

**Proof:** By 2.2(e) and 2.6 we may assume $M = M^*$. We consider the amalgam $(M, P)$. Let $(\alpha, \alpha')$ be a critical pair. As $P \leq ES$, we see that $\alpha$ is of type $M$. Suppose $\alpha'$ is also of type $M$. As $b(M) > 1$, we then would get $[Y_M, Y_{\alpha'}] = 1$, a contradiction. So we have that $b > 2$ is odd. Now we may apply 2.11 - 2.14. The case $[V, R] = 1$ cannot occur, as otherwise $[Y_M, Q] = 1$ and so $Q \leq O_p(M)$, which by 2.4 gives $M \leq \mathcal{C}$, a contradiction.

Lemma 3.21 Let $M \in \mathcal{L}(S)$, $M \not\leq \mathcal{C}$, with $M_0$ maximal. Let $K$ be a $p$-component in $F_p^*(M_0)$. Assume that $(K, S)$ induces a strong (dual) $F$-module in $Y_M$. Then $K/O_p(K) \cong SL(n, q)$, $Sp(2n, q)$ or $\Omega^\pm(2n, 2)$ and $[Y_M, K]$ is the natural module, or $K \cong 3A_6$ and $[Y_M, K] = 64$, or $Y_M$ is a strong $F$-module $K \cong SL(n, q)$ and $[Y_M, K]$ is a direct sum of natural modules, or $n = 3, q > 2$ and there are exactly two such modules involved. Further $C_{YM}(K) = 1$. If $K \cong \Omega^\pm(2n, 2)$, the offender is of order 2.

**Proof:** By 3.6 we have that $K/O_p(K) \cong SL(n, q)$, $Sp(2n, q)$, $\Omega^\pm(2n, 2)$, $3A_6$ and the irreducible modules in $[Y_M, K]$ are as in the assertion, or $K/O_2(K) \cong A_n$ with the natural module or $A_7$ with the four dimensional module, or $K$ is solvable. So let us handle the nonsolvable case first.

By 3.19 we have $[K, Q] \leq K$. Further by 2.4 we have $C_{YM}(K) = 1$. Let $V$ be an irreducible submodule of $Y_M$. If $K/O_2(K) \cong A_7$ then $C_{K/O_2(K)}(x) \cong L_3(2)$ for $x \in V'$, which contradicts 2.4. Let now $K/O_2(K) \cong A_n$ and $V$ be the natural module. Let $T \in \text{Syl}_2(KQ)$, with $Q \leq T$ and $1 \neq x \in C_V(T)$, with $C_{\Sigma_n}(x) \cong \Sigma_m \times \Sigma_r$. By 2.4 we may assume $O_2(\Sigma_r) \neq 1$. Hence $r = 2, 4$. If $r = 2$ and $m > 4$, then $Q/O_2(K) \cap Q \cong \langle 12 \rangle$. But then there is $1 \neq w \in C_V(Q)$ with $Q \not\leq O_2(C_{KT}(w))$, contradicting 2.4. So we have $r = 4$. Let $m > 4$. Then $Q/O_2(K) \cap Q$ is conjugate to a subgroup of $\langle 12 \rangle(34), (13)(24)$ and we get a contradiction as before. So we have $m \leq 4$ in particular $n = 7$. But now there is also some $1 \neq y \in C_V(T)$ with $C_{\Sigma_7}(y) = \Sigma_6$, which again contradicts 2.4.

Hence we have that there is no such $x$. In particular $n = 2^r$ or $2^r + 1$. In the first case there is some $x$ with $C_{\Sigma_n}(x) \cong \Sigma_{n-1}$, again a contradiction to 2.4. Hence $n = 2^r$ and there is some $x$ with $C_{\Sigma_n}(x) \cong \Sigma_{2^{r-1}} \times \mathbb{Z}_2$. By 2.4, we now have $n = 8$, which is the case $O^+(6, 2)$.
So we are left with $K$ solvable. Then by the dihedral argument we have $|A| = p$ and so $A$ induces a transvections on $Y_M$. Now we see with 2.4 that $F^*(M_0/O_2(M_0))$ is a 2– or 3–group, which is generated by subgroups $X$ which are of order three or quaternion groups of order eight and $|Y_M, X| = p^2$. Hence again by 2.4 we have that $Q$ acts irreducibly on $F^*(M_0/O_2(M_0))/\Phi(F^*(M_0/O_2(M_0)))$, hence $M_0S$ is a minimal parabolic, contradicting 3.19.

What is left, is to prove that $[Y_M, K]$ is irreducible or a direct sum of natural modules in case of $SL(n, q)$. If $Y_M$ is a strong dual module this is clear. So we have a strong $F$–module. In case of $3A_6$ and $\Omega^\pm(2n, q)$, we see that $|V : C_V(A)| = |A|$ for an offender $A$ and an irreducible submodule $V$, hence $[Y_M, K]$ is irreducible. If $K = Sp(2n, q)$ and $V$ is the natural module, then for $a \in A$ we have that $[V, a]$ is uniquely determined by $C_V(a)$. Hence $|A| = |V : C_V(A)|$ and so again $[Y_M, K]$ is irreducible. Let $K = SL(n, q)$.

By 3.16 we have the assertion about $[Y_M, K]$ or $q > 2, n = 3$.

**Lemma 3.22** Let $M \in L(S), M \not\leq \tilde{C},$ with $M_0$ maximal. Let $p = 2$ and let $A \leq M$ with $|A| \geq 4$, $C_A(Y_M) = 1$ and $|Y_M : C_{Y_M}(A)| = 4 = |Y_M : C_{Y_M}(a)|$ for $a \in A^2$. Then either $F^*(M_0/O_2(M_0))$ is classical or $3A_6$ and $[Y_M, F^*(M_0/O_2(M_0))]$ is the natural module, or for $R = \langle A^M \rangle$ we have $R/O_2(R) \cong L_n(2)$ and $[Y_M, R]$ is a direct sum of two natural modules for $R$, $R \leq M_0$ and $M_0/O_2(M_0) \cong L_n(2)$ or $L_n(2) \times \Sigma_3$.

**Proof:** Let $R = \langle A^M \rangle$. As $A$ acts quadratically, we see that $R$ has exactly one 2-component $K$ which then is nonsolvable. Set

$$Y = [R, Y_M]/C_{[R, Y_M]}(R).$$

Suppose that $Y$ involves exactly one irreducible module. Then this is a strong $F$–module and we may apply 3.6. Now we see that $C_{M_0}(K)$ just can induce field multiplications on $Y$. This with 2.4 shows $K \leq M_0$ and $C_{[R, Y_M]}(R) = 1$. By 3.19 we have $[K, Q] \leq K$. Suppose $F_p(M_0) \neq 1$, then there is some $t \in F^*(M_0/O_2(M_0)) \setminus E(M_0/O_2(M_0))$ such that $C_Y(t) \cap C_Y(Q) \neq 1$, which shows $[Q, t] = 1$. Hence we get that $F_p^*(M_0) = E_p(M_0)$.

So we have that $Y$ involves two modules. Then $A$ induces transvections and then by 3.4 we have that $R \cong L_n(2)$ for some $n$ and we have two natural modules involved. By 3.15 and 3.14 we have that $Y$ is an extension of the natural module by the natural module. Then by 3.16 we get that $Y$ is a direct sum. Now again 3.14 and 3.15 show that $Y = [Y_M, R]$. Now we see that $C_{M_0}(R)$ induces a subgroup of $\Sigma_3$ on $Y$. So if $R \not\leq M_0$ we get with 2.4
\[ M_0/O_2(M_0) \cong \Sigma_3, \] contradicting 2.7. Hence \( R \leq M_0 \). And so again we have \( M_0/O_2(M_0) \cong L_n(2) \), or \( L_n(2) \times \Sigma_3 \).

**Lemma 3.23** Let \( M \in L(S) \), \( M \not\leq \tilde{C} \), and \( M_0 \) be maximal. Let \( Y_M \) be a strong dual \( F \)-module or a weak strong dual \( F \)-module (where \( p = 2 \) in the latter) with offender \( A \), then \( E_p(M_0) \) has just one component. Further the offender acts nontrivially on \( E_p(M_0) \) and \( F_p^*(M_0) = E_p(M_0) \), or \( K = \langle A^M \rangle \cong L_n(2) \) and \( M_0/O_p(M_0) = \Sigma_3 \times L_n(2) \) and \([Y_M, M_0] \) is the tensor product module. The second possibility just occurs for weak strong dual \( F \)-modules.

**Proof:** We may assume \( M^* = M \). Let now \( K = [F_p^*(M/O_p(M)), A] \). The assumption yields that there are at most two nontrivial \( K \)-modules in \( Y_M \). Suppose there are two such modules. Then we have a weak strong dual \( F \)-module and the assertion follows with 3.22. So we may assume that there is just one module. Assume next \([K, M_0] \leq O_p(M_0) \). So as \( F_p^*(M_0) \) has to act nontrivially on this module. But \([Q, [Y_M, K]] = 1 \), a contradiction. Now we get that \( K^Q \leq M_0 \).

Let \( K_1 \times \ldots \times K_r = K^Q \). Let \( r > 1 \). As in 3.19 we see that \( K \) is a minimal parabolic. Then we have \( K \cong SL_2(q) \) and \([Y_M, K] \) is the natural module. Now we see that \( C_{M_0/O_p(M_0)}(K^Q) = F(M_0/O_p(M_0)) \). Set \( U = [F_p(M_0), Q] \). Then we see that \( U\langle K^Q \rangle = C_U(K^Q) \langle C_{[Y_M,K^Q]}(Q) \rangle \langle K^Q \rangle \). Hence we have \( U\langle K^Q \rangle = \langle K^Q \rangle \langle \tilde{C} \cap (U\langle K^Q \rangle) \rangle \). Hence \( M_0S \) is a minimal parabolic, contradicting 2.7.

So \( K^Q = K \). Now by 2.4 \( K \) is the only component of \( M_0 \) not centralized by \( A \). As in 3.22 we see that \( F_p^*(M_0) = E_p(M_0) = K \), if \( K \) is nonsolvable. So we have that \( K \) is solvable. Because of quadratic action we get that \(|A| = p \) and so \( A \) induces a \( GF(p) \) transvection. Hence we see that \( p = 2 \), or 3. In particular we have that \( K \) is a 3–group or 2–group respectively. Set \( M_0 = M_0/C_{M_0}(Y_M) \). Then we see that \( \tilde{X} = \langle A^K \rangle = \Sigma_3 \) or \( SL_2(3) \). Now we see that \( \langle X^Q \rangle \cong K_1 \times \cdots \times K_r \) with \( K_i \cong \Sigma_3 \) or \( SL_2(3) \). Now with 2.4 we get that \( X = M_0 \). As \([C_{M_0}(Y_M), Q] = 1 \), we see that \( M_0S \) is a minimal parabolic, which contradicts 2.7.
4 The case $b(M) = 1$

The aim of this paragraph is to prove the following proposition.

**Proposition 4.1** Let $M \in L(S)$, $M \not\leq \tilde{C}$, with $M_0$ maximal. Assume further $p = 2$. If $Y_M \not\leq Q$ then one of the following holds

a) $F^*(M_0/O_p(M_0))$ is quasisimple and isomorphic to $3A_6$, $Sp(2n, q)$, $SL(n, q)$, $\Omega^\pm(2n, q)$, $U_n(q)$, $G_2(q)$, $E_6(q)$, $M_{22}$ or $M_{24}$ and $Y_M$ is a $2F$-module with quadratic or cubic offender.

b) $F^*(M_0/O_p(M_0)) \cong L_n(r) \times L_m(r)$ (one of both allowed to be solvable) and $Y_M$ is the tensor product module.

For the remainder we will assume $p = 2$ without any further notice. We will prove 4.1 in a series of lemmas.

**Lemma 4.2** Let $M \in L(S)$, $M \not\leq \tilde{C}$, with $M_0$ maximal. Suppose $Y_M \not\leq Q$. Then one of the following holds.

1. There is some $g \in \tilde{C}$ with $Y_M^g \leq S \leq M$, $Y_M \leq M^g$ and $Y_M^{g^2} = Y_M$. Further $\not\exists Y_M : C_{Y_M}(Y_M^g) = |Y_M^{g^2} : C_{Y_M}(Y_M)|$. In particular $Y_M$ is an $F$-module.

2. There is some $g \in \tilde{C}$ such that for $L = \langle Y_M, Y_M^g \rangle$ we have $L/O_2(L) \cong L_2(q)$, $Sz(q)$, $q$ even or $D_{2u}$, a dihedral group of order $2u$, $u$ odd. Set $q = 2$ in the latter. Further we have that $A = Y_M^g \cap O_2(L) \leq S \leq M$. For the action of $A$ on $Y_M$ we have $|Y_M : A, A, A| = 1$, if $x \in Y_M \setminus O_2(L)$, then $C_A(x) = A \cap Y_M$, and $|Y_M : C_A(A)| \leq q|A/(A \cap Y_M)|$. In particular $Y_M$ is a $2F$-module with offender $A/(A \cap Y_M)$ and an $F + 1$-module in case of $q = 2$.

**Proof:** We find everything in 2.16 where $G = \tilde{C}$ and $A = Y_M$.

For the remainder of this section we will fix the following notation from 4.2. If we are in 4.2(1) then we set $A = Y_M^g$. The structure of $L$ tells us that we may assume that $g^2 \in S$ in any case. By 2.6 we have $M_0 \leq M^g$. Further $A \leq S$. Hence again we may assume $M = M^g$. 

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Lemma 4.3 There is some component $K$ (maybe solvable) of $M_0$ such that $[A, K] \neq 1$.

Proof: Suppose that $[F^A_2(M_0), A] \leq O_2(M)$. By the $P \times Q$–lemma, $A$ acts nontrivially on $C_{Y_M}(Q)$. Suppose first that we have 4.2(1). Choose $x \in C_{Y_M}(Q)$. Then $x^g \in C_A(Q) \leq A \cap Q \leq A \cap O_2(M)$ by assumption. So we have $[x^g, Y_M] = 1$, but then also $[x, A] = 1$, a contradiction.

So we have 4.2(2). Then $C_{Y_M}(Q) \leq O_2(L)$ and so again $C_{Y_M}(Q)^g \leq C_A(Q) \leq A \cap O_2(M)$. Now we get the contradiction as before.

Lemma 4.4 Let $K$ be a 2-component of $M_0$ with $[K, A] \not\leq O_2(M_0)$, then $[K, A] \leq K$.

Proof: Set $K_1 = KO_2(M_0)/O_2(M_0)$. We may assume that $K_1$ is nonsolvable. Let $K_1^A \neq K_1$. Suppose first that we are in 4.2(1). Then by [Cher] we have either $|A : C_A(K_1)| = 2$ or $K_1 \cong L_2(2^n)$. In the first case by quadratic action there is some $x \in Y_M$ which induces a transvection on $A$. Hence there is also some $a \in A$, which induces a transvection on $Y_M$. But this contradicts 3.23. So we have the latter. As we may assume that $K$ is nonsolvable we have $n > 1$, but then $A$ does not offend on $[Y_M, K^A]$. Hence there are other components which are normalized by $A$. Now as $Q$ normalizes $A$, we get that $Q$ acts on the components normalized by $A$ and those which are not. In particular by quadratic action we get some $V \leq Y_M$ with $[V, Q] \leq V$ and $[K, V] = 1$. Then by 2.4 $K \leq \tilde{C}$, a contradiction.

So we may assume that we are in 4.2(2). Suppose first $q > 2$. By 2.16 we know that $Y := (Y_M \cap O_2(L))(Y_M^q \cap O_2(L))/(Y_M \cap Y_M^q)$ is a direct sum of natural modules. So let $A_1$ be contained in the intersection of $A$ with one of this modules, with $|A_1 : A_1 \cap Y_M| = q$. Then we see that $A_1$ acts quadratically on $Y_M$. Hence by [Cher] we have three possibilities

(1) $[K, A_1] \leq K$

(2) $|A_1 : C_{A_1}(K_1)| > 2$ and $[K_1, A_1] \not\leq K_1$ and $K_1 \cong L_2(2^n)$

(3) $|A_1 : C_{A_1}(K_1)| = 2$ and $[K_1, A_1] \not\leq K_1$.

Assume (3). Let $a \in C_{A_1}(K_1)$. Then $K_1^{A_1}$ acts on $[Y_M, a]$. By quadratic action we have $[Y_M, a, K_1] = 1$. In particular we get that $[Y_M, K_1]$ is centralize by $a$ and so by 4.2 $[Y_M, K_1] \leq O_2(L)$. Hence $A$ acts quadratically on $[Y_M, K_1^{A_1}]$. Assume now $|A : C_A(K_1)| = 2$. Then $[Y_M, K_1]$ is centralized.
by a subgroup of index two in $A$. Let $B$ be a complement in $[Y_M, K_1]$ of $C_{[Y_M, K_1]}(A)$. Then all $b \in B$ have the same centralizer in $Y_M^0$. As we have that $|B| > 2$, we see that $B$ has to normalize any 2-component of $M_0^y$. Hence there are also 2-components in $M_0$ normalized by $A$, which gives a contradiction as before.

So we may assume that $K_1^A = \Omega(4, 2^n)$. In particular we may assume that we are in (2). Suppose first $[Y_M, K_1] \leq O_2(L)$. By 3.14 there is some $y \in C_{Y_M}(K_1) \setminus O_2(L)$. Hence we see $[y, A][Y_M \cap A] = Y_M \cap O_2(L)$. Now $[Y_M, K_1, A] \leq [Y_M \cap O_2(L), A] = [y, A, A]$. But $[y, A] \leq C_{Y_M}(K_1)$, a contradiction. So we have $[Y_M, K_1] \not\leq O_2(L)$. Choose $y \in [Y_M, K_1] \setminus O_2(L)$, then we get $[y, A][Y_M \cap A] = [Y_M, A_1]$. Hence we see that $K_1^A$ contains all components not centralized by $A_1$. In particular it is $A$-invariant and so we get $[y, A][Y_M \cap A] = [Y_M, A]$ and so $K_1^A = K_1^A$ contains all components on which $A$ acts nontrivially. As $A$ is normalized by $Q$ we get with 2.4 that $M_0$ is a minimal parabolic, which contradicts 2.7.

So we have shown that $K_1^A = K_1$. Assume first $[K_1, Y_M] \leq O_2(L)$. As $A$ acts quadratically on $O_2(L)$, we see that $A$ acts quadratically and we may argue as before. So we have $[Y_M, K_1] \not\leq O_2(L)$. Let $V_1$ be a minimal nontrivial $K_1$-submodule of $Y_M$ and let $a \in A$ with $K_1^a \neq K_1$. Then $V_1 \cap V_1^a \leq C_{V_1}(K_1)$ and so $V_1 \cap A \leq C_{V_1}(K_1)$. In particular $[V_1 \cap O_2(L), N_A(V_1)] = 1$. Now we have for some $y \in V_1$, $[y, A][Y_M \cap A] = [Y_M, A_1]$. Hence $V_1 = [Y_M, K_1]$. Now for $K_2 = K_1^a$, we have that $[Y_M, K_1, K_2] = 1$. But then $A_1$ acts trivially on $[Y_M, K_2]$, a contradiction. So we have $K_1^A = K_1$.

So now we have $q = 2$ and then $Y_M$ is an $F + 1$-module. If $[Y_M, K_1] \leq O_2(L)$, then again $A$ acts quadratically on $[Y_M, K_1]$. If we have $|A : C_A(K_1)| > 2$, we may argue as before. So let $|A : C_A(K_1)| = 2$. Then as before $[Y_M, K_1]$ centralizes a subgroup of index 2 in $A$ and a subgroup of index 4 in $Y_M$. As before we may assume that $[Y_M, K_1]$ does not normalize any component in $M_0^y$. Hence as we now get that $A$ induces transvections on $[Y_M, K_1]$, a contradiction.

So we have $[Y_M, K_1] \not\leq O_2(L)$. Let $a \in A$ with $K_1^a \neq K_1$. Assume first $[Y_M, a, A] = 1$, then by [Cher] either $|A : C_A(K_1)| = 2$, or $K_1 \cong L_2(r)$ and orthogonal $\Omega(4, r)$-modules are involved. Suppose the latter. Then as before, we get some $y \in C_{Y_M}(K_1)$ with $[A, y][A \cap Y_M] = [Y_M, A]$, a contradiction. So we have $[A : C_A(K_1)] = 2$. Now we have that $[K_1, Y_M, C_A(K_1)] = 1$, but as $[Y_M, K_1] \not\leq O_2(L)$ this shows $|A : A \cap Y_M| = 2$. Now $A$ induces transvections on $Y_M$ and the assertion follows with 3.23.

So we have that $[Y_M, a, A] \neq 1$. Now $A$ induces transvections on $[V, a]$ to a hyperplane. Let $b \in C_A(K_1)$ and assume first that $[Y_M, b, K_1] = 1$. 34
Then also \([Y_M, K_1, b] = 1\) and so \([Y_M, K_1] \leq O_2(L)\), a contradiction. Hence \(K_1\) acts nontrivially on \([Y_M, b]\). But \(b\) induces a transvection on \([Y_M, a]\), a contradiction. So we have that \(C_A(K_1) = 1\). Further \(K_1^A = K_1 \times K_1^a\). As \([Y_M, a, A] \neq 1\), we see that \(K = C_{K_1 \times K_1^a}(a) \cong K_1\) acts faithfully on \([Y_M, a]\), and so as \(A\) induces transvections to a hyperplane, we get with \([GBSM]\) \(K_1 \cong L_\sigma(2), Sp(2n, 2), \Omega^±(2n, 2)\) or \(A_n\). We have \(|A| > 2\). So assume first \(|A| = 4\). Then \(|[Y_M, a]| \leq 4\), but \(K\) has to act nontrivially on \([V, a]\).

So we have \(|A| > 4\) and then \(K_1 \cong L_\sigma(2)\). Further \(|[Y_M, a]| \geq 2^\alpha\). As \(|A| \leq 2^\alpha\), we get equality everywhere and \(N_A(K_1)\) induces the full transvection group on \([Y_M, a]\). This shows that \([Y_M, K_1]\) just involves the natural module. By 3.14, we get \([Y_M, K_1 K_1^a] = [Y_M, K_1] \oplus [Y_M, K_2]\). We see that \(Q\) normalizes \(K_1 K_1^a\) and so by 2.4 we get a contradiction.

**Lemma 4.5** Let \(K\) be a component of \(M/O_2(M)\) with \([K, A] \neq 1\). If \([K, Y_M]\) is a 2F-module with offender \(A\), then \([K, Q] \leq K\), or \(KS\) is a minimal parabolic.

**Proof:** First of all we may assume that \(K \leq M_0 O_2(M)/O_2(M)\) and \(K\) be nonsolvable. Suppose that \(KS\) is not a minimal parabolic. Set \(U = (KQ)\). Then by 2.4 we get that \(C_{Y_M}(U) = 1\). Let \(T\) be a Sylow 2-subgroup of \(UQ\). Let \(V_1\) be some irreducible \(K\)-submodule in \(Y_M\) and \(V = \langle V_1^{UQ} \rangle\). If \(V_1\) is not an \(F\)-module, then we see that \(V\) is a direct sum of images of \(V_1\) under \(Q\). The structure of \(V_1\) is given by 3.7 - 3.10. Now choose \(x \neq C_V(T)\). Then we see that \(C_K(x)\) is not a 2-group. But this contradicts 2.4. So we have that \(V_1\) is an \(F\)-module. Now we may assume that \(V\) is a tensor product, but then for \(x\) as before we see that \(C_{UQ}(x)\) contains \(C_K(x)^Q\), which is not a 2-group.

**Lemma 4.6** Let the assumption be as in 4.5. Then \([K, Q] \leq K\).

**Proof:** By 4.5 we may assume that \(KS\) is a minimal parabolic. Let \(K_1\) be some component in \(M_0/O_2(M)\) not in \(KS\). If \([[K, Y_M], K_1] = 1\), then also \([K^Q, [K^Q, Y_M]] = 1\) and so by 2.4 we get \(K \leq C \cap M/O_2(M)\) and then \([K, Q] = 1\). So we have \([[K, Y_M], K_1] \neq 1\). By 4.4 we have \([K, A] \leq K\). If \([K, Y_M]\) is irreducible, then with 3.7 - 3.10 we see that \(K_1\) has to induce field multiplications on \([K, Y_M]\). But then we find some \(t \in K_1 KS\), which centralizes some element in \(C_{[Y_M, K^2]}(Q)\), but does not normalize \(Q\), a contradiction to 2.4. Hence we have that \([K, Y_M]\) involves at least two nontrivial modules. Further we may assume that \(K_1\) acts on these modules nontrivially. This shows \(K \cong L_2(q)\) and \([Y_M, K]\) is a direct sum of two modules on
which $A$ acts quadratically. Suppose first that $A$ acts quadratically on $Y_M$, then with 4.2 we see that $Y_M$ is an $F$–module with offender $A$. But this is not possible. Hence $A$ does not act quadratically and so $[Y_M, K^S] \leq O_2(L)$. But then $A$ has to induce an $F$–module, i.e. $B = C_A(K^S) \neq 1$. Now we see that $[B, Y_M, K] = 1$, as $[B, Y_M \cap Y^g_M] = 1$. Let $K_2$ be a 2-component of $M$ with $[B, K_2] \neq 1$, then $[K_2, [Y_M, K^S]] = 1$. But this contradicts 2.4. So we have seen that there in no $K_1$, in particular $M_0S$ is a minimal parabolic contradicting 2.7.

Lemma 4.7 Let $K$ be a component of $M/O_2(M)$ with $[K, A] \neq 1$. Then $[K, Q] \leq K$.

Proof: If $[K, Y_M]$ is a 2$F$–module with offender $A$ we are done. So let $[K, Y_M]$ not be a 2$F$–module. Set $E_1 = K^Q$. Now as $A$ induces a 2$F$–module on $Y_M$ there is some component $L$ such that $[L, Y_M]$ is a 2$F$–module. By 2.4 we have that $[E_1, [Y_M, L]] \neq 1$. But now $[Y_M, E_1]$ involves a 2$F$–module invariant under $Q$, hence as in 4.5 and 4.6 we get a contradiction.

Lemma 4.8 Let $K_1$, $K_2$ be two components (possibly solvable) of $M/O_2(M)$ with $[K_1, Q] \leq K_i$, $i = 1, 2$. Suppose there are $V_i \leq Y_M$, $V_i^Q = V_i$ and $[V_i, K_{3-i}] = 1$, $i = 1, 2$. Then $K_i \leq (M \cap C)/O_2(M)$.

Proof: This follows immediately from 2.4

Lemma 4.9 Assume 4.2(2). Let $K$ be a component of $E(M_0/O_2(M_0))$, which is not centralized by $A$. Suppose $K_1$ is a second component, maybe solvable, with the same property. Then these are all such components, $K \cong L_n(r)$, $K_1 \cong L_m(r)$ for some some $n, m$ and $r = 2^k$ (including the solvable variant) Further $Y_M$ is the tensor product module for $K \times K_1$.

Proof: Let $KA = K \times T$. By abuse of notation we denote with $K$ also $KO_{Out_M}(K)$. Let $V_1$ be some irreducible submodule of $K$ in $Y_M$. By 2.4 and 4.7 we get that $V_1$ is nontrivial. Let $W = V_1 \oplus V_2 \oplus \cdots \oplus V_s = V_1^T$. If $s = 1$, then we get $[V_1, T] = 1$. By 4.8 this shows $T = 1$. Now we have that there is some $a \in A$ which induces an outer automorphism on $K$. Let now $V_1$ be an irreducible $E(K)$–module. Assume further that $V_1^a = V_2 \neq V_1$. Then we have that $|V_1| < 2|A|$. In particular we see that $A \cap E(K) \neq 1$. If $V_1 \leq O_2(L)$, then we get $[A \cap K, V_1, a] = 1$, a contradiction. So we have $V_1 \not\leq O_2(L)$. Let $|V_1| = r^t$ and $|V_1 : C_{V_1} (A \cap E(K))| = r^{t-e}$. Then we have $|A \cap E(K)| \leq r^e$, as $V_1 \cap Y^g_M = 1$. Now we see $|V_1 \oplus V_2 : C_{V_1 \oplus V_2}(A)| \geq r^{2t-e}$. On the other hand
\[ |V_1 \oplus V_2 : C_{V_1 \oplus V_2}(A)| \leq |A : A \cap E(K)||A \cap E(K)|^{r^{t-e}} \leq |A : A \cap E(K)|^{r^t}. \]

Hence we see \( r^{t-e} \leq |A : A \cap E(K)| \leq 4 \). So we get \( t = e + 1 \) and then \( A \cap E(K) \) induces the full transvection group to a point, i.e. \( K \cong L_t(r) \). But as \( |A/A \cap E(K)| = r \), we have that \( A \) contains some element which does not act trivially on the Dynkin diagram. But this is not possible, as \( A \cap E(K) \) has to induce transvections to a point on \( V_2 \) too. So we see that \( V_1 \) is invariant under \( K \). Now we see that \( K_1 \) has to induce field multiplications on \( V_1 \) and then we get some \( 1 \neq t \in K_1 \times K \setminus K \) which centralizes some nontrivial element in \( C_{V_1}(Q) \), in particular \( t \in C \) by 2.4, a contradiction. So we have \( s > 1 \).

We first assume \( C_A(K) = 1 \). Further we may assume that \( C_T(V_1) = 1 \). Finally we may choose notation such that \( V_i = V_i^t \) for certain \( t_i \in T \), \( i = 1, \ldots, s \).

Assume first \( V_1 \leq O_2(L) \). We have \( W = V_1^A \), so \( W \leq O_2(L) \). In particular \( A \) acts quadratically on \( W \). But then we must have \( N_A(V_1) = 1 \). But for \( x \in O_2(L) \) there is always some \( 1 \neq a \in A \), with \([a, x] = 1 \), a contradiction.

So we have \( V_1 \not\leq O_2(L) \). As \( V_1^a = V_2 \) for some \( a \in A \), we have that \( V_1 \cap Y^g_M = 1 \). Assume first \( O_2(L) \cap V_1 \neq 1 \). Then \( V_1 \cap V_2 \cap Y^g_M \neq 1 \). Now let \( t_1 \in T \setminus \{t\} \). We see that \( V = (V_1 \oplus V_2)^{t_1} \cap (V_1 \oplus V_2) \neq 1 \). As \( V \) is a \( K \)-module, we have that either \( V = V_1 \oplus V_2 \) or \( V \cong V_1 \). Now always \( V \geq V_1 \oplus V_2 \cap Y^g_M \). Let \( b \in A \) then \( b = ut_2 \), \( u \in K \), \( t_2 \in T \). Then \( V^{t_2} \cap V \neq 1 \). Suppose \( V \neq V_1 \oplus V_2 \), then \([V, t_2] = 1 \), i.e. \([V, T] = 1 \). This shows \( V = [W, T] \) and \( T \) acts quadratically on \( W \). If \( V = V_1 \oplus V_2 \), then \([V, t] = [V_1, t] \) and so \([W, t, t_1] = 1 \) and again \( T \) acts quadratically on \( W \).

Now by construction we have \( W = [V_1, T]V_1 \) and so \( T \) induces some group of transvections over some field \( GF(r) \), where \( V_1 \) is a module over \( GF(r) \). Let \( |V_1| = r^x \). As \( a \) acts on \( V_1 \oplus V_2 \) we see that it centralizes on \([V_1, t] \) exactly an half dimensional subspace, which means that \( x = 2y \). Hence we now get that \( |W : C_W(A)| \geq r^{y^s} \). Further \(|T| \leq r^{s-1} \).

Suppose first \( A \cap K = 1 \). Then \(|A| = |T| \). Further \( r^{y(s-1)} \leq |W, T| : C_W(T) \geq |A| \leq r^{s-1} \). This shows \( y = 1 \). Further \(|A| = r^{s-1} \). But as now \( E(K) \leq L_2(r) \), we see that \(|A| \leq r \) and so \( s = 2 \). Then \( W = V_1 \oplus V_2 \) and \(|W| = r^4 \). Now get that \(|Y_M : Y_M \cap O_2(L)| = r = |A| \), which shows that \(|Y_M : C_{Y_M}(A)| = r \), a contradiction.

So we have \( A \cap K \neq 1 \). Let \( v \in V_1 \setminus O_2(L) \). Then as \( V_1 \cap Y^g_M = 1 \), we see that \([v, A \cap K] \cong A \cap K \). This shows that \(|V_1 \cap O_2(L)| \geq |A \cap K| \). Now let \( a \in A \setminus K \). Then \([V_1, a] \leq O_2(L) \) and \(|[V_1, a] \cap Y^g_M| = |V_1 \cap O_2(L)| \geq |A \cap K| \).
We have $W = V_1(W \cap O_2(L))$, so

$$|W : C_W(A)| = |A||V_1 : V_1 \cap O_2(L)| \leq |A||V_1|/|A \cap K| = |T||V_1|$$

Now again for $|V_1| = r^{2y}$, we have $|W : C_W(A)| \geq r^{sy}$. So we get

$$ys \leq 2y + (s - 1).$$

Let first $s = 2$. Then we see that $|W, A|/W \cap Y^2_M \cong V_1$, hence $|A| = |V_1|$, which shows that $|A \cap K| = |V_1 \cap O_2(L)|$ and $|V_1/V_1 \cap O_2(L)| = r$. Hence $A \cap K$ induces the full group of transvections to a hyperplane on $V_1$. In particular $C_K(A \cap K) = A \cap K$. Now for $a \in A \setminus K$, we have $a = ut$ with $u \in K$ and so $[u, A] = 1$, a contradiction. So we have $s > 2$ and so $y = 1$ or $y = 2$. If $y = 1$, we get again $E(K) \cong L_2(r)$ and a contradiction as before. So let $y = 2$. Now $s = 3$. We have $|A| \leq r^4$ and $|W : C_W(A)| \geq r^6$. We see that $|V_1 : V_1 \cap O_2(L)| = r^2$ and $|A| = r^4$. Hence we have $|W : C_W(A \cap K)| = r^6$. As $A \neq A \cap K$, this is not possible.

So we have $V_1 \cap O_2(L) = 1$. In particular $A \cap K = 1$. Let $t \in T$. Then $K$ acts on $[V_1, t]$. Let $a \in A$ such that $t = ua$ with $u \in K$. Hence $[[V_1, t], a] \neq 1$. In particular $[V_1, t] \cap O_2(L) \neq 1$. But then as before we get $|[V_1, t], T| = 1$, in particular $T$ acts quadratically. Now $|T| = |A| \leq r^{s-1}$. Further again $|V_1| = r^{2y}$ and $|W : C_W(A)| \geq r^{sy}$. So we see that $sy \leq 2(s - 1)$, which gives $y = 1$. But then $E(K) \leq L_2(r)$ and so $|A| \leq r$, which shows $s = 2$. But $|A| \geq |Y^1_M : Y_M \cap O_2(L)| \geq |V_1| = r^2$, a contradiction.

Hence we have $C_A(K) \neq 1$. In particular we may assume $A \cap T \neq 1$. Let $1 \neq a \in T \cap A$ and $[V_1, a] \leq V_1$, then we see that $[V_1, a] = 1$. Hence $V_1 \leq O_2(L)$ and then $W \leq O_2(L)$. Now $A$ acts quadratically and $W = V_1(W \cap Y^2_M)$. Let $t = bu \in T \setminus A$ with $u \in K$ and $b \in A$. Then $b$ acts on $[V_1, t]$ nontrivially and so $[V_1, t] \cap Y^2_M \neq 1$. This shows that $[V_1, t, T] = 1$ and so also $T$ acts quadratically. But now $(b, u)$ acts quadratically and then $V_1 = V_1$, a contradiction. So we again see that $T$ acts regular on $V_1^T = V_1^A$. In particular $V_1 \leq O_2(L)$. Then as above we get that $T$ acts quadratically on $W$ and $T$ induces transvections. Hence $|T| \leq r^{s-1}$ again.

Suppose first $A \cap K \neq 1$. Then $V_1 \cap O_2(L) \neq 1$ and we get $|W : C_W(A)| = |V_1/V_1 \cap O_2(L)|^{s-1}|V_1|$. We have $|A| \leq |A \cap K|^{r^{s-1}}$. Further we see that $W = V_1(W \cap O_2(L))$ and so $|W : C_W(A)| \leq |V_1/O_2(L)||A|$. Now $|V_1/V_1 \cap O_2(L)|^{s-1}|V_1| \leq |V_1/V_1 \cap O_2(L)||V_1 \cap O_2(L)|^{r^{s-1}}$. This shows $|V_1/V_1 \cap O_2(L)| = r$ and $|T| = r^{s-1}$. Finally $|A \cap K| = |V_1 \cap O_2(L)|$ and so $A \cap K$ is the full transvection group to $V_1 \cap O_2(L)$, which shows $K \cong L_m(r)$. But as $C_K(A \cap K) = A \cap K$ we get $T = T \cap A$. Let $K_1$ be
a maybe solvable component with $[T, K_1] \neq 1$ and $m \in K_1$ with $V_1^n \not\subseteq W$. If $v \in V_1 \setminus O_2(L)$, then we have $[v, A](Y_M \cap Y^n_M) = Y_M \cap O_2(L)$. Hence by $|Y_M \cap O_2(L) : C_{Y_M \cap O_2(L)}(A)| = |A| = |W \cap O_2(L) : C_{W \cap O_2(L)}(A)|$ we get that $V_1^n \not\subseteq O_2(L)$ and $(V_1^n)^T = V_1^n$. But then $[V_1^n, T] = 1$. This is impossible. So we have that $[W, K_1] = W$ and then because $T$ is the full transvection group, we get that $K_1 = L_s(r)$ or $r = 2$ and $K_1/C_{K_1}(W) \cong \Sigma_3$ and $W$ is the tensor product module. Suppose $W \neq Y_M$. By 2.4 $C_{Y_M}(K) = 1$. Further the same argument as above shows that $W$ is the socle of $K$ in $Y_M$. So let $W_1/W$ be an irreducible module in $Y_M/W$. If $K$ acts trivially then by 3.14 and 3.15 we get a contradiction, or $K \cong L_2(r)$. So assume $K \neq L_2(r)$. So $W_1/W$ is a nontrivial module. Again we see that $W_1/W$ is covered by $Y_M \cap Y^n_M$, which shows that $A$ acts trivially, a contradiction, so $W = Y_M$. Hence we are left with $K \times K_1 \cong L_2(r) \times L_2(r)$. and $W = [K \times K_1, W]$. But then $W$ is the $O^+(4, r)$–module and again $W = Y_M$ follows with 3.14.

If $A \cap K = 1$, then as above we get $K \cong L_2(r)$ and $|T : T \cap A| \leq r$. We have $Y_M \cap O_2(L) = (W \cap O_2(L))(Y_M \cap Y^n_M)$. This shows that $[Y_M, T] \leq W$. Let $K_1$ be another component of $M/O_2(M)$ with $[K_1, A] \neq 1$. Then we see that $W \geq [K_1, Y_M]$. Now as $T$ is a transvection group on $W$, we see that $K_1 \cong L_s(r)$ and $W$ is the tensor product module. Further we have $s > 2$ and so as above $W = Y_M$.

We just have to show that in the solvable case for $K_1$ we have $K_1 \cong Z_3$. But we have $C_{K_1}(W) = C_{K_1}(Y_M) \leq \bar{C}$, so $[Q, C_{K_1}(W)] = 1$. As $[Q, K_1] = K_1$, we see that $[K_1, Q]$ is cyclic and so of order three.

**Lemma 4.10** Assume 4.2(2). Let $K$ be a component of $E(M_0/O_2(M_0))$, which is not centralized by $A$. Then $K = E(M_0O_2(M)/O_2(M))$, or there is a second maybe solvable component $K_1$ which is also not centralized by $A$, and $E(M_0O_2(M))/O_2(M)) = K \times K_1 \cong L_n(r) \times L_m(r)$ for some $n, m$ and $r = 2^t$ (including the solvable variant). Further $Y_M$ is the tensor product module for $K \times K_1$.

**Proof:** Suppose first that there are at least two components $K, K_1$ with $[K, A] \neq 1$ and $[K_1, A] \neq 1$. Then by 4.9 we have that $Y_M$ is the tensor product module for $K \times K_1$. Now as $Y_M$ is irreducible any other component has to centralize $Y_M$ and so we have the assertion.

So we may assume that there is a unique component $K$ which is not centralized by $A$. Let $K_1$ be a further component in $M_0O_2(M)/O_2(M)$. Let $V_1$ be an irreducible module of $KA$ in $Y_M$. By 4.8 we see that $[V_1, K_1] \neq 1$. Let first $[V_1, K_1] \not\subseteq V_1$, hence $W = V_1^{K_1} = V_1 \oplus V_2 \oplus \cdots \oplus V_s$, $s > 1$.
Suppose first that \( V_1 \not\leq O_2(L) \). Choose \( v \in V_1 \) with \( C_A(v) = 1 \). Then \( Y_M \cap O_2(L) = (V_1 \cap O_2(L)) (Y_M \cap Y_M^2) \) and so \([A,Y_M \cap O_2(L)] \leq V_1\). In particular \( A \) acts quadratically on any irreducible submodule besides \( V_1 \). If \( V_1^{K_1Q} \not= V_1 \), it acts quadratically on \( V_1 \) too, as \([A,K_1Q] \leq A\). We have \([Q,Y_M] \leq O_2(L), \) so \([Q,Y_M,A] = 1\). Suppose there is some \( u \in Q \) and \( V \) an irreducible module with \( V^u \not= V \). Let \( t^u \in V^u \). Then \([tt^u,A] = 1\). But this implies that \([t,a] = [t^u,a]\) for all \( a \in A\). Then \([t,a] \in V \cap V^u = 1\), a contradiction. Hence \( Q \) fixes any irreducible submodule of \( K \). As in 4.9 we see that \( V_1 \) as an \( K \)-module is irreducible and then there is some element \( 1 \not= t \in K_1K \) not in \( K \) which centralizes some nontrivial element in \( C_{V_1}(Q) \), a contradiction.

So we have that \( W \), the sum of all irreducible \( K \)-submodules, is contained in \( O_2(L) \). Now let \( \tilde{W} \) be maximal such that \([\tilde{W},K \times K_1] \leq \tilde{W}\) and \( \tilde{W} \leq O_2(L) \). Let \( \tilde{W} \) be such that \( \tilde{W}/\tilde{W} \) is an irreducible \( K \times K_1 \)-module. Let first \([\tilde{W},K_1] \not\leq \tilde{W}\). Suppose further \([K,\tilde{W}] \leq \tilde{W}\). Then the action of \( A \) on \( Y_M \) shows \( \tilde{W} \cap O_2(L) \leq \tilde{W} \). As \( A \) is normalized by \( Q \), we see that \([\tilde{W}Q,A] \leq \tilde{W} \) and so \([W,Q] \leq \tilde{W} \). Then as \([Q,K_1] \not= 1\), we get the contradiction \([\tilde{W},K_1] \leq \tilde{W}\). So we have \([K,\tilde{W}] \not\leq \tilde{W}\). Now we write \( \tilde{W}/\tilde{W} = V_1 \oplus \ldots \oplus \tilde{V}_i \). If \( K_1Q \) does not fix \( \tilde{V}_i \) we get that \( A \) acts quadratically and then as above we get that \( Q \) fixes any irreducible submodule, a contradiction as before. So either \([\tilde{V}_1,K_1Q] \leq \tilde{V}_1 \) or \([\tilde{V}_1,K_1] = 1\). Assume the latter. Then we have \([K_1,\tilde{W}] \leq \tilde{W}\). Now choose \( \omega \in K_1, o(\omega) \) odd. Let \( x \in \tilde{W} \setminus O_2(L) \), with \([\omega,x] = 1\), this exists. Now as \([A,\omega] = 1\), we see that \([x,A] \leq C_{\tilde{W}}(\omega)\). We know that \([x,A] \) covers \( V_1 \), i.e. \( V_1 \leq [x,A] (Y_M \cap Y_M^2) \). So let \( v_1 \in V_1 \), then there is \( w \in W \cap [x,A] \) and \( \tilde{w} \in Y_M \cap Y_M^2 \) with \( v_1 = w\tilde{w} \). Now \([w,A] \leq C_{\tilde{W}}(\omega)\) and \([w,A] = [w\tilde{w},A] = [v_1,A] \in V_1 \). But we may choose \( \omega \) such that \( C_{V_1}(\omega) = 1\). This shows \([V_1,A] = 1\), a contradiction.

So we are left with \([\tilde{V}_1,K_1Q] = \tilde{V}_1\). Then we have that \( K_1 \) induces field multiplications on \( \tilde{V}_1 \), in particular \( K_1 \) is solvable. Further \( Q \) has to induce an outer automorphism on \( K \). We have that \( W \leq O_2(L) \) and so \( W \) is an \( F \)-module with offender \( A \). We have \([V_1,A] \not= 1\). So \( W \) is better than \( F \)-module. By 3.4 we see that \( K \) is a Lie type group. Set \( W_1 = V_1^Q \). Then we have that \( Q \leq O_2(C_K(C_{W_1}(Q))) \). But then we see that \( K \cong \Omega^+(6,q) \) and then \( W_1 \) is not better than \( F \)-module.

So we are left with \([V_1,K_1] \leq V_1 \) for any irreducible \( K \)-submodule \( V_1 \). In particular \( K_1 \) has to induce field multiplications on \( V_1 \). If \( V_1 \) is \( Q \)-invariant, we get a contradiction as before. So set \( W_1 = V_1^Q \). Now \( V_1 \) is an \( F \)-module and \( Q \) induces outer automorphism on \( K \). So again we see that \( K \cong \Omega^+(n,q), n = 6,8,10, \) and \( V_1 \) is the half spin module. But
then in \( C_{W} (S \cap K)^{t} \) we have exactly two \( K \)-orbits. So again there is some \( t \in K_{1}K \setminus K \), which centralizes some element in \( C_{W} (Q) \), a contradiction.

**Lemma 4.11** Assume 4.2(2). Assume \( [A, E(M_{0}O_{2}(M)/O_{2}(M))] \leq O_{2}(M) \), Then \( M_{0}/O_{2}(M_{0}) \cong \Sigma_{3} \times \Sigma_{3} \) and \( |Y_{M}| = 4 \).

**Proof:** By 4.3 we know that \( [F_{2}(M_{0}), A] \not\leq O_{2}(M) \). Let \( P \) be a Sylow \( p \)-subgroup of \( F(M_{0}/O_{2}(M_{0})) \), which is normalized by \( A \). Let \( B \leq A \) be a quadratic fours group on \( Y_{M} \), with \( [B, P] \neq 1 \). Then by 2.4 we have \( C_{B}(P) = 1 \). By 3.1 we have \( Y_{M} = \langle y \mid |B : C_{B}(y)| \leq 2 \rangle \). But any such element is contained in \( O_{2}(L) \cap Y_{M} \), a contradiction. So for any quadratic fours group \( B \) we have \( [P, B] = 1 \). In particular \( q = 2 \) and \( Y_{M} \) is an \( F + 1 \)-module with offender \( A \). Let \( a \in A^{2} \) with \( [a, P] = 1 \). Then \( A \) induces transvections on \( [a, Y_{M}] \) and so \( p = 3 \) and \( |A : C_{A}(P)| = 2 \). Set \( A = \langle b \rangle \times B \), where \( B = C_{A}(P) \). Hence \( B \) induces transvections to a hyperplane on \( [a, Y_{M}] \). Let \( K \) be any component of \( M/O_{2}(M) \) with \( [K, B] \neq 1 \), then we get \( K \cong L_{n}(2) \) and \( [K, Y_{M}] \) is a direct sum of two natural modules. Further by 4.10 we have that \( K \not\leq M_{0} \). So \( M_{0}/C_{M_{0}}(Y_{M}, K) \cong \Sigma_{3} \) and by 2.4 \( C_{M_{0}}(Y_{M}, K) \leq \tilde{C} \). This shows \( M_{0}/O_{2}(M) \cong \Sigma_{3} \) contradicting 2.7.

So we have \( C_{A}(P) = 1 \). Again \( p = 3 \). Let \( a \in A^{2} \). Then again we have \( A = \langle a \rangle \times B \), where \( B \) induces transvections to a hyperplane on \( [Y_{M}, a] \). This shows \( |A| \leq 4 \). If \( |A| = 2 \), then \( |[Y_{M}, A]| = 2 \) and so \( |[P, a]/C_{P[a]}(Y_{M})| = 3 \).

Now as before we see that \( M_{0}/O_{2}(M_{0}) \cong \Sigma_{3} \), contradicting 2.7.

So let \( |A| = 4 \). Then \( |Y_{M} : C_{Y_{M}}(A)| = 8 \) and \( |Y_{M} : C_{Y_{M}}(a)| = 4 \) for \( a \in A^{2} \). Now we have \( X = [P, A]/C_{P[a]}(Y_{M}) \leq L_{6}(2) \). If \( |X| = 27 \), then \( X \) is uniquely determined and it exist exactly three subgroups \( T \) in \( X \) with \( |[T, Y_{M}]| = 4 \). As \( Q \) is a \( 2 \)-group it has to normalize one of these groups \( T \). Now \( C_{[T, Y_{M}]}(Q) \neq 1 \) and so \( Q \) has to normalize any such \( T \) by 2.4. But then again by 2.4 we get \( X \leq \tilde{C} \), a contradiction.

So we have \( |X| = 9 \). If \( |[X, Y_{M}]| = 64 \), then \( X \) contains some \( \rho \) with \( |(\rho, Y_{M})| = 64 \). Let \( a \in A \) with \( \rho^{a} = \rho^{-1} \), then \( |a, Y_{M}]| = 8 \), a contradiction.

So we have \( |[X, Y_{M}]| = 16 \) and this is the orthogonal module for \( AX \). We see that \( O_{2}^{2}(M_{0}/C_{M_{0}}([X, Y_{M}])) = Z_{3} \times Z_{3} \). In particular \( P \) is 2-generated. Let now \( U \) be a critical subgroup in \( P \), then \( U \) is also 2-generated and so \( U = P \). If \( U \) is elementary abelian, we are done by 2.7. Let \( U \cong 3^{1+2} \).

Again by 2.7 we have that \( O_{2}(M)^{Q} \leq O_{2}(M)A \). Then by 2.4 we have that \( Q \) has to centralize \( Z(U) \) and so \( |Q : Q \cap M_{0}| = 2 \). Now \( Q \) inverts \( U/Z(U) \) and so \( Q \) inverts some \( \rho \) with \( |(\rho, Y_{M})| = 4 \), a contradiction as before.

\[ \text{Funique} \]
Lemma 4.12 Assume 4.2(1). Let $K$ be a component of $E(M_0/O_2(M_0))$, which is not centralized by $A$. Then there is no second component $K_1$ (maybe solvable) with the same property.

Proof: As in the previous lemma set $KA = K \times T$. Further set $W = V_1 \oplus \cdots \oplus V_s$, where $V_i$ is a nontrivial irreducible $K-$submodule of $Y_M$, and $W = V_1^T$. Suppose first $s > 1$. As $A$ is a quadratic offender as an $F-$module, we have that $[V_1, A] \cap V_1 = 1$ and so $|V_1| \leq |A|$. Further by quadratic action we see that $A$ and $T$ both induce transvection groups on $W$. Again $|V_1| = r^{2g}$ for some $r$, we see that $|W : C_W(A)| = r^{sg}$, which shows $s = 2$ and then $|T : T \cap A| \leq r$. Let $T \cap A \neq 1$. Then we may assume that $[K_1, T \cap A] \neq 1$. Now quadratic action and 4.7 and 4.8 yield a contradiction. So we have that $A \cap T = 1$. Now we see that $W$ centralizes a subgroup of index $r$ in $A$ and so $W^g$ centralizes a subgroup of index $r$ in $Y_M$. In particular $[W, W^g] = 1$. Hence $[W^g, Y_M] = 1$ and then $[A, W] = 1$, a contradiction.

So we are left with $s = 1$. Let $T \neq 1$. Then $[V_1, T] \leq V_1$ for all irreducible $K-$submodules in $Y_M$. This shows that $[K_1, V_1] = 1$ for all irreducible $K$-submodules in $Y_M$. But then again 4.8 yields the contradiction.

So we have $T = 1$. By quadratic action we have that $A \cap E(K) = 1$ and so $|A| \leq 4$. In particular $V_1$ is an irreducible $E(K)-$module. Then we get $|A| = 2$ and $A$ induces a transvection on $Y_M$. But then $[A, K_1] = 1$, a contradiction.

Lemma 4.13 Assume 4.2(1). Let $K$ be a component of $E(M_0/O_2(M_0))$, which is not centralized by $A$. Then $K = E(M_0O_2(M)/O_2(M))$, or there is a second component $K_1$ with the same property, and $E(M_0O_2(M)/O_2(M)) = K \times K_1 \cong L_n(r) \times L_m(r)$ for some some $n, m$ and $r = 2^i$ (including the solvable variant). Further $Y_M$ is the tensor product module for $K \times K_1$.

Proof: We have $[K, A] \neq 1$ and $[K_1, A] = 1$. Let $V_1$ be an irreducible $K-$submodule of $Y_M$ and $W = V_1 \oplus \cdots \oplus V_s = V_1^{K_1}$. Clearly by 4.8 we may assume that $s > 1$. Now as $Y_M$ is an $F-$module with offender $A$ we have that $V_1$ is better than an $F-$module which with 3.4 gives that $K \cong L_n(r)$, $Sp(2n, r)$, $U_n(r)$ or $\Omega^+(2n, r)$ and $V_1$ is the natural module. As $K \leq M_0$, we have $[K, Q] \neq 1$. Let $v \in V_1$ such that $v$ is centralized by $S \cap K$, then we have that $Q$ is normalized by $N_K(\langle v \rangle)$. As also $K_1 \leq M_0$, we have that $Q \not\leq K$. Now choose $x \in Q \setminus K$. Then $[x, N_K(Q)] \leq O_2(N_K(Q))$. But this shows that $x$ induces an inner automorphism on $K$ and so $KQ = K \times U$. Suppose
$U \cap Q = 1$. Then we have $K \cong Sp(2n,2)$ or $U_n(2)$ and $|Q : Q \cap O_2(M)| = 2$. Then $|Q : C_Q(Y_M)| = 2$. Hence also $|Q : C_Q(A)| = 2$. Now we may assume that $[V_1, U] \neq 1$. Then we see that $[V_1, x] \cong V_1$ as an $N_K(Q)$-module. As $[V_1, x] \leq Q$ this shows that $A$ induces a transvection group on $V_1$ and so $|A| = 2$, which contradicts the fact that $V_1$ is better than $F$-module. Hence we have $U \cap Q \neq 1$ and we may assume that $x \in U$. Now $[W, x]$ is a nontrivial $K$-module. Let $W_1 \leq [W, x]$ be an irreducible nontrivial $K$-module, which is centralized by $U$. In particular $A$ acts nontrivially and so $W_1 \not\subseteq Y_M$. We have that $W_1^g \leq Q$. If $[W_1^g, W_1] = 1$, then we have $W_1^g \leq O_2(M)$ and then we see that $[A, W_1] = 1$, a contradiction. Hence we have that there is an offender on $W_1$ in $A \cap Q$, which means that there is an offender on $V_1$, which is contained in $O_2(N_K(Q))$. Now we see that $K \cong L_n(r)$ or $Sp(2n, r)$. Further by 2.4 we get that $|C_{W_1}(Q)| = r$ and so $Q \cap K = O_2(N_K(Q))$. Now $[W, x]^g \leq Q \cap K \cap A$. But $Q \cap K$ is an exact offender on $W_1$, which shows that $W_1 = [W, x]$. The same argument also implies that $[W, y] \leq [W, x]$ for all $y \in U$. Hence $U$ is a transvection group to a point.

We have $|[W_1, A], Q]| \leq r$, hence $|[W_1^g, Y_M], Q]| \leq r$. As $[W_1^g, V_1] \neq 1$, we get $|[W_1^g, Y_M], Q]| = r$ and $|[W_1^g, V_1]| = r$, hence $|[W_1^g, W]| = r^s$. and so $|[W_1, A]| \geq r^s$. Let $y \in C_{W_1}(W_1)$ with $[y, A] \neq 1$, then $[y^g, Y_M] \neq 1$. But $y^g \in C_{W_1}(W_1) \leq O_2(M)$, a contradiction. So we have $|C_{W_1}(W_1), A| = 1$. If $K \cong Sp(2n, r)$, then we get $A = W_1^g$ is of order at most $r$, which contradicts the fact that $A$ is an offender and $s > 1$. So we have $K \cong L_n(r)$ and $A$ is contained in a transvection group to a hyperplane. Now we see that the largest completely reducible $K$-submodule of $Y_M$ is a direct sum of natural modules. So without loss we may assume that this is $W$. As $U$ is a transvection group to a point, we see with 2.4 that $K \cong L_n(r)$ and $U$ is the full transvection group. Finally $W$ is the tensor product of the corresponding natural modules.

What is left is to show that $W = Y_M$. Otherwise by 3.14 we get a submodule $\tilde{W} > W$ such that $W/\tilde{W} \cong W$ as $K \times K_1$-module. In particular there is $x \in U$ such that $\tilde{W}/x$ is an extension of the natural $K$-module by the natural $K$-module. Now we know that $|[\tilde{W}, x]^g : [\tilde{W}, x]^g \cap O_2(M)]| \leq r$. Hence $|[\tilde{W}, x] : C_{\tilde{W}, x}(A)| \leq r$, a contradiction. So we have $W = Y_M$.

**Lemma 4.14** Assume 4.2(1). Then $[A, E(M_0O_2(M)/O_2(M))] \not\leq O_2(M)$.

**Proof:** Suppose false. By 4.3 we have that $[A, F_2(M_0)] \not\leq O_2(M)$. As $A$ acts quadratically on $Y_M$ we see with 2.4 that $A$ acts nontrivially on exactly on Sylow subgroup of $P$ of $F(M_0/O_2(M))$. As $A$ is an $F$-module.
offender, we see that $P$ is a 3–group. By 3.1 we get $Y_M = \langle y \mid A : C_A(y) \rangle \leq 2$. So we see $[P, A]A/C_{[P, A]}(Y_M) \cong \Sigma_3 \times \cdots \times \Sigma_3$. As $Q$ normalizes $[P, A]$ we get with 2.4 that $P = [P, A] = F^*(M_0/O_2(M_0))$. Now $Q$ acts transitively on the elements $\rho$ in $P$ with $|[\rho, Y_M]| = 4$. This shows that $M_0S$ is a minimal parabolic, contradicting 2.7.

Proof of 4.1 By 4.13, 4.14, 4.11 and 4.10 we just have to treat the case that $M_0/O_2(M_0)$ just has one component $K$. By 4.3 $[K, A] \neq 1$. If we are in 4.2(1) we get with 2.4 that $A$ acts faithfully on $K$, i.e. $[K, Y_M]$ is an $F$–module. Then according to 3.4 we just have to investigate $K \cong A_n$. Now 2.4 and 3.12 imply that we have $K \cong A_7$ and $Y_M$ involves the 4-dimensional module. But then a subgroup $L_3(2)$ in $A_7$ centralizes some element centralized by $Q$, which gives $Q \leq O_2(M)$, a contradiction.

So we are in 4.2(2). Let $t \in C_A(K)$. Then $A$ acts quadratically on $[Y_M, t]$. Further $K$ acts nontrivially as $C_{Y_M}(K) = 1$. There is $B \leq A$ with $|B| = q$ and $B$ acts quadratically on $Y_M$. So $C_B(K) = 1$. Now $|[Y_M, t] : C_YM(B)| \leq |B|$ and so $K$ induces an $F$–module. Then we may argue as above. Hence we may assume that $A$ acts faithfully on $K$ again.

Now we have an irreducible submodule $W$ which is a 2$F$–module which is not an $F$–module. Then according to 3.7, 3.8, 3.9 and 3.10 we get that we just have to handle $K \cong E_7(r), A_9$ and $M_{23}$. We have that $A_9$ is a minimal parabolic, which contradicts 2.7. In case of $M_{23}$ there is some $M_{22}$ centralizing some element in $C_W(Q)$ and so again we have the contradiction $Q \leq O_2(M)$. Hence it just remains $E_7(r)$. Then $W = V(\lambda_7)$ is the 56–dimensional module. Now we get that $C \cap K \geq r^{27}E_6(r)$ and so $Q : Q \cap O_2(M)$ $\geq r^{27}$. The action of $Q$ on $W$ shows that $|[W, Q]| = 5^{55}$. As $[W, Q] \leq Q$, we see that $[W, Q]$ is an $F$–offender on $[W, Q]^g$. But the action of $Q$ on $[W, Q]$ shows that we have that $|[W, Q], [W, Q]^g| = 1$. But then even $[W, [W, Q]^g] = 1$ and then $[Y_M, [W, Q]^g] = 1$. This now implies that $[A, [W, Q]] = 1$ and then $[A, Q] = 1$, a contradiction. This proves 4.1.
5 The case $b(M) \geq 2$

Proposition 5.1 Let $M \in \mathcal{L}(S)$, $M \not\leq \bar{C}$, with $M_0$ maximal. Let $b(M) > 1$ and $Y_M \leq O_p(C_G(x))$ for all $1 \neq x \in Y_M$. Assume further that there is no $g \in G$ such that $1 \neq [Y_M, Y_M^g] \leq Y_M \cap Y_M^g$. Then $F^*(M_0/O_p(M_0))$ is quasisimple and isomorph to $3A_6$, $Sp(2n,q)$, $SL(n,q)$ or $\Omega^\pm(2n,2)$. Further $[Y_M, E_p(M)]$ is the natural module, or $F^*(M_0/O_p(M_0)) \leq SL(n,q)$ and $[Y_M, E_p(M)]$ is the direct sum of two natural modules.

**Proof:** By 2.2(c) and 2.6 we may and will assume that $M = M^*$. Let $P$ be a minimal parabolic in ES with $P \not\leq M$. Set $U = \langle M, P \rangle$. Assume first $O_p(U) \neq 1$. Now $M_0 \leq U_0$. Hence by assumption $U_0 = M_0$. So $P \leq N_G(M_0)$. As $E \leq N_G(M_0)$ there is some $P$ with

$$O_p(\langle M, P \rangle) = 1.$$  

By assumption, we have that the parameter $b$ of the amalgam is odd and at least three. By 3.20 $Y_M$ is a strong $F$-module or a strong dual $F$-module. Let $A$ be an offender and $M_1 = [E_p(M_0), A]$. Let $K$ be a $p$-component of $M_1$. Then application of 3.21 yields the structure of $K$ and $[Y_M, K]$, besides in the case of $K \cong SL(n,q)$, we have to show that there are exactly two such modules. Let $Y_M, K = V_1 \oplus V_2 \oplus \cdots \oplus V_r$. We have $A = V^- \cap O_p(L^+)$ in the notation of 2.13. Further we see that $|V_i, A| \geq |V^- \cap Q^+/C_{V^-}(V^+)|$. This shows $|[Y_M, V^- \cap Q^+]| \geq |V^- \cap Q^+/C_{V^-}(V^+)|^r$. Now we have that $|V^- / C_{V^-}(V^+)| \leq |V^- \cap Q^+/C_{V^-}(V^+)|^2$, so as $[Y_M, V^- \cap Q^+] \cap Z(L^-) = 1$, we get $r = 2$.

Suppose now first that $[Y_M, K]$ is irreducible. Now we see that any other component just can induce a field multiplication on $[Y_M, K]$, in particular is solvable. This shows that here is some $t \in Q$ which induces an outer automorphism on $K$ and so $K \cong SL(n,q)$ or $Sp(2n,q)$. But as $Q \leq O_p(C_{M_0}(x))$ for $1 \neq x \in C_{[Y_M, K]}(Q)$, this gives $K \cong SL(2,q)$ but then as $M_0S$ is not a minimal parabolic, there must be some $s \in F^*(M_0/O_2(M_0)) \setminus K$ with $[s, x] = 1$, $x$ as above, a contradiction.

So we have that $K \cong L_n(q)$ and $[Y_M, K]$ is the direct sum of two natural modules. Let $K_1K = M_0$. Let $V$ be an irreducible $KS$-submodule in $Y_M$, and $R$ be a 1-dimensional subspace $[V, A]$, where $A = V^- \cap O_p(L^+)$. Now $R$ is normalized by some Sylow $p$-subgroup $S^g$ in $KS$. In particular $W = (R^{K_1S^g})$. Hence $(K_1S^g, V^-) \leq N_G(W) = H$. We have $|C_H(W), (\langle q^g \rangle^H)| \leq O_p(H)$. Further $O_p(K_1) \leq [K_1, Q] = K_1Q^g$.
and \( O(K_1) = O^p(K_1O_p(H)) \). Hence \( V^- \leq C_H(W) \leq N_G(O^p(K_1)) \). But \( M_0 \leq N_G(O^p(K_1)) \) now the maximality yields \( V^- \leq N_G(M_0) \), a contradiction. This contradiction shows \( F^*(M_0/O_p(M_0)) = K \).

Hence we may assume that \( [A, E_p(M_0)] = 1 \). Suppose \( [A, M_0] \leq O_p(M_0) \). If we are in the situation of 2.12 or 2.14 then \( A \) is a dual offender, and so there is exactly one module involved, which has to be centralized by \( F^*(M_0) \), but then \( M_0 \leq \tilde{C} \) by 2.4, a contradiction. Hence we are in the situation of 2.13. Basically as above we get a contradiction. We have \( A \leq V^- \), but \( V^- \not\leq M \). Further we have that \( |A| > p \). Now by 3.6, we see that there is some \( 1 \neq x \in [Y_M, A] \), which is centralized by a Sylow \( p \)-subgroup \( S^* \) in \( M \). Now \( V_1 = \langle x^{F^*(M_0)} \rangle \) is normalized by \( F^*(M_0)S^* \). As \( V_1 \leq [Y_M, A] \leq V^- \), we see that \( \langle V^-, F^*(M_0)S^* \rangle \leq N_G(V_1) \). But \( F^*(M_0)S^* \) is conjugated in \( M \) to \( M_0S \), so we see by maximality of \( M_0 \) that \( E_p(N_G(V_1)) = E_p(M_0) \). Then \( N_G(E_p(M_0)) \geq \langle M, V^- \rangle = \langle M, P \rangle \), a contradiction.

So we have \( [A, F_p(M_0)] \not\leq O_p(M) \). By quadratic action and 2.4 we see that \( A \) normalizes a unique Sylow \( r \)-subgroup \( R \) in \( F(M_0/O_p(M_0)) \). Again \([Y_M, R]\) is a strong \( F \)-module or strong dual \( F \)-module with offender \( A \). Then we easily see that \( p = 2 \) or \( 3 \) and \( |A| = p \). Further \([A, R]/A/C_{[A, R]}(Y_M) \cong SL_2(p) \). Further with 2.4 we see \( R = F^*(M_0/O_p(M_0)) \). Set \( \tilde{R} = R/C_R(Y_M) \). Then with 2.4 we see that \([\tilde{R}, Q]Q \) is a direct product of groups \( SL_2(p) \) on which \( Q \) acts transitively. Further we have \( \tilde{R} = R \) and then we see that \( M_0S \) is a minimal parabolic, which contradicts 2.7.

**Proposition 5.2** Let \( M \in \mathcal{L}(S) \), \( M \not\leq \tilde{C} \), with \( M_0 \) maximal. Suppose \( b(M) = 2 \) and there is some \( g \in G \) such that \( 1 \neq [Y_M, Y_M^g] \leq Y_M \cap Y_M^g \). Then \( F^*(M_0/O_p(M_0)) \) is quasisimple and isomorphic to \( 3A_6, Sp(2n, q), SL(n, q) \) or \( \Omega^\pm(2n, 2) \). Further \([Y_M, E_p(M_0)]\) is the natural module.

**Proof:** Set \( Y = C_{Y_M}(Q) \) and \( \bar{Y} = C_{Y_M}(\bar{Q}) \), where \( \bar{Q} = Q^g \). By assumption \( Y_M^g \leq N_G(M_0) \) and \( Y_M \leq N_G(M_0^g) \). If \([Y_M, M_0] \leq O_p(M_0) \) and \([Y_M, M_0^g] \leq O_p(M_0^g) \), then by the \( P \times Q \)-lemma we have that \( Y \) acts nontrivially on \( \bar{Y} \). But now \( 1 \neq [Y, \bar{Y}] \) is centralized by \( \langle Q, Q^g \rangle \), which by 2.4 gives \( Q = Q^g \) and so \( Y_M^g \leq Q \) and then \( Y_M^g \leq O_p(M_0) \), so \([Y_M, Y_M^g] = 1\), a contradiction.

Assume first that \( Y_M^g \) centralizes \( M_0 \). Then \( Y_M^g \) acts on some component of \( M_0^g \) nontrivially. Let \([Y_M, L] \neq 1 \) for some component \( L \) of \( M_0^g \). Then by 2.4 we have that \( Y_M \) acts faithfully. Let \( L_1 \times \cdots \times L_r = L^Q \), with \( L = L_1 \). Then we see that \([Y_M, L] \) is an \( F \)-module. Suppose \( r > 1 \). Suppose \( L_1 \times L_r \) induces some tensor product on \( Y_M \). Then \([Y_M, L_r] = 1 \). So we may assume
that $Y_M$ induces on $[Y^g_M, L_1]$ an $F$–offender. Let $V_1$ be some irreducible submodule in $[L_1, Y^g_M]$ such that $V_1 \otimes V^h_1$, for some $h \in Q^g$, is the tensor product module. Now we have that $|V_1| = r^t$ and $|C_{V_1}(Y_M)| = r^x$. So we get $|Y_M : C_{Y_M}(V_1)| \geq r^{t(t-x)}$. But the largest elementary abelian subgroup of $GL(t, r)$ centralizing a subspace of dimension $x$ is of order $r^{x(t-x)}$. So we have $x = t$, a contradiction. Hence we have shown that $[L^Q, Y_M]$ is a direct sum of certain modules for the components $L_i$. But the we get as in 3.19 that $L$ is a minimal parabolic. There is just one orbit and so we see that $M_0 S$ is minimal parabolic, contradicting 2.7. Hence we have $L^Q = L$. Let $L = K^g_1$ for some component $K_1$ in $M_0$. Then we have that $[Y^g_M, K^g_1]$ acts trivially on $K_1$ and nontrivially on some component $K_2$ of $M/O_p(M)$. Now by 3.17 we see that $[K_1 \times K_2, Y_M]$ is a direct sum. But then we have a contradiction with 2.4. Hence we may assume that there is a unique component $K_1$ in $F^*(p_0)$ which is normalized by $Y^g_M$ and that $K^g_1$ is the unique component in $M^g$ normalized by $Y_M$. Further we may assume that $Y_M$ is an $F$–module with offender $Y^g_M$. Recall that component here always includes the solvable case. Now we may apply 3.4.

Let first $K_1$ be solvable. Then $K_1$ induces a direct product of $SL_2(p)$, $p = 2, 3$, on $Y_M$. Hence we see that $[C_{M_0}(K_1), [K_1, Y_M]] = 1$. So we get that $K_1 = F^*(M_0/O_p(M_0))$. This, as we have seen before, contradicts 2.7.

So let now $K_1 \cong A_n$ and $Y_M$ involves the natural module $V$ as a submodule. Then we get a contradiction with 3.12.

Let $n = 7$ and $V$ be the 4–dimensional module. Then for $1 \neq x \in C_V(S)$, we have $C_{K_1}(x) \cong L_3(2)$, which has no normal 2–subgroup, a contradiction.

Suppose $K_1$ is classical or $G_2(q)$ and we have one of the natural modules. Suppose that $Q$ normalizes this module. Assume further that $K_1$ acts transitively on the vectors and there is some $1 \neq x \in [Y_M, Y^g_M]$, which is centralized by $\langle Q, Q \rangle$. Hence by 2.4, we get $Q = Q^g$ and so $Y^g_M \leq Q$. Now we see that we do not have he unitary group. Let $K_1$ be classical. Then we see $|Y_M : C_{Y_M}(Y^g_M)| = |C_{Y_M}(Y^g_M)|$. Hence there is exactly one nontrivial $K_1$–module involved and so by 2.4 we have that $K_1 = E(M_0/O_p(M_0))$. If $K_1 \cong G_2(q)$, then as $Y^g_M \leq \cap_{1 \neq x \in [Y_M, Y^g_M]} O_p(C_{K_1}(x))$, we get a contradiction to $|Y^g_M : Y^g_M \cap O_p(M)| = q^3$.

So we have that $Q$ does not normalize this module. In particular there is some $h \in Q$, $h$ induces an outer automorphism on $K_1$. We have that $Q \leq O_p(C_{KQ}(x))$ for $1 \neq x \in C_Y(M)$. This shows that we have $K_1 \cong L_4(q)$ and $K_1$ induces the natural module $V$. Hence we have $|Y_M : Y_M \cap O_2(M^g)| = q^4$ and so $Y_M = V \oplus V^h$. Now there are two orbits in $[Y_M, Y^g_M]$ of length
2(q^2 - 1) and q^4 - 2q^2. As q^4 - 2q^2 > 2q^2 - 2, we see that there is some 1 \neq x \in Y_M, Y_M^g, centralized by \langle Q, Q^q \rangle. Then Y_M^g \leq Q^g, a contradiction.

Let now K_1 \cong L_n(q) and V be V(\lambda_2). We may assume n > 4. Then by [GBSM] we have a unique offender. Hence |Y_M^g : C_{Y_M^g}(Y_M)| = q^{n-1}
and ||Y_M, Y_M^g|| = q^{n-1}. In particular Y_M^g is not in O_p(C_{K_1}(x)) for any 1 \neq x \in V, which is centralized by a Sylow 2-subgroup of K_1. Take u \in Y_M^g \setminus C_{Y_M^g}(Y_M). Then all elements in [Y_M, u] are conjugate. Hence as above we get Y_M^g \leq Q, a contradiction.

Now let K_1 \cong Sp(6,2) and V be the spin module. Then any offender is contained in the largest normal 2-subgroup of the point stabilizer in the natural representation. Hence \[[V, Y_M^g]|| = q^4. Now any element in [V, Y_M^g] is centralized by a Sylow 2-subgroup and so again Y_M^g \leq Q. As |Y_M^g : Y_M^g \cap O_2(M)| \geq q^4, a contradiction.

Let now K_1 \cong \Omega^+ (2n, q) and V be the natural module. If Y_M^g contains elements 1 \neq u \in K_1, then [V, u], contains some x, whose centralizer contains Q. As Y_M^g \subseteq Q, the point stabilizer on the natural module does not contain an offender. So we see that x is centralized by Sp(2n - 2, q) in K_1^g. But then as Y_M \leq Q we get |Y_M : C_{Y_M^g}(Y_M)| = 2, a contradiction. So we have |Y_M^g| = 2, and so q = 2. By 2.4, we see that E_p(M_0) = K_1 and V = [Y_M, K_1].

Let finally K_1 \cong \Omega^+ (10, q) and V be the half spin module. Let H \leq K_1 with |O_2(H)| = q^8 and |C_V(O_2(H))| = q^8. If 1 \neq u \in O_2(H) is contained in Y_M^g, and [u, V] = [O_2(H), V] then O_2(H) = Y_M^g / Y_M^g \cap O_2(M). But any x \in V centralizes some nontrivial element in O_2(H), while there are elements in O_2(H) whose centralizer in V is exactly C_V(O_2(H)), a contradiction. Now we see that for any such u we have that [u, V] contains just elements, which are centralized by a Sylow 2-subgroup. Hence again Y_M^g \leq Q. We have that there is R \leq K_1 with R/O_2(R) \cong L_5(q) and |O_2(R)| = q^{10}, which stabilizes a point in V and so Y_M^g / Y_M^g \cap O_2(M) \leq O_2(R). But R has the following series in V

1 \leq V_1 \leq V_2 \leq V

where V_1 is a trivial module V_2/V_1 is the module isomorphic to O_2(R) and V/V_2 is the natural R/O_2(R)-module. As V_1 = C_V(O_2(R)), we see that O_2(R) contains no offender on V.

Hence what is left, is K_1 \cong 3A_6. Then again we have a unique offender, which is of order 4. In particular there is just one module involved and so by 2.4 we have K_1 = E_p(M_0).

We have seen that E(M_0/O_p(M_0)) is quasisimple. Further [E_p(M_0), Y_M] is irreducible. So F(M_0/O_p(M_0)) just can induce field multiplications. But
then we get $F(M_0/O_p(M_0)) = 1$, as seen several times before. This is the assertion.

Then next proposition will be proved by a series of lemmas.

**Proposition 5.3** Let $M \in L(S)$, $M \not< \bar{C}$, with $M_0$ maximal. Suppose $Y_M \leq Q$ but there is some $1 \neq x \in Y_M$ with $Y_M \not< O_p(C_G(x))$. Then $F^*(M_0/O_p(M_0))$ is quasisimple and isomorph to $3A_6$, $Sp(2n,q)$, $SL(n,q)$ or $\Omega^\pm(2n,q)$. Further $[Y_M, E_p(M_0)]$ is the natural module.

Until further notice we will assume that 5.3 is false. Then we choose some $L \in L$, i.e. any subgroup with $O_p(L) \neq 1$, with

(a) $Y_M \not< O_p(L)$

(b) $C_S(Y_M) \leq L$

(c) $|L \cap M|_p$ is maximal with respect to (a),(b)

(d) $L$ is minimal with respect to (a) - (c)

This is possible as, we have some $x \in Y^*_M$ with $Y_M \not< O_p(C_G(x))$.

Denote by $T$ a Sylow $p$–subgroup of $M \cap L$.

**Lemma 5.4** $T \in Syl_p(L)$.

**Proof:** By 2.2(e) and 2.6, we may assume that $M = M^*$. Suppose now that $N_L(T) \not< M$. Then by maximality of $M_0$, we see that $O_p((M, N_L(T))) = 1$. Hence $(M, N_L(T))$ is an amalgam. Let $(\alpha, \alpha')$ be a critical pair. Then both $\alpha$ and $\alpha'$ are of type $M$. But now $1 \neq [Y_M, Y^g_M] \leq Y_M \cap Y^g_M$ for some $g$. This contradicts 5.2.

**Lemma 5.5** We have

a) $L/O_p(L)$ is a minimal normal subgroup with respect to $T$

b) $Y_M$ normalizes any component of $L/O_p(L)$

c) Let $P$ be a proper parabolic in $P(T)$ in $L$, then $\langle Y^P_M \rangle$ is an elementary abelian $p$–group.

d) Let $P$ be a proper parabolic in $P(T)$ in $L$. Then $A = \langle Y^P_M \rangle$ acts quadratically on $Y_L$.
Proof: a) This is just the minimal choice of \( L \)
b) As \( Y_M \leq T \) and \( Y_M \) acts quadratically this follows from [Cher]
c) Let \( P \) be some parabolic. By the minimal choice of \( L \), we have that \( Y_M \leq O_p(P) \). Now \( (Y_M^P) \) is abelian by 5.2
d) We have \([Y_L, A] \leq A\). By c) \( A \) is abelian, so \([Y_L, A, A] = 1\).

Lemma 5.6 If \( L \) is not a minimal parabolic and \( K \) is a component, which is of Lie type in characteristic \( p \), then \( Y_L \) is a strong quadratic module.

Proof: Let first \( Y_L = \Omega_1(Z(T)) \). Then by the maximality of \( T \), we have that \( T \in \text{Syl}_2(G) \). But now by 2.4 we would get \( Y_M \nleq Q \), a contradiction. So \( Y_L \) is a nontrivial module. Now suppose false. Then \( \langle Y_M^P \rangle O_p(L)/O_p(L) \) is contained in a root subgroup for any parabolic \( P \), which is impossible.

Lemma 5.7 Let \( K \) be a component of \( L/O_p(L) \) in \( \text{Chev}(p) \), then one of the following holds

a) \( K \) is a rank 1 Lie group
b) \( K \cong L_n(q), Sp(2n, q), U_n(q) \) and \( Y_M O_p(L)/O_p(L) \) is in a transvection group.
c) \( K \cong G_2(q) \) or \( 3D_4(q) \) and the natural module is in \( Y_L \)
d) \( K \cong Sp(2n, q), Y_L \) contains the spin module and \( Y_M O_p(L)/O_p(L) \) is in a short root group.

Proof: We will apply 5.6 and [Str]. Let first \( K \cong L_n(q) \). Then \( Y_L \) contains some \( V(\lambda_i) \). Let \( K_i \) be the parabolic corresponding to \( \lambda_i \). Let \( A = \langle Y_M^{K_i} \rangle \). Then \( A = O_p(K_i) \). But this is just quadratic for \( i = 1, n - 1 \). Let now \( K \cong U_n(q) \) and \( V(\lambda_i) \) a submodule. Now as before, we get \( i = 1 \). As \( O_p(K_1) \), \( K_1 \) the point stabilizer is not abelian, we see that \( Y_M \) is in the transvection group.

Let \( K \cong Sp(2n, q) \). Let \( K_1 \) be the point stabilizer in the natural representation. Suppose \( O_p(K_1) \) acts quadratically. Then we have the spin module. Now consider \( K_2 \), the normalizer of a short root group. But \( Z(O_2(K_2)) \) does not act quadratically on the spin module, so we have that \( Y_M \) is in the short root group. If \( O_p(K_1) \) is not quadratic, we have that \( Y_M \) is in a long root group. Now \( (Y_M^{K_2}) \) acts quadratically and so \([N_K(Y_M), [V, Y_M]] = 1\), which implies that we have the natural module.
Let $K \cong F_4(q)$ and $V = V(\lambda_1)$ or $V(\lambda_4)$. Then either $Z(O_2(K_1))$ or $Z(O_2(K_4))$ has to act quadratically, where $K_1, K_4$ are the maximal parabolics related to the roots. But both is not true.

Let $K \cong \Omega^\pm(2n,q)$. Then we have the natural or half spin module. But on the natural module $O_p(K_1)$ does not act quadratically. On the half spin module $O_2(K_n)$ does not act quadratically.

For $K \cong E_6(q)$, $E_7(q)$, or $2E_6(q)$ and $V(\lambda_i)$ we just consider $Z(O_2(K_i))$, which does not act quadratically.

**Lemma 5.8** $L$ is nonsolvable

**Proof:** Let $L$ be solvable. Then by quadratic action we have

$$Y_L = \langle x \mid [Y_M : C_{Y_M}(x)] \leq p \rangle$$

Let $x \in Y_L$ with $|[Y_M, x]| = p$. Then $\langle x \rangle$ is a strong dual offender on $Y_M$. But this contradicts 3.23 and 3.6.

**Lemma 5.9** Let $K$ be a component of $L/O_p(L)$. Then $[Y_M, K] \neq 1$. Define $\tilde{Y}$ by $\tilde{Y}C_{Y_M}(K) = Y_M$ and $\tilde{Y} \cap C_{Y_M}(K) = 1$. Then $|\tilde{Y}| \geq p^2$.

**Proof:** Let $|\tilde{Y}| = p$. Then there is a transvection on $Y_M$ and 3.23, 3.6 yield a contradiction.

**Lemma 5.10** Let $\tilde{Y}$ as in 5.9. Let $W \leq Y_L$ minimal with $1 \neq [W, K] \leq W$, then $C_W(K) \neq 1$.

**Proof:** Suppose $C_W(K) = 1$. By 5.8 we have $|\tilde{Y}| > p$. Then by [MeiStr1], [MeiStr2] $K$ is a Lie type group in characteristic $p$, $A_n$, $U_4(3)$ or some sporadics.

Let $K \cong A_n$. As $\tilde{Y}$ is in the normal 2-subgroup of any parabolic, we see that $\tilde{Y} \sim \langle(12)(34), (13)(24)\rangle$. But this group is not quadratic on the natural module and so we have the spin module. Then $[x, W] = [\tilde{Y}, W]$ for all $1 \neq x \in \tilde{Y}$. Now we may apply 3.23 and 3.6.

As $\langle\tilde{Y}^P\rangle$ is an abelian 2-subgroup for any nontrivial parabolic $P$, we see that $K \cong 3M_{22}$ or $M_{24}$ if $K$ is sporadic. If $K$ is $M_{24}$, then for $P = 2^4A_8$, we see that $\langle\tilde{Y}^P\rangle = O_2(P)$. But this group does not act quadratically on any module, contradicting 5.5. So let $K \cong 3M_{22}$. Then let $P_1 = 2^4\Sigma_5$ and $P_2 = 2^43A_6$. Now $\tilde{Y} \leq O_2(P_1) \cap O_2(P_2)$. Hence $|\tilde{Y}| = 4$, $\tilde{Y} \leq U$, $U \cong SL(3,4)$ and $\tilde{Y}$ is a root group in $U$. Now $U$ induces two natural
modules in $W$. As $\tilde{Y}$ is contained in some $L_2(4)$ in $U$ which has a submodule in $W$ which is the natural one, we see that $Y_M$ is a strong dual $F$–module, and so the assertion follows with 3.23 and 3.6.

If $K \cong 3U_4(3)$ or $3A_6$, then there is some parabolic $P$ such that $\langle \tilde{Y}^P \rangle$ is non abelian, contradicting 5.5.

Now we have that $K$ is of Lie type in characteristic $p$. Then by 5.7 we have that $K \cong L_n(q)$, $Sp(2n,q)$, $U_n(q)$, $G_2(q)$ or $^3D_4(q)$ or $K$ is a minimal parabolic. If $\tilde{Y}$ is in a root group, then it is also in some subgroup $U \cong L_2(q)$, which has a natural module in $W$. Now again 3.23 and 3.6 gives a contradiction. So assume now that $K \cong G_2(q)$ or $^3D_4(q)$ and $W$ is the natural module. Then again $\tilde{Y}$ is in a root group and so $W$ is a strong offender and we may apply 3.23 again.

So let finally $K$ be a minimal parabolic. If $K \cong L_2(q)$ or $Sz(q)$, then with 3.3 we get that $W$ is the natural module and we can apply 3.23 again. So let $K \cong U_3(q)$. Then by [Str] we have that $W$ is the natural module or the 8–dimensional one. In the case of the natural module, we again have 3.23. If $W$ is the 8-dimensional $GF(q)$–module, we have $||x,W|| = ||\tilde{Y},W|| = q^4$ for any $1 \neq x \in \tilde{Y}$. Hence $W$ is a dual strong offender and we may apply 3.23 again.

**Lemma 5.11** $T$ is not a Sylow $p$–subgroup of $G$. Further $Y_L$ is an $F$–module.

**Proof:** If $T$ is a Sylow $p$–subgroup we may assume $T = S$. By 5.10 we have $C_W(K) \neq 1$. But then also $C_{Y_L}(E_p(L)) \neq 1$. This gives $\Omega_1(Z(S)) \cap Z(L) \neq 1$ and so by 2.4 $L \leq \tilde{C}$. But this contradicts $Y_M \leq Q$. So we have that $T$ is not a Sylow $p$–subgroup of $G$.

Let now either $\Omega_1(Z(T)) \leq L$ or $J(T) \leq L$. Let $X$ be the one of these two groups which is normal. Then, as we may assume $T \leq S$, we get $N_S(X) > T$. Further $Y_M \not\leq O_p(N_G(X))$ and $C_{Y_M}(Y_M) \leq T \leq N_G(X)$. As $|N_G(X) \cap M|_p > |L \cap M|_p$, we have a contradiction to the choice of $L$. So neither $Z(T)$ nor $J(T)$ are normal in $L$ and then $Y_L$ is an $F$–module.

**Lemma 5.12** If $K$ is a component of $L/O_p(L)$ then $K \cong L_n(q)$, $U_n(q)$, $Sp(2n,q)$ or $G_2(q)$.

**Proof:** By 5.11 and 5.7 and 3.4, we just have to handle the case of $K \cong A_n$. Then by 5.5 and 5.9 we see that $\tilde{Y} \sim \langle (12)(34), (13)(24) \rangle$ and $Y_L$ involves the natural module. But this group does not act quadratically on the natural module, a contradiction.
Lemma 5.13 We have $p = 2$ and $|\bar{Y}| \leq 4$.

Proof: By 5.12, 5.7 and 5.5 we have that $\bar{Y}$ is in some root group. So $\bar{Y} \leq U$, $U \cong SL_2(r)$, where $U$ is minimal.

Let $p$ be odd. Then there is a submodule $\bar{W} \leq \bar{Y}$ which is the natural module. Hence we get that $\bar{W}$ induces a strong dual $F$–module on $Y_M$, a contradiction again. So we have $p = 2$.

By 3.2 and 3.3 there is $\bar{W} \leq \bar{Y}$ such that $\bar{W}/C_{\bar{W}}(U)$ is the natural module. By 3.2 we see that $Y_M$ is a weak strong dual $F$–module. Now application of 3.23 gives the assertion. Recall that in the case of $M_0/O_2(M_0) \cong \Sigma_3 \times L_n(2)$ we have $|\bar{Y}| = 4$.

Lemma 5.14 If $p = 2$, then $|\bar{Y}| > 4$.

Proof: By 5.9 we may assume that $|\bar{Y}| = 4$. By 5.12 and 5.7 we have that $\bar{Y}$ is in some root group. So $\bar{Y} \leq U$, $U \cong L_2(r)$, where $U$ is minimal. By 3.2 and 3.3 there is $\bar{W} \leq \bar{Y}$ such that $\bar{W}/C_{\bar{W}}(U)$ is the natural module.

Now $\bar{W}$ is an offender as a strong $F$–module. Now we can apply 3.22. This yields that there are $x_1, x_2 \in Y_M$ with $\bar{Y} = \langle x_1, x_2 \rangle$ and $[W, x_1] \cap [W, x_2] = 1$. Just choose $x_i$ in the different natural modules. As $R$ is the only component normalized by $\bar{W}$ but not centralized $R$ is also normalized by $W$. By 5.10 we have $C_W(K) \neq 1$. By 5.12 and 3.14 we have $K \cong Sp(2n, q)$ or $G_2(q)$ and $W/C_W(K)$ is the natural module.

Let $K \cong G_2(q)$. By 3.14 we have $|C_W(K)| \leq q$. Further we have that $[x_1, W][C_W(K)/C_W(K)] = [x_2, W][C_W(K)/C_W(K)]$. As we know that $|[x_1, W][C_W(K)/C_W(K)]| = q^2$, we see $[x_1, W] \cap [x_2, W] \neq 1$, a contradiction.

We have $K \cong Sp(2n, q)$. From 3.22 we see that $[Q, R] \leq R$ and then $C_W(Q) \neq 1$. Hence by 2.4 we get $L = K(\bar{C} \cap L)$ and so $K = E_p(L)$.

As $\bar{W}$ is an indecomposable $K$–module and as $Y_M$ is not a strong dual $F$–module, we see that $Y_M \cap C_{\bar{W}}(U) \neq 1$. In particular $Y_M \cap C_W(K) \neq 1$. Further as $Z(L) \neq 1$, we have $Q \neq L$. In particular no $1 \neq x \in Y_M \cap C_W(K)$ is centralized by $Q$. Hence from 3.22 we see that we may choose $Q$ in such a way that $|Q : C_Q(x)| = 2$. Hence by the choice of $L$ we have $C_Q(x) \leq L$ and so for $S_1 = TQ$ we get $|S_1 : T| = 2$.

Now we may quote 2.15 which gives us that either $O^2(L)$ is an extension of $W$ by $Sp(2n, q)$ or $O^2(L/O_2(L)) \cong Sp(6, q)$ and $O_2(L)$ is as in the corresponding parabolic of $F_4(q)$.

We first show $q = 4$ in any case. We have $|Q : Q \cap T| = 2$. Further we have that $|C_{[W, Y_M]}(Q \cap T)| \geq q$. On the other hand we know that
$M_0/O_2(M_0) \leq \Sigma_3 \times L_n(2)$ and $R \cap Q$ is the point stabilizer. So we see that $|C_{[W,Y]}(Q \cap R)| = 4$. If $Q \cap R \leq T$ we get $q = 4$. If $Q \cap R \not\leq T$, then we see $|C_{[W,Y]}(Q \cap R \cap T)| \leq 16$ and so again $|C_{[W,Y]}(Q \cap T)| \leq 4$ and then $q = 4$.

As $|Q,C_{W}(T)| \neq 1$, we see that there is some $\Sigma_3 \leq R$ which acts on $C_{W}(T \cap K)$. Hence there is some $\rho$, $o(\rho) = 3$ and $x \in Q \setminus T$ with $\rho^x = \rho^{-1}$. Further we have $\rho \in \langle Q,Q^g \rangle$, where $C_{W}(T \cap K) = \langle C_{W}(S_1),C_{W}(S_2^g) \rangle$. Finally $\rho$ acts transitively on $W/W \cap O_2(M)$.

Let $P = C_K(C_{W}(T \cap K))$. As $C_{W}(T \cap K) \cap C_{W}(Q) \neq 1$ we get with 2.4 that $[P,Q] \leq Q$ and $[P,Q^g] \leq Q^g$. As $P$ normalizes $Q \cap T$ which is of index two in $Q$, we see $[P,Q] \leq Q \cap T$. Hence $\langle Q,Q^g \rangle \leq N_G(PT) \leq N_G(O^2(P))$.

Suppose first that $K \not\geq L_2(q)$. Then by 2.15 $P$ is nonsolvable and so $[P,P] \leq O_2(P)$. We have that $O_2(P)/O_2(L)$ is an indecomposable module. As $P/O_2(P)$ is centralized by $\rho$, we see that either $[O_2(P),\rho]O_2(L) = O_2(P)$ or $[O_2(P),\rho] \leq O_2(L)$. But $[\bar{Y},\rho] = 1$ and so $O_2(P),\rho \leq O_2(L)$.

We have $|W : (O_2(P)\cap W)C_{W}(T\cap K)| \leq q$. Hence $|W^xO_2(L) : O_2(L)| \leq q$, as $O_2(P)' \leq O_2(L)$. Let $O_2(L)$ be as in $F_4(q)$. As $W^xO_2(L) \leq P$, we have $W^xO_2(L) \leq M_0O_2(L)$. As $[\bar{Y},\rho] \leq O_2(L)$ and $\rho$ acts transitively on $W/W \cap O_2(P)'$, we get $W^x \leq O_2(L)$. Now $W^x$ is normalized by $x$ and as $[K,O_2(L)] = W$, we also have that $WW^x$ is normal in $L$. But as $Y_M \not\leq O_2(L)$, we have a contradiction to the choice of $L$.

So assume that $O_2(O^2(L))$ is as in $F_4(q)$. Now again $W^x \leq O_2(L)$. But then the action of $Sp(4,q)$ immediately shows that $W^x = W$, and so $W \leq \langle L,Q \rangle$, again a contradiction.

So we are left with $K \cong L_2(4)$. Now again $\rho$ normalizes $T$. We have that $\rho$ acts transitively on $W/C_{W}(T \cap K)$. Further $\rho$ centralizes $\bar{Y}$. This shows $W^x \cap O_2(L) > C_{W}(K \cap T)$ and so $|W^x : C_{W^x}(W)| < 4$ and then $|W : C_{W}(W^x)| < 4$, which again gives $W^x \leq O_2(L)$ and a contradiction as above.

From 5.13 and 5.14 we get a contradiction. This proves 5.3.

We collect the results of this section.

**Proposition 5.15** Let $M \in \mathcal{L}(S)$, $M \not\leq \bar{C}$, with $M_0$ maximal. If $b(M) \geq 2$ then $F^*(M_0/O_0(M_0))$ is quasisimple and isomorph to $3A_6$, $Sp(2n,q)$.
$SL(n,q)$ or $\Omega^\pm(2n,q)$. Further $[Y_M, F^*(M_0/O_2(M_0))]$ is the natural module, or $F^*(M_0/O_p(M_0)) \cong SL_n(q)$ and $[Y_M, F^*(M_0/O_p(M_0))]$ is a direct sum of two natural $SL(n,q)$–modules.

**Proof:** The assertion follows with 5.1 and 5.2 and 5.3.

So up to now we have shown

**Theorem 5.16** Let $p = 2$ and $M \in \mathcal{L}(S)$, $M \not\leq \hat{C}$, with $M_0$ maximal. Then one of the following holds

i) $Y_M \not\leq Q$ and one of the following holds
   a) $F^*(M_0/O_p(M_0))$ is quasisimple and isomorph to $3A_6$, $Sp(2n,q)$, $SL(n,q)$ or $\Omega^\pm(2n,q)$, $U_n(q)$, $G_2(q)$, $E_6(q)$, $M_{22}$ or $M_{24}$ and $Y_M$ is a $2F$–module with quadratic or cubic offender.
   b) $F^*(M_0/O_p(M_0)) \cong L_n(r) \times L_m(r)$ (one or both allowed to be solvable) and $Y_M$ is the tensor product module

ii) $Y_M \leq Q$ and one of the following holds and $F^*(M_0/O_p(M_0))$ is quasisimple and isomorph to $3A_6$, $Sp(2n,q)$, $SL(n,q)$ or $\Omega^\pm(2n,q)$. Further $[Y_M, F_p^*(M_0)]$ is the natural module.

iii) $Y_M \leq Q$ and $F^*(M_0/O_p(M_0)) \cong SL(n,q)$ and $[Y_M, F^*(M_0/O_p(M_0))]$ is a direct sum of two natural $SL(n,q)$–modules.
6 The geometry for $M$

In this section we will prove the main theorem. We are going to show that $M_0$ always has a geometry. In fact we show a little bit more. We will prove that the case that $M_0$ involves a direct product of two linear groups and $Y_M$ is the tensor product module does not occur. This depends on the fact that we will have not $E$-uniqueness. Further we show that there are never diagram automorphisms involved. In particular if $\Omega^+(2n,q)$ is involved, then $O^+(2n,q)$ is not involved. We take the notation of 5.16. Until further notice we will assume that $b(M) = 1$.

Lemma 6.1 Suppose that $M_0$ does not have a geometry, then $M_0 S/O_2(M_0 S)$ has a normal subgroup $O^+(2n,q)$ or $L_n(q) \times L_n(q)$, where both components are conjugate in the second case.

Proof: Suppose false. Then by 4.1 and 5.1 we have that $M_0$ involves a Lie type group extended by a diagram automorphism. So the possible cases are $L_n(q)$, $n \geq 5$, or $E_6(q)$. Further we have that $Y_M$ has a submodule $V = V(\lambda_1) \oplus V(\lambda_{n-1}), V(\lambda_2) \oplus V(\lambda_{n-2}), V(\lambda_3), \text{for } n = 6$, and $V(\lambda_1) \oplus V(\lambda_6)$. In all cases we see that $Q$ is inside the simple group. If $V = V_1 \oplus V_2$ this shows that $C_{V_i}(Q) \neq 1$ for $i = 1, 2$. Hence by $E$-uniqueness for $1 \neq v_i \in C_{V_i}(Q)$, $i = 1, 2$, we have $C_M(v_i) \leq \tilde{C}$. But then $M_0 \leq \langle C_{M_0}(v_1), C_{M_0}(v_2) \rangle \leq \tilde{C}$, a contradiction. Hence we are left with $L_6(q)$ and $V(\lambda_3)$. Now we see $|QO_2(M_0)/O_2(M_0)| = q^9$. Hence we see $|V : V \cap Q| \leq q$. We know that $Y_M$ is a 2F-module with cubic offender $A$, as $V$ is not an $F$-module. Suppose first $V \leq Q$. As $A = Y_M^g \cap M_0$ for suitable $g$ and $V$ is not an $F$-module we see $|V, V^g| = 1$. Now also $|V, Y^q_M| = 1$ and so $|V, A| = 1$. But then $|Y_M, A| = 1$, a contradiction. So we have $V \leq Q$. Then set $B = A \cap V^g$. Now we see that $|V \cap Q : C_{V \cap Q}(B \cap Q)| \leq |B \cap Q|$. In $V$ we have the following series

$$1 < C_V(Q) < H < |V, Q| < V$$

where $|C_{V}(Q)| = q$, $|H, Q| = C_{V}(Q)$ and $H/C_{V}(Q) \cong QO_2(M_0)/O_2(M_0)$ as $N_{M_0}(QO_2(M_0))-module$, $H = [Q, V, Q]$ and $|V/[V, Q]| = q$. So we see that in any case $|V \cap Q : C_{V \cap Q}(B \cap Q)| > |B \cap Q|$, besides $B \cap Q = 1$. In the latter $|B| \leq q$ and $B$ induces a 2F-module on $V$, which shows $|V : C_V(B)| \leq q^2$, but in $L_6(q)$ there are no such elements.

Lemma 6.2 Assume that $M_0S$ involves $O^+(2n,q)$, then $Y_M$ contains a module $V$ which is the natural module.
Then we have a set if the dimension of $O$ of $A$ is of order $q$.

Let now first $n = 4$. In the group $\Omega^+(8,q)$ there are the following types of involutions $a_2$, $a_4$, $c_2$, $c_4$. This means that on the natural module $W$, we have that the dimension of $[W,t]$ is $2$, $4$, $2$, $4$, respectively. Further in case of type $a$ this is singular and in case of type $c$ this is not. Let $V = V_1 \oplus V_2$. Then we have $|V_i : C_{V_i}(t)| = q^2$ for both $i$ if $t$ is of type $a_2$. If $t$ is of type $c_2$ or $a_4$ then for one of the two modules $V_i$ we have $|V_i : C_{V_i}(t)| = q^2$ while for the other this index is $q^4$. If $t$ is of type $c_4$ we have $|V_i : C_{V_i}(t)| = q^4$ for $i = 1, 2$. So we see that $|V : C_V(t)| = q^4, q^6, q^6, q^8$ for $t$ of type $a_2$, $c_2$, $a_4$, $c_4$, respectively. Now as $V$ is not an $F$-module, we see that $V \not\leq Q$. As in 6.1 set $B = A \cap V^g$. Then we see that $|V : C_V(B \cap Q)| \leq q|B \cap Q|$. If $B \cap Q$ is not contained in the simple group, then $|V : C_V(B \cap Q)| \geq q^8$, a contradiction. So we have $|B \cap Q| \leq q^6$. This shows that $B \cap Q$ does not contain elements of type $a_4$. Suppose it consists not entirely of elements of type $a_2$, then we see that $|B \cap Q| \geq q^5$. Now we may assume that $|V_1 : C_{V_1}(B \cap Q)| \geq q^4$ and $|V_2 : C_{V_2}(B \cap Q)| \leq q^2$. As not all elements in $B \cap Q$ can have the same centralizer in $V_2$, we see that $|V_2 : C_{V_2}(B \cap Q)| = q^3$ and so $|B \cap Q| = q^6$ and $B \cap Q$ is uniquely determined. But in that case we have $|C_{V_2}(B \cap Q)| = q$, a contradiction. So all elements are of type $a_2$. But the largest subgroup of that type is of order $q^3$, so we see that $|C_{V_1}(B \cap Q)| = q^6$ and $|B \cap Q| = q^3$. Now all elements in $(B \cap Q)^g$ have the same centralizer in $V_1$, which is impossible as the centralizers of involutions in $B \cap Q$ are maximal subgroups of $O^+(8,q)$. So we are left with $B \cap Q = 1$. Then $|B| \leq q$ and then $V$ would be an $F$-module with offender $B$ a contradiction.

Let $n = 3$. Now $V$ is an $F$-module. So let first $V^g$ be an $F$-offender on $V$. Then the offender $A$ is of order $q^4$. Again as $|V : V \cap Q| \leq q$ we see that
\[ A \cap Q \geq q^3 \]. This shows \( QO_2(M_0) = O_2(M_0 \cap C) \). In particular \( A \not\leq Q \) and so \( V \not\leq Q \). Then we have \( V \not\leq Q \). Now the action of \( Q \) on \( V \) shows that for \( x \in V \setminus Q \), we have \(|[Q', x]| = q^3\). So we get \(|[Q, x] : [Q', x]| = 2\). In particular \( x \) is a transvection on \( Q/Q' \). This shows that \( V/V \cap Q \) is a transvection group on \( Q/Q' \). If \( |V : V \cap Q| = 2 \), we get the situation of a cubic \( 2F \)-module, which will be handled later. So we may assume \(|V : V \cap Q| > 2\). Hence we have a foursgroup of \( GF(2) \)-transvections to the same hyperplane. In particular \( VQ/Q \) is contained in some \( L_n(2) \) in \( C \). But then we see that there is some conjugate \( V^h \) of \( V \) such that \( \langle V, V^h \rangle/O_2(V^h) \cong \Sigma_3 \). Hence again we get a \( 2F \)-module, which we will handle right now.

So assume now that \( V \) is a \( 2F \)-module with cubic offender \( A \), where \( A = V^g \cap M_0 \) for suitable \( g \). We have \(|V : C_V(A)| \leq q|A|\), as \(|V : V \cap Q| \leq q\). Suppose that \( A \) is not contained in the \( \Omega^+(6, q) \). Then we see \(|V : C_V(A)| \geq q^4\). As \(|V : V \cap Q| \leq q\), we see that \( A \cap Q \neq 1 \), so \(|V : C_V(A)| \geq q^5\). As \(|A| \leq 2q^3\), we get \( q = 2 \) and equality everywhere. But then we see easily that \(|V_1 : C_{V_1}(A)| \geq 4\) and then \(|V : C_V(A)| \geq 64\), a contradiction. So we have that \( A \) is in the simple group. Suppose next that \( A \) is a group of transvections on the natural module. Then \( A \) acts quadratically on \( V \). But by construction in that case \( A \) is an \( F \)-offender on \( V \), a contradiction. So we have that \( A \) contains elements which do not act as transvections. Hence \(|V_i : C_{V_i}(A)| \geq q^2\) for \( i = 1, 2 \). If we would have equality on both modules, then again we would have quadratic action, in which case \( A \) would be an \( F \)-offender, which has to centralize \( V \cap Q \), a contradiction. So we may assume \(|C_{V_1}(A)| = q\). In particular \(|A| \leq q^3\). As \(|V : C_V(A)| \leq |A|q \leq q^4\), we get the contradiction \(|V_2 : C_{V_2}(A)| \leq q|A|\).

What is left is the case that \( V \leq Q \) and \([V, V^g] = 1\). But then we get \([V, Y^g_M] = 1\) and so also \([Y^g_M \cap M_0, Y_M] = 1\), a contradiction.

Let \( M_0/O_2(M_0) = \Omega^+(2n, q) \), then we denote with \( P_0 \) the unique minimal parabolic of \( M_0S \) containing \( S \) which is not contained in \( C \).

**Lemma 6.3** If \( M_0/O_2(M_0) \) has a normal subgroup \( \Omega^+(2n, q) \), then we have that \( M_0S/O_2(M_0S) \) does not contain \( O^+(2n, q) \).

**Proof:** Suppose false. As \( M_0 \) is not a minimal parabolic we have \( n \geq 3 \). By 6.2 we have that \( V \) is the natural module. If \( V \leq Q \), then as \( QO_2(M) \) does not contain an \( F \)-offender we get that \([V^g, V] = 1\) for all \( g \in C \). But then also \([V^g, Y_M] = 1\) and so \([Y_M, Y^g_M] = 1\), a contradiction. So we have \( V \not\leq Q \). This shows \(|V : V \cap Q| = q\). We have that \( VQ/Q \) is
centralized by some $\Omega^+(2n - 2, q)$ and normalized by some $O^+(2n - 2, q)$ in $\bar{C}$.

Let $K$ be a component of $\bar{C}/Q$ which contains this $\Omega^+(2n - 2, q)$. Suppose first that $VQ/Q$ projects injectively onto some subgroup of $K$. So we may assume that $VQ/Q \not\leq K$. Let first $K$ be a group of Lie type in characteristic two. Because of the existence of $O^+(2n - 2, q)$, we have that $K$ admits a diagram automorphism, so $K \cong E_6(q)$, $L_m(q)$ or $\Omega^+(2m, q)$. In the first and second case we get $2n - 2 = 4$ or $6$. Hence for $t \in VQ/Q$ an involution we have $||(t, Q)|| = q^4$, $q^6$, respectively. But in $E_6(q) : 2$, we have that $t$ is centralized by $L_6(q) : 2$, so $||(t, Q)|| > q^6$. Let $K \cong L_m(q)$. Then $t$ is centralized by $L_{m-2}(q) : 2$. This shows that we must have $K \cong L_6(q)$, and $M_0/O_2(M_0) \cong \Omega^+(8, q)$. Now as $||(Q, t)|| = q^6$, we see that $Q$ involves exactly one nontrivial irreducible $K$–module $W$. We have $[t, Q, O_2(M_0)] = 1$. So as $O_2(M_0)Q/Q \neq VQ/Q$ we see that $W$ is strong quadratic and with [Str] we get that $W \cong V(\lambda_3)$. Let $P$ be a minimal parabolic in $K : 2$ which is not in $M \cap \bar{C}$. Then $P$ centralizes $C_W(S)$. As $\Omega_1(Z(S)) \cap V$ is normalized by $M_0 \cap \bar{C}$ we see that $[K, \Omega_1(Z(S)) \cap V] = 1$. But then the action of $\Omega^+(8, q)$ on $V$ shows that $[O^2(P), \langle \Omega_1(Z(S))^{P_h} \rangle] = 1$. As $O_2((P_0, P)) \neq 1$, we get $\langle P, P_0 \rangle = PP_0$. Let now $S_1$ be the subgroup of a Sylow 2-subgroup of $M \cap \bar{C}$ which normalizes all minimal parabolics in $\Omega^+(8, q)$. Then this also normalizes all minimal parabolics in $K$. So $S_1$ is a Sylow 2–subgroup of $H = \langle F_2^+(M_0), K, S_1 \rangle$. This group is generated by 6 minimal parabolics which belong to the diagram

![Diagram]

By [Tits] we have that $H$ is an automorphism group of $E_6(q)$. Hence $HS$ contains $E_6(q) : 2$. Let now $u$ be some involution in $HS$, which induces an automorphism on $E_6(q)$, which is not type preserving. Then it centralizes in $E_6(q)$ either $F_4(q)$ or $2E_6(q)$. In particular $|O_2(C_G(u))| \leq q^{12}$ or $q^{18}$. But both groups do not have representations of such small dimension, which contradicts $C_G(O_2(C_G(u))) \leq O_2(C_G(u))$.

So we are left with $K \cong \Omega^+(2m, q)$. We have $||(Q, t)|| = q^{2n-2}$. Again there is exactly one nontrivial irreducible module $W$ involved, which has to be a strong quadratic one. As $O_2(C_K(t)) \leq O_2(M)Q/Q$ we see with [Str] that $W$ is the natural module. But then we get $2n - 2 = 2$, so $n = 2$, a contradiction.

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Suppose next that $K$ possesses no quadratic fours group on $Q$. Then we see that $O_2(C_K(t))$ is cyclic or a quaternion group. The structure of the alternating groups, sporadic groups or groups of Lie type in odd characteristic shows that we have $VQ/Q = O_2(C_K(t))$ is of order two. In particular by 3.14 $V = O_2(M)$ and $q = 2$. Then $M = 2^6O^+(8,2)$ and $K = A_{10}$ or $A_{11}$. This further shows that $Q$ is extraspecial of order $2^{13}$. But then $Q/Z(Q)$ can just involve the natural $K$–module, which contradicts the action of $\Sigma_8$ on $Q/Z(Q)$, which induces two permutation modules. So we have that $K$ possesses a quadratic fours group on $Q$. By [MeiStr1] and [MeiStr2] we now get that $K$ is alternating and just natural or spin-modules are involved, $K/Z(K) = U_4(3)$ and 12–dimensional modules are involved, or certain sporadic groups are possible.

Let first $K$ be alternating. Then $\Omega^+(2n - 2, q)$ corresponds to $\Omega^+(4,2)$ or $\Omega^+(6,2)$. In particular $||Q, t|| \leq 64$. Let $W$ be the natural module. Suppose we have that $t$ has at most 7 2-cycles in the cycle decomposition. In particular the $\Omega^+(2n - 2, 2)$ centralizes each cycle, but then it also centralizes $[Q, t]$, a contradiction. So we have exactly 8 2–cycles and so $K \cong A_{16}$. So we see that on the natural module $W$ we have, $||W, t|| = 2^6$. In particular $Q$ involves exactly one nontrivial irreducible $A_{16}$–module. Let $U$ be a minimal normal subgroup which involves $W$. Then we see that $|V \cap U| = 2^7$. This shows that $\Omega_1(Z(S)) \cap V$ is centralized by $A_{16}$, as the $\Sigma_6$ does not have a fixed point on $W$. We know that $V \cap O_2(M)' = 1$. Now choose $U_1$ maximal in $Z(U)$ with $\Omega_1(Z(S)) \cap V$ not in $U_1$. Then $U/U_1$ is either extraspecial or elementary abelian. As $|U/U_1| = 2^{15}$, we see in the former that $V \cap \Omega_1(Z(S)) \leq O_2(M)'$. So we have the latter. But $[Q, V \cap U] = \Omega_1(Z(S)) \cap V$, a contradiction.

So we have the spin module. But then quadratic groups have just involutions which have two 2-cycles. So $t$ centralizes exactly half of the dimension of $W$, i.e. $|W| \leq 2^{12}$. But this is impossible.

Let now $K/Z(K) \cong U_4(3)$ or one of the sporadics of [MeiStr2]. As $\Omega^+(2n - 2, q)$ is involved in the centralizer of a 2-central involution, we get $K/Z(K) \cong U_4(3)$, $Co_2$ or $Co_1$. As $K$ has to have an automorphism, which induces the $O^+(2n - 2, q)$, we get $K/Z(K) \cong U_4(3)$. Now we have $2n - 2 = 4$ and so $||Q, t|| = 16$. As $Q$ just involves 12–dimensional modules, we see that just one such module is involved. But we have an automorphism, which induces $O^+(4,2)$ on the $\Omega^+(4,2)$ in $K$. This means that this automorphism has to interchange the two maximal parabolics in $K$ which involve $A_6$. But the preimage of one of them in $K/C_K(W)$ contains $A_6$, while the other contains $3A_6$. This shows that the automorphism cannot
act on $W$, a contradiction.

Assume next that $2n - 2 = 4$ and so $\Omega^+(4, q) = L_2(q) \times L_2(q)$. These two groups could be contained in two components $K_1 \times K_2$. We will further assume that $VQ/Q \leq K_1 \times K_2$. Further we now know that there is some element in $C$ which interchanges the two components. As $O_2(C_K(t)) \neq 1$, $t$ as before, we get a quadratic fours group $U$ with $|U \cap K_i| = 2$, $i = 1, 2$. This shows that $Q$ cannot involve tensor products for $K_1 \times K_2$. But we have for the $\Omega^+(4, q)$ that $Q$ involves a tensor product. This shows that $q = 2$ and $K_1 \times K_2$ involves $\Sigma_3 \times \Sigma_3$ but this is not the $\Omega^+(4, 2)$ from the $\Omega^+(6, 2)$ in $M_0$. Let $P$ be some minimal parabolic in $(K_1 \times K_2)S$, which is not in $M_0$. Then $\langle P_0, P \rangle = PP_0$ or by 4.1, 5.1 we have $\langle P_0, P \rangle/O_2(\langle P, P_0 \rangle) \cong \Sigma_8$. Suppose the latter. Then we may apply [RoStr] to $\langle M_0, P \rangle$. By maximality of $M_0$ we have $O_2(\langle M_0, P \rangle) = 1$, so we get $|S| \leq 2^{14}$. Further we have $|Q| \geq 2^9$, which comes from $|V \cap Q| = 2^5$ and $|QO_2(M_0)/O_2(M_0)| = 16$. This now shows $|S/Q| \leq 2^5$. Hence $|K_1|_2 = 4$. As $K_1 \times K_2$ has to have a 8–dimensional representation we see $K_1 \cong L_2(4)$, but there is no 2-central involution in $L_2(4)$ centralized by some $\Sigma_3$. So we have $\langle P, P_0 \rangle = PP_0$ for all minimal parabolics in $(K_1 \times K_2)S$ which are not in $M_0S$. Now we see that $[O^2(P_0), O^2(P)] \leq O_2(P) \cap O_2(P_0)$. Now there is a subgroup $S_1$ of $S$ of index two which normalizes $O^2(P_0)$ and both components $K_1, K_2$. This of course contains $Q$ and $O_2(O^2(P_0))$. But both groups are in $O^2(M_0S)O_2(M_0S)$. Hence then the $\Omega^+(4, 2)$ in $K_1 \times K_2$ is the same as those in $\Omega^+(6, 2)$, a contradiction.

So we have shown, that $VQ/Q$ centralizes any component $K$ of $\tilde{C}/Q$ which involves $\Omega^+(2n - 2, q)$. Let $K$ or maybe $K = K_1 \times K_2$ be such a component. Set $W = Q/Q'$. As $Q' \leq O_2(M_0)$, we see that $[t, W] \neq 1$. Further $K$ acts nontrivially on $[W, t]$. As $[O_2(M), [W, t]] = 1$, we see that $O_2(M)Q/Q \cap K = 1$. In particular $K$ contains a subgroup $\tilde{K} \cong \Omega^+(2n - 2, q)$ or $O^+(2n - 2, q)$, which contains a Sylow 2–subgroup of $K$. Let now $\tilde{K}$ be a component on which $t$ acts nontrivially. Assume further that $\tilde{K}$ is normalized by $O_2(M)$. As $\tilde{K}$ and so $K$ acts irreducibly on $[W, t]$, we get that $[W, \tilde{K} \times \tilde{K}]$ is a tensor product $W_1 \otimes W_2$, where $W_2$ is an irreducible module for $\tilde{K}_2$ on which $t$ induces $GF(q)$ transvections. As $t$ is in the center of $S/Q$, we get $\tilde{K}_2 \cong L_m(q), Sp(2m, q)$ or $A_m$.

Suppose first that $V \not\leq O_2(M)'$. Then we see that $O_2(M)$ is elementary abelian, i.e. $Y_M = O_2(M)$. In particular $\tilde{K}_2$ has an elementary abelian Sylow 2–subgroup. This shows $\tilde{K}_2 \cong L_2(q)$. So let us first assume that this is not the case. Then $V \leq O_2(M_0)'$. In particular $\tilde{K} \cong A_m$. Let $K_P$ be the point stabilizer in $\tilde{K}$. As $t$ centralizes $Q'$ we have that $\tilde{K}$ centralizes $Q'$.
and so \(K_P\) centralizes \(\langle \Omega_1(Z(S)) \cap V \rangle^{P_0}\). This shows \(O_2(\langle K_P, P_0 \rangle) \neq 1\) and so \(\langle K_P, P_0 \rangle = K_P P_0\). As \(K \not\cong L_2(q)\), we see that \([O^2(K_P), O^2(P_0)] \leq O_2(P_0) \cap O_2(K_P)\). So we get that \(\langle K_P, M_0 \rangle = K_P M_0\). In particular \(K_P\) acts on \(\langle \Omega_1(Z(S)) \cap V \rangle^{M_0} = V\). So \([O^2(K_P), V] = 1\), which shows that we must have \(\hat{K} \cong Sp(2m, q)\). Now in \(O_2(M_0)\) there are at most \(2n - 1\) nontrivial composition factors for \(K_P/O_2(K_P)\). These all are natural \(Sp(2m - 2, q)\)–modules. As the \(\Omega^+(2n - 2, q)\) in \(M_0\) acts nontrivially on them, we see that we must have a tensor product of some \(\Omega^+(2n, q)\)–module with the natural \(Sp(2m - 2, q)\)–module. This shows that \(\Omega^+(2n, q)\) must have an irreducible module of dimension at most \(2n - 1\), as \(O^+(2n, q)\) is involved, this is impossible.

So we are left with \(\hat{K} \cong L_2(q)\). Suppose that \([O_2(M), \hat{K}] \not\cong \hat{K}\). Then by [Cher] we also have \(\hat{K} \cong L_2(q)\). Suppose next that \([\hat{K}, Y_M] \not\cong \hat{K}\). Then we may choose \(g\) such that \([Y^g_M : Y^g_M \cap M] = 2\). Now as \([V, V^g] \neq 1\), we would get that \(V^g\) is an \(F_1\)–offender on \(V\). Now \(q = 2\) and so \(\hat{K} \cong Z_3\). But now as \(VQ/Q \leq Z(Q/S)\) and induces transvections, we see that \(S\) induces a foursgroup on \(K^{O_2(M)}\). But then \(Y_M\) normalizes \(\hat{K}\). So in any case we have that \([Y_M, \hat{K}] \leq \hat{K}\). Now we may choose \(g \in \hat{K}\). This shows that for \(A = Y^g_M \cap M\), we have \(A \leq O_2(M)Q\). But as then \([V : C_V(A)] = \langle A, q \rangle\), we see that \([Y_M, A] \leq V\). This then with 3.14 shows that \(O_2(M) = Y_M = V\), or \(q = 2\), \(|V| = 2^6\) and \(|Y_M| = 2^7\). Suppose the latter. Then we may apply the same procedure to \(Y_M\) and get that \([Y_M : C_{Y_M}(A)] = 2|A|\). Then \(|C_{Y_M}(A)| = 4\), a contradiction. So we have the former. Further we get that \(|Q| = q^{1+2(2n-2)}\) and \(\hat{K}\) has no fixed points on \(Q/Q'\). We see that \(\langle P_0, \hat{K} \rangle/O_2(\langle \hat{K}, P_0 \rangle) \cong L_3(q), Sp(4, q), 3A_6, 3\Sigma_6\) or \(G_2(q)\), maybe extended by field automorphisms.

Further \(U = \langle \Omega_1(Z(S))^{P_0, \hat{K}} \rangle\) is the natural module. Now we have that the \(\Omega^+(2n - 4, q)\) in \(M_0\) permutes with \(\langle P_0, \hat{K} \rangle\). This shows that it centralizes \(U\). So if \(n > 3\), we get that \(|U| = q^3\) and then \(\langle \hat{K}, P_0 \rangle/O_2(\langle \hat{K}, P_0 \rangle) \cong L_3(q)\), maybe extended by field automorphisms.

So let \(n = 3\). Then \(\hat{K}\) induces exactly 4 nontrivial modules in \(O_2(\hat{K})\). The same applies for \(P_0\). Suppose that \(H = \langle \hat{K}, P_0 \rangle\) induces \(3A_6\) or \(3\Sigma_6\) on a \(6\)-dimensional module in \(O_2(H)\). As \(|O_2(H)| = 2^9\) or \(2^{10}\), we see that we must have the latter and a \(4\)-dimensional module is involved too. But on the \(6\)-dimensional module both \(\hat{K}\) and \(P_0\) induce two nontrivial modules, so we get that \(P_0\) induces 5 nontrivial modules on \(O_2(P_0)\), a contradiction. Let next \(H\) induce \(G_2(q)\) on \(O_2(H)\). As all nontrivial \(\hat{K}\)–modules are in \(Q\), we see that \(QO_2(H)/O_2(H)\) is of index maybe two in \(O_2(\hat{K})/O_2(H)\). But this group is not elementary abelian, a contradiction.
Let now $H/O_2(H) = Sp(4, q)$ or $A_6$. Further there are just 4-dimensional modules involved. Because of the number of nontrivial modules in the two parabolics, they have to be dual to each other. Now we see again, that $Q$ has to cover one of these modules. In particular $Q$ has to induce $GF(q)$-transvections on this module, which shows that $|QO_2(H)/O_2(H)| \leq q$, which contradicts $Q \not\leq O_2(P_0)$.

So we have shown that in general $H/O_2(H) \cong L_3(q)$. Let $S_1$ be a subgroup of index two in $S$ with $O_2(M_0S) \leq S_1$ and $S_1/O_2(M_0S)$ is a Sylow 2-subgroup of $\Omega^+(2n, q)$. Then the structure of $C$ shows that $K$ contains a subgroup of index two with Sylow 2-subgroup $S_1$. Further if $n > 3$, then $H$ permutes with some $\Omega^+(2n - 4, q)$ and so $H$ contains a normal subgroup of index two with Sylow 2-subgroup $S_1$. If $n = 3$, then we see that $O_2(H)$ just involves 3-dimensional modules. If we do not have a subgroup of index two in $H$, then $q = 2$ and there is a nonsplit extension of the 3-dimensional module by the trivial module involved, further this is extended by a 3-dimensional module which also does not split, i.e. we have $U \leq U_1 \leq U_2$, where $U$ and $U_2/U_1$ are 3-dimensional modules and $U_1/U$ is one dimensional. Neither $U_1$ nor $U_2/U$ are split. But then we see that $U_1 \leq [Q, U_2]$ and so $U_1 \leq Q$. But $K$ induces just one trivial module in $Q$, a contradiction. So we have that $H$ also possesses a subgroup of index two with $S_1$ as a Sylow 2-subgroup. Hence $(M_0, K) = NS$, where $N$ has a Sylow 2-subgroup $S_1$ and belongs to the diagram

\[ \cdots \quad \circ \quad \circ \quad \circ \quad \cdots \quad \circ \quad \circ \quad \circ \]

Application of [Tits] shows $N \cong \Omega^+(2n + 2, q)$ extended by some field automorphisms, and $NS = O'2n + 2, q)$ extended by some field automorphisms. Now take a transvection $u$. We have that $C_{\Omega^+(2n + 2, q)}(u) \cong Sp(2n, q)$. So we see that $|O_2(C_G(u))| \leq 2q^n$. But $Sp(2n, q)$ does not have such small representations, so $O_2(C_G(O_2(C_G(u)))) \not\leq O_2(C_G(u))$, a contradiction.

Next we handle the case that $M_0/O_2(M_0)$ contains $L_m(q) \times L_n(q)$ as a normal subgroup. Then let $P_0$ be the minimal parabolic which is not contained in $C$. Now $P_0$ projects onto a direct product $P_{01} \times P_{02}$ in this group.

**Lemma 6.4** Never $M_0/O_2(M_0)$ contains $L_m(q) \times L_n(q)$ as a normal subgroup inducing on $Y_M$ the tensor product module.
Proof: Suppose false. Let first \( m \leq 3 \geq n \). Set \( L = \tilde{C} \cap M \). By \( E \)-uniqueness we have \( QO_2(M_0) = O_2(L) \). Further \( L/O_2(L) \) has a normal subgroup isomorphic to \( \tilde{L} = L_m(q) \times L_{n-1}(q) \). This group induces on \( Y_M \) the following series

\[
1 < V_1 < V_2 < Y_M,
\]

where \( V_1 \) is the trivial module, \( V_2/V_1 \) is a direct sum of the natural \( m-1 \)-dimensional module with the natural \( n-1 \)-dimensional module and \( Y_M/V_2 \) is the tensor product of the two natural module. Further we see that \( V_2 = [Y_M, Q] \). This now shows that \( \tilde{L} \) is in exactly one component \( K \) of \( \tilde{C}/Q \), or \( m = n = 3, q = 2 \) and there are two components interchanged by \( \tilde{L} \), which belongs to the fact that \( \Sigma_3 \wr Z_2 \) contains two \( \Omega^+(4,2) \), one which induces a tensor product and one which induces a direct sum. Assume first that we have just one component. We see that \( Q \) involves a \( F \)-module with offender \( Y_M/Y_M \cap Q \). Now 3.4 shows \( K \cong L_t(q), Sp(2t,q), \Omega^\pm(2n,q), U_t(q) \), or \( A_t \). Then we see because of the structure of our parabolic \( \tilde{L} \), that we have \( K \cong L_t(q) \) or \( n = m = 3, q = 2 \) and \( K \cong \Omega^+(2t,2) \), or \( A_t \), further \( M_0 S/O_2(M_0 S) \cong L_3(2) \wr Z_2 \) (recall that \( \Sigma_3 \wr Z_2 \) contains two \( \Omega^+(4,2) \), one which induces a tensor product and the other a direct sum).

Assume first \( K \cong L_t(q) \). Suppose \( [\Omega_1(Z(S)), K] \neq 1 \). Then by 3.4 we have \( \langle \Omega_1(Z(S)) \rangle^K \) is the \( V(\lambda_2) \)-module. But on that module we have a tensor product for \( \tilde{L} \), which shows that we must have \( m = n = 3, q = 2 \), and further a diagram automorphism in \( \tilde{C} \), which now implies \( t = 4 \), so \( K \cong \Omega^+(6,2) \), a case we will handle later. So we may assume that \( [\Omega_1(Z(S)), K] = 1 \). Now in \( Q/\Omega_1(Z(S)) \) centralizes \( S \cap K \) a group of order at least \( q^2 \), which is in \( V_2/V_1 \). By [smith] we see that there are two nontrivial irreducible modules involved, which then by 3.4 have to be the natural module and its dual. Choose \( P \) a minimal parabolic of \( K \), which is not in \( M_0 \). Then we see that it acts trivially on \( C_{Q/\Omega_1(Z(S))}(S) \). As \( P_{01} \) and \( P_{02} \) both act on this group nontrivially we see that \( \langle P_0, P \rangle = P_0 P \). Let \( T = N_S(P_{01}) \). Assume that \( T \) is a Sylow 2-subgroup of \( TK \). Then we see that \( \langle K, P_{01} \rangle \) has a diagram of \( L_{t+1}(q) \). But the order of \( T \) and \( [Tits] \) show that then \( O_2(\langle K, P_{01} \rangle) \neq 1 \). But any other minimal parabolic in \( \tilde{C} \), which has \( T \) as a Sylow 2-subgroup, normalizes this group, so we see that \( E \) is contained in that group. Now the same applies for \( \langle K, P_{02} \rangle \), which contradicts \( E \)-uniqueness. So we have that \( T \) is not a Sylow 2-subgroup of \( KT \), in particular \( S \neq T \). This gives \( E(M_0/O_2(M_0)) = L_3(2) \times L_3(2) \), a case we will handle just now.
So let us assume \( q = 2, \ n = m = 3 \) and \( K \cong L_6(2), \Omega^+(2t, 2), \) or \( A_t. \) In all cases a diagram automorphism is involved. As we should not have a tensor product in \( K \) involved, we cannot have \( L_t(2). \) Further as we must have a nontrivial irreducible \( \hat{L} \)-module, we see that also \( \Omega^+(2t, 2) \) is not possible. So we are left with \( A_t. \) Then again because of \( F \)-module we see that exactly one irreducible nontrivial module, the natural one is involved in \( Q. \) As \( [Q, Y_M \cap Q] \neq 1, \) we see that \( [Z(S), K] = 1. \) The action of \( Y_M \) on \( Q \) shows that \( |Q : C_{\overline{Q}/Z(S)}(t)| = 4 \) for all \( t \in Y_M \setminus Q. \) But there is no group of order 16 with these property on the natural module.

So we have to handle the case that \( n = m = 3, \ q = 2 \) and \( \hat{L} \) is in two components \( K_1 \times K_2. \) Now \( \hat{L} \cap K_1 \times K_2 \) induces a tensor product in \( Q, \) which shows that the same is true for \( K_1 \times K_2. \) As \( Y_M \cap K_1 \) centralizes a subgroup of index 4 in \( Q, \) we see that \( K_1 \) induces at most two nontrivial modules, on which \( K_2 \) acts nontrivially, a contradiction.

So let now \( n = 2, \ m \geq 3. \) Then we see that \( M_0 \cap \overline{C}/Q \) is an extension of the natural module with \( L_{m-1}(q). \) Further we have a quadratic foursgroup. Hence by [MeiStr1] and [MeiStr2] we see that \( K = 3U_4(3), A_t, \) some sporadic group or a group of Lie type in characteristic two. Suppose we do not have \( A_t \) or a Lie type group in characteristic two. As for \( A = Y_M/Y_M \cap Q \) we have \( |Q : C_Q(\mathfrak{a})| \leq |\mathfrak{a}|q \) and \( q \leq 4 \) in that cases, we see with 3.9, 3.10 and the fact that \( A \) acts quadratically that \( K \cong M_{22}, \) or \( M_{24} \) and we have the natural module. By [MeiStr2] we have that \( |\mathfrak{a}| = 4, \) so \( |Q : C_Q(\mathfrak{a})| \leq 4 \) for all \( \mathfrak{a} \in A, \) a contradiction on the natural module.

Let next \( K = A_t. \) Then by [MeiStr2] we have that \( Q \) involves just natural or spin modules. In the latter \( |A| = 4 \) and so \( q = 2. \) But then \( |Q : C_Q(\mathfrak{a})| \leq 4 \) for \( \mathfrak{a} \in Q, \) and then \( t = 7. \) But we have that \( [[Q, M_0 \cap \overline{C}], A] \) is centralized by \( O^2(M_0 \cap \overline{C}), \) which is not the case for the 4–dimensional module for \( A_7. \) So we have that just permutation modules are involved. Now \( q \leq 4. \) If \( t > 12, \) then we see that \( a \in A \) has at most 4 cycles. But there is no such group of order 16 on which \( L_2(4) \) acts. So we have \( t \leq 12 \) for \( q = 4. \) Now as \( A \) is normal in a Sylow 2–subgroup of \( K, \) we see that there is no factor \( L_2(4) \) in \( K \cap M_0. \) This shows \( q = 2. \) If \( t \neq 8, \) we see that \( a \in A \) has at most two cycles. Then \( A \) corresponds to \( (12)(34), (12)(56). \) But now on the permutation module \( V \) we see that the element in \( [V, A] \) which is centralized by \( N_K(\mathfrak{a}) \) is not a commutator, which contradicts the action of \( Q \) on \( Y_M. \) So we get \( t = 8, \) but then we have a Lie type group.

So assume finally that \( K \) is a group of Lie type in characteristic two. As \( Z(S) \leq Q', \) and \( Y_M \) centralizes \( Q', \) we see that \( [K, \Omega_1(Z(S))] = 1. \) Then as before we see that two nontrivial modules are involved in \( Q. \) Hence we
get that $Y_M$ induces an $F$-module and so it induces transvections on this modules. Hence we get $K = L_t(q)$. Let $P_{01}$ be the minimal parabolic such that $P_{01}$ projects onto some component of $M_0/O_2(M_0)$. Then we see that $P_{01}$ commutes with the point stabilizer in $K$. By maximality we may assume that this point stabilizer is in $M_0 S$. Hence we have $L_2(q) \times L_t(q)$ normal in $M_0/O_2(M_0)$. Let $P_1$ be the parabolic in $K$ not in $M_0$. Suppose that $P_1$ permutes with $P_{02}$. Then by $E$–uniqueness we have that $O_2(\langle K, P_{02} \rangle) = 1$. But now we see that this group has a diagram of type

\[
\begin{array}{cccccccc}
\circ & \circ & \cdots & \circ & \circ \\
\circ & \circ & \cdots & \circ & \circ \\
& & & & \\
\end{array}
\]

or

\[
\begin{array}{cccccccc}
\circ & \circ & \cdots & \circ & \circ \\
\circ & \circ & \cdots & \circ & \circ & \cdots & \circ & \circ \\
& & & & \\
\end{array}
\]

So by [Tits] this group either is $L_{t+1}(q)$ or $\Omega^+(2t, q)$ extended by some field automorphisms. In both cases we see that this group is not compatible with the structure of $K$ acting on $Q$. So we have that $\langle P_1, P_{02} \rangle \neq P_1 P_{02}$. As $P_{01}$ cannot commute with $K$, we see that also $\langle P_{01}, P_1 \rangle \neq P_{01} P_1$. Set $U = \langle P_0, P_1 \rangle$. If $O_2(U) \neq 1$, then with 5.16 we have the structure of $U$ and $Y_U$. As $P_1$ fixes a point in $Y_U$, we now see that this has to be the $V(\lambda_2)$–module and so $E(U/O_2(U)) \cong \Omega^+(6, q)$ or $Sp(6, q)$. Now we see that $\langle K, P_{01} \rangle$ has a diagram of type

\[
\begin{array}{cccccccc}
\circ & \circ & \cdots & \circ & \circ \\
\circ & \circ & \cdots & \circ & \circ & \cdots & \circ & \circ \\
& & & & \\
\end{array}
\]

or

\[
\begin{array}{cccccccc}
\circ & \circ & \cdots & \circ & \circ \\
\circ & \circ & \cdots & \circ & \circ & \cdots & \circ & \circ \\
& & & & \\
\end{array}
\]

By [Tits] we get again that this group is $L_{t+1}(q)$ or $Sp(2t, q)$ extended by field automorphisms. Again the action of $K$ on $Q$ gives a contradiction.

So we are left with $n = m = 2$. Further by 2.7 we have that $S$ normalizes both components. Let $P_1$ be a minimal parabolic of $\tilde{C}$ with $U = \langle M_0, P_1 \rangle \neq P_1 M_0$. Then by the maximality of $M_0$, we have that $O_2(U) = 1$. In particular neither $P_{01}$ nor $P_{02}$ commutes with $P_1$. Set $U_i = \langle P_{0i}, P_1 \rangle$, $i = 1, 2$. Then we have that $E(U_i/O_2(U_i))$ is a rank two Lie group, $3A_6$, $M_{22}$ or $3M_{22}$. As $M_0$ does not induce an $F$–module on $Y_M$, we get with the amalgam method, that both $U_i$ have to induce $F$–modules. Hence we have that $M_{22}$ and $3M_{22}$ are not possible by 3.4. Assume that
\[ E(U_1/O_2(U_1)) \cong 3A_6 \text{ and } Z_1 = \langle \Omega_1(Z(S))^{U_1} \rangle \text{ is the 6–dimensional module. Then } q = 2 \text{ and if } Z_1 \not\leq O_2(M_0S), \text{ then there are elements in } Y_M \text{ inducing transvections on } Z_1, \text{ a contradiction. So } [Y_M, Z_1] = 1 \text{ and then } Y_M \leq O_2(U_1). \text{ Set } V_1 = \langle Y_M^{U_1} \rangle. \text{ Then } V_1 \text{ is elementary abelian. We consider the coset graph for } U_1 \text{ and } M_0S. \text{ Let } (1, \alpha) \text{ be a critical pair. Then we see that } [V_1, Z_0] = [Z_1, Z_0]. \text{ But then } [V_1, U_1] \leq Z_1 \text{ and so } Z_1 Y_M \not\leq U_1. \text{ But the element of order three in } M_0 \cap U_1 \acts \text{ fixed point freely on } Y_M, \text{ so } Y_M \leq Z_1. \text{ But then } \langle Q^N U_1 \rangle \text{ induces } \Sigma_3 \times \Sigma_3 \text{ on } Y_M, \text{ which shows that } M_0 \leq U_1, \text{ a contradiction.}

So assume next that \( E(U_1/O_2(U_1)) \cong G_2(q) \). As before \( Y_M \leq O_2(U_1) \). As an offender on the natural module has order \( q^3 \), we get again that \( Z_1 Y_M \leq U_1 \). But some element of order \( q+1 \) in \( M_0 \cap C \) acts fixed point freely on \( Y_M \), so \( Y_M \leq Z_1 \) again. Then \( O_2(U_1) \leq O_2(M_0S) \). This implies that we have a parabolic in \( G_2(q) \) which possesses a factor group \( q \times L_2(q) \). This implies \( q = 2 \). But then we see that \( C_{Z_1}(O_2(M_0)) \) is of order 4, a contradiction.

So assume now that \( E(U_1/O_2(U_1)) \cong U_3(q), U_5(q) \) or \( Sp(4, q) \) and also \( E(U_2/O_2(U_2)) \) is of that type. We may look at the amalgam \( (U_1, U_2) \). Then we see that \( Z_1 \) is an \( F \)-module with offender in the largest normal 2–subgroup of the point stabilizer. Now 3.4 and 3.5 show that \( U_1 \) is of type \( Sp(4, q) \). In particular, as now \( M_0/O_2(M_0) \) has the normal subgroup \( L_2(q) \times L_2(q) \), we see that also \( U_2 \) is of type \( L_2(q) \). Again we consider the amalgam \( (U_1, M_0S) \). Let \( (1, \alpha) \) be a critical pair. Suppose first \( \alpha \) of type 1. Then \( Y_M \leq O_2(U_1) \) and so \( V_1 = \langle Y_M^{U_1} \rangle \) is elementary abelian. If \( [Z_0, V_1] = [Z_0, Z_1] \), we get again \( Y_M \leq Z_1 \), but then as \( Z_1 \) is an \( F \)-module for \( U_1 \) we see with 3.4 that \( O_2(M_0S) = O_2(U_1) \), a contradiction. So \( [Z_0, V_1] \neq [Z_0, Z_1] \). Hence \( Z_1 \) and vice versa \( Z_0 \) induces transvections on \( Z_0, Z_1 \), respectively. Now \( [C_{Z_0}(Z_1), V_1] \neq 1 \). We have \([O_2(U_1), Y_M] \leq Z_1\), so we see that \( [C_{Z_0}(Z_1), V_1] = q^2 \) and so \( [C_{Z_0}(Z_1), V_1] = Z_1 \cap Y_M \). But then we see that \( [Z_0, V_1] \leq Z_1 \), a contradiction again. This shows that \( \alpha \) is of type \( M_0 \). Then by quadratic action \( Z_0 \) induces a transvection group on \( Z_1 \). In particular \( |Z_0 : Z_0 \cap O_2(U_1)| = q \). Now \( Z_1 \) induces a transvection group on \( Z_0 \), which contradicts the fact that \( Z_0 \) is the tensor product module.

So we are left with the case of a Dynkin diagram. Then by [Tits] we have that \( U \) is an extension of a rank three Lie group by field automorphisms. Now as \( M_0 \) induces a tensor product module, we immediately see that \( E(U) \cong L_4(q) \). Now this implies that \( |Q| \leq q^5 \). Suppose \( |Q| = q^5 \). Further we see that \( N_U(Z(Q))/Q = L_2(q) \times Z_{q-1} \times Z_{q-1} \) extended by field automorphisms. This shows that \( C_G(Z(Q)) \leq U \) or \( q = 2 \). As \( \tilde{C} = N_G(Z(Q)) \), we see by \( E \)-uniqueness, that it cannot be contained in \( U \), as it would be
contained in some 2-local of $U$. So we have $q = 2$ and $\bar{C}/Q$ is of order 18. But now $\bar{C}$ is generated by minimal parabolics, which have fixed point on $Q/Z(Q)$. This means that these commute with $M_0$, but then $M_0$ commutes with $C$, a contradiction. So we have that $|Q| < q^5$ and so $|Q| = q^3$, elementary abelian. Now there is some minimal parabolic $P_2$ in $\bar{C}$ which does not normalize $Z(O_2(P_1))$. Hence we get that $\langle M_0 S, P_1 \rangle = M_0 P_1$. But then $Y_M \leq O_2(P_1)$, and then $Y_M = O_2(\langle M_0, P_1 \rangle)$, and then $C_{\langle M_0, P_1 \rangle}(Y_M) \not\leq Y_M$, a contradiction.

So we now have $b(M) \geq 2$. Further we may assume that $M_0/O_2(M_0) \cong O^+(2n,q)$ and $Y_M$ is the natural module.

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