A characterization of $\text{Aut}(G_2(3))$

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Let $p$ be prime and $G$ a finite group. We say that $G$ has characteristic $p$ if $C_G(O_p(G)) \leq O_p(G)$ and that $G$ has local characteristic $p$ if all $p$-local subgroups of $G$ have characteristic $p$. $G$ is a $K_p$–group, if any simple section of any $p$-local subgroup of $G$ is a known finite simple group, that is an abelian, an alternating group, a group of Lie type or one of the 26 sporadic groups. This paper is part of a program to investigate $K_p$–groups of local characteristic $p$. See [MeStStr1] for an overview.

Of fundamental importance to theory of groups of local characteristic $p$ are large subgroups: A $p$–subgroup of a group $G$ is called large if

(i) $C_G(Q) \leq Q$ and

(ii) $N_G(U) \leq N_G(Q)$ for all $1 \neq U \leq C_G(Q)$.

For example, if $G$ is simple group of Lie-type in characteristic $p$, $S \in \text{Syl}_p(G)$ and $Q = O_p(C_G(Z(S)))$, then $Q$ is almost always a large subgroup of $G$. Indeed this is true exactly when $Z(S)$ is a root group, that is if $G$ is neither $Sp_{2n}(2^k)$, $n \geq 2$, $F_4(2^k)$ nor $G_2(3^k)$.

If $Q$ is a large subgroup of $G$, then it easy to see that also $O_p(N_G(Q))$ is a large subgroup of $G$. For a finite group $L$ let $Y_L$ be the unique maximal elementary abelian normal $p$–subgroup of $L$ with $O_p(L/C_L(Y_L)) = 1$. Such a group exists (see for example [MeStStr1, Lemma 2.0.1(a)]).

Let $G$ be a finite $K_p$-group of local characteristic $p$, $S$ a Sylow $p$-subgroup of $G$ and $Q$ a large $p$-subgroup of $G$ with $Q \leq S$ and $Q = O_p(N_G(Q))$. Let $M$ be a $p$-local subgroup of $G$ with $S \leq M$ and $Q \not\leq M$. The Structure Theorem (see [MeStStr2]) determines the pair $(M/C_M(Y_M), Y_M)$. The proof of the Structure Theorem is subdivided into the cases $Y_M \leq Q$ and $Y_M \not\leq Q$. Put $M^\circ = (Q^M)$, $\overline{M} = M/C_M(Y_M)$ and $V = [Y_M, M^\circ]$. For the case that $Y_M \not\leq Q$ the Structure Theorem asserts that one of the following holds:

1. [a] There exists a normal subgroup $K$ of $\overline{M}$ such that $K = K_1 \circ K_2$ with $K_1 \cong SL_{m_i}(q)$, $Y_M \cong V_1 \otimes V_2$, where $V_i$ is a natural module for $K_i$, and $M^\circ$ is one of $K_1, K_2$ or $K_1 \circ K_2$. 

1
2. \( \mathcal{M}^0, p, V \) is as in the following table:

<table>
<thead>
<tr>
<th>( \mathcal{M}^0 )</th>
<th>( p )</th>
<th>( V )</th>
<th>( \mathcal{M}^0 )</th>
<th>( p )</th>
<th>( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \text{SL}_n(q) )</td>
<td>( p )</td>
<td>( V_{\text{nat}} )</td>
<td>( \text{O}^+_3(2) )</td>
<td>2</td>
<td>( V_{\text{nat}} )</td>
</tr>
<tr>
<td>( \text{SL}_n(q) )</td>
<td>( p )</td>
<td>( \wedge^2(V_{\text{nat}}) )</td>
<td>( \Omega^-_{10}(q) )</td>
<td>2</td>
<td>( \text{halfspin} )</td>
</tr>
<tr>
<td>( \text{SL}_n(q^2) )</td>
<td>( p )</td>
<td>( S^2(V_{\text{nat}}) )</td>
<td>( E_6(q) )</td>
<td>( p )</td>
<td>( q^{27} )</td>
</tr>
<tr>
<td>( 3\text{Alt}(6), 3\text{Sym}(6) )</td>
<td>2</td>
<td>( 2^6 )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \Gamma\text{SL}_2(4), \Gamma\text{GL}_2(4) )</td>
<td>2</td>
<td>( V_{\text{nat}} )</td>
<td>( M_{12} )</td>
<td>3</td>
<td>( 3^6 )</td>
</tr>
<tr>
<td>( \text{Sp}_{2n}(q) )</td>
<td>2</td>
<td>( V_{\text{nat}} )</td>
<td>( M_{24} )</td>
<td>2</td>
<td>( 2^{11} )</td>
</tr>
<tr>
<td>( \Omega_n^+(q) )</td>
<td>( p )</td>
<td>( V_{\text{nat}} )</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Here \( q \) is a power of \( p \) and \( V_{\text{nat}} \) denotes the natural module of a classical group.

A priori there is no reason why one could not have that \( Y_M \not\leq Q \) and \( [Y_M, M^0] \leq Q \). Indeed this does happen, but a corollary in [MeStStr2] states that its only possible if \( M/C_M(Y_M) \cong \text{SL}_3(2) \) and \( [Y_M, M] \) is a natural \( \text{SL}_3(2) \)-module for \( M \); and

\( Y_M \not\leq Q \) and \( [Y_M, M] \leq Q \).

Then \( G \) is isomorphic to \( \text{Aut}(G_2(3)) \).

We remark that the proof of this theorem is independent from the Structure Theorem. In a forthcoming paper we will determine the structure of \( G \) in the remaining cases for \( Y_M \not\leq Q \) in the Structure Theorem.

1 Preliminaries

In this section we collect some results on modules for quasisimple groups, which will be needed in the proof of the theorem.

As the three dimensional module for \( \text{SL}_3(2) \) will play a prominent role, we start with collecting some facts about this module:

**Lemma 1.1. [132]** Let \( M = \text{SL}_3(2) \) and \( V \) a corresponding natural \( \mathbb{F}_2 M \)-module. Let \( W_1 \) be the transvection group in \( M \) to a point in \( V \) and \( W_2 \) the transvection group to a hyperplane in \( V \).
Lemma 1.2. 

(a) Let \( \tau_i \) be an element of order 3 in \( G \) normalizing \( W_i, i = 1, 2 \), then \( [[W_1, V], \tau_1] = 1 \), while \( [[W_2, V], \tau_2] = [W_2, V] \).

(b) Let \( V_1 \) be a \( \mathbb{F}_2M \)-module with \( [V_1, M] = V \), \( C_{V_1}(M) = 0 \) and \( V_1 \neq V \). Then \( |V_1/V| = 2 \), \( V = [V_1, W_1] \) and \( [V_1, W_2] = [V, W_2] \). In particular, \( W_1 \) does not act quadratically on \( V_1 \).

(c) Let \( V_1 \) be as in (b) and \( v \in V_1 \setminus V \). Then \( |C_M(v)| = 21 \) and \( M \) acts transitively on \( V_1 \setminus V \).

Proof. (a) is clear. To prove (b) let \( t \in W_2^2 \). Then \( [V_1, t] \leq C_V(t) = C_V(W_2) = [V, W_2] \) and so \( [V_1, W_2] = [V, W_2] \). Put \( V_2 = V + C_{V_1}(t) \) and note that \( V_2 \) is a \( \mathbb{F}_2M \)-submodule of \( V_1 \) and \( |V_2/C_{V_1}(t)| = 2 \). Let \( U = N_M(W_2) \). So \( U \cong \text{Sym}(4) \) and we may generate \( U \) by three conjugates of \( t \). Hence \( |V_2/C_{V_2}(U)| \leq 2^4 \). Since \( C_V(U) = 0 \), we get \( V_2 = V \oplus C_{V_2}(U) \). Gaschütz' Theorem [Hu, (I.17.4)] now shows that \( V_2 \) splits over \( V \). Since \( C_V(M) = 0 \) we conclude that \( V = V_2 \) and \( C_{V_1}(t) \leq V \). From \( [V_2, t] \leq C_V(t) \) we have \( |V_1/C_{V_1}(t)| = |[V_2, t]| \leq 4 \). Since \( V_1 \neq V \) this implies that \( |V_1/V| = 2 \) and \( [V_1, t] = C_V(t) \). Note that \( V = \langle C_V(t) \mid t \in W_1^4 \rangle \) and so \( V = [V, W_1] \). Thus (b) holds.

Let \( v \in V_1 \setminus V \). Since \( C_{V_1}(t) \leq V \), \( C_M(v) \) has odd order. Thus \( 8 \leq |M/C_M(v)| = |v^M| \leq |v + V| = 8 \) and (c) holds. \( \square \)

A finite group is a \( \mathcal{CK} \)-group if all of its composition factors are known finite simple groups.

Lemma 1.2. [kleinlie] Let \( H \) be a finite \( \mathcal{CK} \)-group, \( V \) a faithful \( \mathbb{F}_2H \)-module and \( x \) a 2-central involution in \( H \). Put \( L = F^*H \). Suppose that

(i) \( L \) is quasisimple and \( V \) is a simple \( \mathbb{F}_2L \)-module; and

(ii) \( H = L \langle x \rangle, \langle [V, x] \rangle \leq 4 \) and \( x \) is contained in a quadratic fours group of \( H \) on \( V \).

Then one of the following holds:

1. \( H \cong SL_n(2), SL_n(4), Sp_{2n}(2) Sp_{2n}(4), SU_n(2), G_2(2)' \) or \( \Omega_{2n}^+(2) \) and \( V \) is a corresponding natural module.

2. \( H \cong Sp(6, 2), V \) is the spin module and \( x \) is a short root element.

3. \( H \cong 3\text{Alt}(6) \) and \( V \) is the 6-dimensional module.

4. \( L \cong \text{Alt}(n) \) and \( V \) is the permutation module. Moreover, either \( H \cong \text{Sym}(n) \) and \( x \) is 2-cycle or \( H \cong \text{Alt}(n) \) and \( x \) is a double 2-cycle.

5. \( H \cong \text{Alt}(7) \) and \( V \) is the four dimensional module.
Proof. Suppose first that $L/Z(L)$ is a group of Lie type in characteristic 2. Since $O_p(L) = 1$ we conclude from [Gr] that $L$ itself is a group of Lie-type. Since $x$ is 2-central we have $x \in L$ and so $H = L$. Then by [PaRo, 14.25] either (1) holds or $L \cong Sp_6(2)$ and $V$ is the spin module.

Consider the latter case and let $S$ be a Sylow 2-subgroup of $L$ with $x \in Z(S)$. Let $W$ be the natural module for $L$. Then $[Z(S), W]$ is 2-dimensional and singular. So there exists $u \in W$ such that $\langle [W, Z(S)], u \rangle$ is a 3-dimensional singular space. Denote by $y$ the transvection to $u$. Then we have that $C_L(y)$ acts irreducibly on $V_y = C_V(O_2(C_G(y)))$ by [Sm1]. So $V_y$ is the natural module for $C_L(y)/O_2(C_L(y)) \cong Sp_4(2)$. As $[V, y] \cap V_y \not\subseteq 1$, we see that $V_y \leq [V, y] \leq C_V(y)$. In particular, $V/C_V(y)$ involves a natural module isomorphic to $V_y$. Further this natural modules is not isomorphic to $O_2(C_L(y))/\langle y \rangle$ as $C_L(y)$-module. By the choice of $y$, we have that $Z(S) \cap O_2(C_L(y)) = 1$ and $Z(S)/O_2(C_L(y))/O_2(C_L(y)) = Z(S)/O_2(C_L(y))$. Since $|[V, y]| = 4$, $x$ has to induce a transvection on $V_y$ and so does not act as a transvection on $O_2(C_L(y))/\langle y \rangle$. Hence $x$ is a short root element in $C_L(y)/O_2(C_L(y))$ and then also in $L$. Thus (1) holds.

So we may assume from now on that $L/Z(L)$ is not a group of Lie-type in characteristic 2. Since $|[V, x]| \leq 4$, [PaRo, 15.3] shows that $L/Z(L)$ is not a sporadic group.

Suppose $L \cong Alt(6), 2F_4(2)'$ or $G_2(2)'$. Since $x$ is 2-central either $H = L$ or $H \cong Sp_4(2)$. In the first case we are done by [PaRo, 14.29] and in the second by [PaRo, 14.25].

Suppose now $L/Z(L) \cong Alt(n)$ but $Z(L) \neq 1$. Then by [Gr] $n = 6$ or 7 and $|Z(L)| = 3$. As $|V, x| \leq 4$ this forces $|Z(L), x| = 1$. Thus $x \in L$, $H = L$ and $H$ can be generated by three conjugates of $x$. Therefore $|V| \leq 64$ and so $n = 6$, the assertion (3).

Suppose next that $L \cong Alt(n)$, $n = 7$ or $n \geq 9$. If $V$ is the permutation module, then $|V, x| \leq 4$ implies that $x$ is a 2-cycle or a double 2-cycle and (4) holds.

If $V$ is not the permutation module, then since $M$ contains a quadratic fours-group on $V$, $V$ is the spin-module (see [MeiStr2]). In particular, the 3-cycles in $M$ act fixed-point freely on $V$. If $x$ is not a fixed-point free permutation, then $x$ inverts a three cycle $d$ and so $|V| = |[V, d]| \leq |[V, x]|^2 = 16$. Thus (5) holds. So suppose that $x$ is a fixed-point free permutation. Then $n$ is even, $n \geq 10$ and $x$ inverts a double 3-cycle. Since a 3-cycle is the product of two double 3-cycles we conclude that $|V| \leq |[V, x]|^4 = 2^8$, a contradiction to $n \geq 10$.

Suppose finally that $L/Z(L)$ is a group of Lie-type in odd characteristic. Since $M$ contains a quadratic fours group, [MeiStr1] show that $L \cong 3.U_4(3)$. Since $x$ is 2-central, $x \in L$ and since $L$ has a unique conjugacy class of involutions, we see that $x$ is contained subgroup $K$ of $L$ with $K \cong 3.Alt(6)$. Let $U$ be any composition factor for $K$ on $V$. Since $Z(K) \leq Z(L)$, $U$ is a faithful $K$-module. By the 3.Alt(6)-case, $|U| = 2^6$ and since $[U, x]$ is $Z(K)$-invariant, $|[U, x]| \geq 4 = |[V, x]|$. Thus $U$ is the only composition factor for $K$ on $V$ and $|V| = 2^6$, a contradiction, since $3^7$ divides $|L|$ but not $|GL_6(2)|$. \qed

Lemma 1.3. [char irr] Let $H$ be a group, $F$ a field, $W$ an $FH$-module and $A \trianglelefteq B \trianglelefteq H$. Suppose that there exist a simple $FB$-submodule $Y$ of $W$ with $[W, A] \leq Y$ and $W = Y^H$. Then every proper $FH$-submodule of $W$ is centralized by $\langle A^H \rangle$. In particular, $W/C_W(\langle A^H \rangle)$
is a simple $\mathbb{F}H$-module.

Proof. Let $U$ be a submodule of $\mathbb{F}H$-module of $W$ with $U \neq W$. Since $W = \langle Y^H \rangle$ we have $Y \not\subseteq U$. Hence $[U, A] \leq Y$ and since $Y$ is a simple $B$-module, $[U, A] = 1$. Thus also $[U, \langle A^H \rangle] = 1$. \qed

2 Proof of the Theorem

In this section we prove Theorem 1. So let $G, M, S$ and $Q$ be as there. We set $\hat{C} = \mathbb{G}_Q(Q)$, $V = [Y_M, M], \hat{M} = \mathbb{G}_Q(V), M^0 = \langle Q^M \rangle$, $Z = \mathbb{G}_1Z(S)$ and $Q_M = O_2(M)$.

Let $L$ be minimal in $\hat{C}$ such that $L$ is $M \cap \hat{C}$-invariant and $Y_M \not\subseteq O_2(LY_M)$. Set $W = \langle V^L \rangle$, $B = (M \cap \hat{C})(L \cap \hat{M})$, $M_1 = MB$, $M_2 = LB$, $Q_1 = O_2(M_1)$, $H = LY_M$ and $T = O^2(M \cap \hat{C})$. Note here that $M \leq M_1 \leq \hat{M}$, $L \leq M_2 \leq \hat{C}$ and $B \leq M_1 \cap M_2$. For $X \leq M_2$ put $\hat{X} = XQ_2/Q_2$ and for $X \leq W$ put $\hat{X} = XZ(W)/Z(W)$.

Lemma 2.1. [M]

(a) [f] $C_{G}(M^0) = 1$, $Z(M) = 1$ and $Y_M = \Omega_1Z(Q_M)$.

(b) [a] $|Z| = 2$, $M \cap \hat{C} = C_M(Z)$, $QQ_M = O_2(M \cap \hat{C})$, $M = M^0Q_M$ and $[Y_M, Q] = V$.

(c) [b] $\hat{M} = M^0C_G(V)$ and $[M^0, C_G(V)] \leq O_2(M^0) \leq O_2(\hat{M}) \leq Q_M$.

(d) [g] $M_1 = M^0B = M^0(L \cap \hat{M})$ and $M_1$ is a subgroup of $\hat{M}$.

(e) [c] $Y_M \leq \hat{M}$ and $C_G(V) = C_G(Y_M)$.

(f) [d] $O_2(M^0) = M^0 \cap Q_1$, $B = (M^0 \cap B)C_B(V)$, $C_{M_1}(V) = C_B(V)$ and $M_1/Q_1 = M^0Q_1/Q_1 \times C_B(V)/Q_1$.

(g) [e] $O_2(B) = Q_1Q_2 = Q_1Q$.

Proof. (a) If $Q \leq Q_M$, then $Y_M \leq C_G(Q) \leq Q$, a contradiction to the assumptions. Thus $Q \not\subseteq Q_M$. Suppose $C_G(M^0) \neq 1$. Then since $Q$ is large, $M \leq C_G(C_{G}(M^0)) \leq C_G(Q) = \hat{C}$ and so $Q = O_2(\hat{C}) \leq O_2(M) = Q_M$, a contradiction. Hence $C_G(M^0) = 1$ and so also $Z(M) = 1$. Clearly $Y_M \leq \Omega_1Z(Q_M)$. Since $Q_M \leq \Omega_1Z(Q_M)$ and $M/Q_M(\cong S_{L_3}(2))$ is simple, $C_M(\Omega_1Z(Q_M)) = Q_M$ and so $O_2(M/C_M(\Omega_1Z(Q_M))) = 1$. The definition of $Y_M$ now implies that $Y_M = \Omega_1Z(Q_M)$.

(b) By Gaschütz' theorem, $Z \leq [Y_M, M]Z(M) = V$. Since $V$ is a natural $S_{L_3}(2)$-module for $M$ we get that $|Z| = |C_V(S)| = 2$. Since $Q \not\subseteq Q_M$ and $M/Q_M$ is simple, $M = M^0Q_M$. Since $Z \leq C_G(Q) \leq Q$ and $Q$ is large, $C_M(Z) \leq M \cap \hat{C}$. So $C_M(Z)$ normalizes $C_V(Q)$ and thus $C_V(Z) = Z$. Since $M \cap \hat{C}$ normalizes $C_V(Q)$ this implies $C_M(Z) = M \cap \hat{C}$. Thus $M \cap \hat{C}/Q_M \cong \text{Sym}(4)$ and since $Q_M/Q_M$ is a non-trivial normal 2-subgroup of $M \cap \hat{C}/Q_M$, $Q_M = O_2(M \cap \hat{C})$. Hence by 1.1(b), $[Y_M, Q] = V$.

(c) Since $M^0$ induces $\text{Aut}(V)$ on $V$, $\hat{M} = M^0C_G(V)$.
Since $Q$ is large, $C_G(V) \leq C_G(C_V(Q)) \leq N_G(Q)$ and thus $[Q,C_G(V)] \leq Q$. So $[Q,C_G(V)] \leq O_2(C_G(V)) \cap M^o \leq O_2(M^o)$. Conjugation under $M$ gives, $[M^o,C_G(V)] \leq O_2(M^o)$ and so (c) holds.

(d) By (b), $M = M^oQ_M$ and since $Q_M \leq B$, we have $M_1 = MB = M^oB$. As $B = (M^o \cap B)(L \cap M)$, $M_1 = M^o(L \cap \tilde{M})$. By (c) , $\tilde{M}$ normalizes $M^o$. Since $B \leq \tilde{M}$, we conclude that $M_1 = M^oB$ is a subgroup of $\tilde{M}$.

(e) Put $D := \langle Y_M^\tilde{M} \rangle$. Since $\tilde{M}$ normalizes both $M^o$ and $V$ we get $[D,M^o] = V$ and $[B, O_2(M^o)] = 1$. By (a) $C_D(M^o) = 1$. Since $[D,V] = 1$ we have $[D, M^o, D] = 1$ and the Three Subgroups Lemma implies $[D, D, M^o] = 1$ and $D' \leq C_D(M^o) = 1$. So $D$ is abelian and thus elementary abelian. Hence by 1.1, $|D/V| \leq 2$ and so $Y_M = D$. Hence $Y_M \leq \tilde{M}$. Since $|Y_M/V| = 2$ we get $[Y_M, \tilde{M}] \leq V$ and so $[Y_M, O^2(C_G(V))] = 1$. Since $Q_M = C_S(V) \in \text{Syl}_2(C_G(V))$ and $[Q_M, Y_M] = 1$, this gives $[Y_M, C_G(V)] = 1$ and so $C_G(V) = C_G(Y_M)$.

(f) Since $M^o \leq \tilde{M}$ and $M_1 \leq \tilde{M}$, $O_2(M^o) \leq M_1$. Also $Q_1 \cap M^o \leq M^o$ and so $O_2(M^o) = Q_1 \cap M^o$. Since $\tilde{M} = M^oC_G(V)$, $M_1 = M^oC_M_1(V)$. As $B$ normalizes $C_V(Q)$ we have $B \leq N_{M_1}(Z) = \langle M^o \rangle \cap C_M_1(V)$ and so $B = (M^o \cap B)C_B(V), M_1 = M^oC_B(V)$ and $C_M_1(V) = C_B(V)C_M(V) = C_B(V)$.

(g) Note that $O_2(C_M(V)) \leq Q_1$ and $O_2(M^o \cap B) = O_2(M^o)Q \leq Q_1Q$. Since $B/Q_1 = (M^o \cap B)Q_1/Q_1 \times C_B(V)/Q_1$, this implies $O_2(B) = Q_1Q$. Since $Q \leq Q_2 \leq O_2(B)$, we get $O_2(B) = Q_1Q_2$.

Lemma 2.2. [elem]

(a) [e] $L = O^2(L) = [L, Y_M]$ and $H = \langle Y_M^L \rangle = \langle Y_M^{L^2} \rangle$

(b) [f] $W \neq V$, $[W, L] \neq 1$ and $C_{Q_2}(L) = C_{Q_2}(H) = C_{Q_2}(W) \leq Q_1$.

(c) [b] $[Q_2', L] = 1$ and $[Q_2, L] \leq W$.

(d) [z] $WQ_1 = O_2(B)$, $[Y_M, W] = V$ and $V \cap Z(W) = Z$.

(e) [a] $[W, Q_2] = W' = Z = \Phi(W)$.

(f) [c] $[W, L] = W$ and $C_W(L) = Z(W)$.

(g) [d] $\hat{W}$ is a selfdual, simple $F_2M_2$–module and homogeneous $F_2H$–module.

Proof. By the minimal choice of $L$, $L = O^2(L)$ and $L = [L, Y_M]$. In particular, $\langle Y_M^L \rangle = Y_M[L, Y_M] = LY_M$. Together with 2.1(e) this is (a).

Suppose $W = V$. Then $L \leq N_G(V) = \tilde{M}$ and $Y_M \leq O_2(LY_M)$, a contradiction to the choice of $L$. If $[W, L] = 1$, then $W = \langle V^L \rangle = V$, a contradiction.

Thus $W \neq V$. Set $D = C_{Q_2}(L)$. Suppose $D \not\leq Q_1$. Since $B$ normalizes $D$ and acts simply on $O_2(B)/Q_1$ we get $DQ_1 = O_2(B)$ and so by 1.1(b), $V = [Y_M, D] \leq D$ and $[V, L] = 1$, a contradiction. Thus $D \leq Q_1$ and $D \leq C_{Q_2}(LY_M) = C_{Q_2}(\langle Y_M^L \rangle)$.

Since $C_{Q_2}(V) = Q_2 \cap Q_M = C_{Q_2}(Y_M)$ we have $C_{Q_2}(W) = C_{Q_2}(\langle V^{L^2} \rangle) = C_{Q_2}(\langle Y_M^{L^2} \rangle) \leq C_{Q_2}(L) \leq D$ and so (b) holds.
As \(Q_2 \leq Q_M\), we have that \([Q_2^2, Y_M] = 1\). Since \(L \leq \langle Y_M^2 \rangle\), we get \([L, Q_2^2] = 1\). Further as \([Y_M, Q_2] \leq V \leq W\), we also get \([Q_2, L] \leq W\), which is (c).

If \([W, V] = 1\), then \(W = Z(W)\). Thus (b) gives \(W \leq C_{Q_2}(W) = C_{Q_2}(L)\), a contradiction. Hence \([W, V] \neq 1\) and \(W \not\leq Q_1\). Since \(B\) normalizes \(WQ_1\) this gives \(WQ_1 = O_2(B)\) and so \([Y_M, W] = V\) and \(V \cap Z(W) = C_V(W) = Z\). Thus (d) holds. Moreover, \([W, V] = [Q_2, V] = Z\). By (c) \(Z\) is centralized by \(L\) and so since \(W = \langle V^{M_2} \rangle = \langle V^L \rangle\), \([W, W] = [W, Q_2] = Z\), which is (e).

By (b) \(Z(W) = C_W(L) = C_W(H)\) and since \(L = O^2(L)\), \(Z(W)/Z = C_{W/Z}(L) = C_{W/Z}(\langle Y_M^{M_2} \rangle)\). Since \(M \cap \hat{C}\) acts simply on \(V/Z\) we conclude from 1.3 that \(Z(W)/Z\) is the unique maximal \(M_2\)-submodule of \(W/Z\). If \([W, L] \leq Z(W)\), then \(W = \langle V^L \rangle = VZ(W)\) and \(W\) is abelian, a contradiction. Thus \([W, L] \not\leq Z(W)\) and so \(W = [W, L]Z\). By (e) \(Z \leq [W, L]\) and so \(W = [W, L]\). So (f) is proved and \(\hat{W}\) is a simple \(\mathbb{F}_2 M_2\)-module.

The commutator map \(\hat{W} \times \hat{W} \to Z, [xZ(W), yZ(W)] \to [x, y]\) is a non-degenerate bilinear form on \(\hat{W}\) and so \(\hat{W}\) is a selfdual \(\mathbb{F}_2 M_2\)-module. Suppose that \(\hat{W}\) is not homogeneous as an \(\mathbb{F}_2 H\)-module and let \(\hat{W}_i, 1 \leq i \leq n\), be the Wedderburn components of \(H\) in \(\hat{W}\). Then \(\hat{W} = \bigoplus_{i=1}^n \hat{W}_i\) and so \(\hat{V} = [\hat{W}, Y_M] = \bigoplus_{i=1}^n [\hat{W}_i, Y_M]\). It follows that the action of \(B\) on \(\hat{V}\) is imprimitive. But \(\hat{V} \cong V/V \cap Z(W) = V/Z\) as \(B\)-module and so \(|\hat{V}| = 4\) and \(B\) acts transitively on \(\hat{V}^\ast\), a contradiction. \(\square\)

**Lemma 2.3. [Wquad]**

(a) [a] \(W\) acts quadratically on \(Q_M/V\). In particular, any non-trivial composition factor for \(M\) on \(Q_M/V\) is a natural \(SL_3(2)\)-module.

(b) [b] \(N_{M_2}(Y_M Q_2) = N_{M_2}(V) = B\).

(c) [c] If \(g \in L\) with \([Y_M, Y_M^g] \leq Q_2\), then \([Y_M, Y_M^g] = 1\) and \(Y_M Y_M^g\) acts quadratically on \(Q_2\) and \(\hat{W}\).

(d) [d] \(C_{M_2}(\hat{W}) = Q_2\).

**Proof.** We have that \([Q_M, W, W] \leq [W, W] = Z \leq V\), by 2.2(c),(e). So \([Q_M/V, W, W] = 1\).

Since \(WQ_M/Q_M\) has order 4, \(W\) does not act quadratically on the Steinberg module. Since the only simple \(\mathbb{F}_2 SL_3(2)\) modules are the trivial module, the two natural modules and the Steinberg module, we have (a).

(b) Let \(g \in N_L(Y_M Q_2)\). Then \(g\) normalizes \([W, Y_M Q] = [W, Y_M] = V\) by 2.2(d).

(c) By (b) we have that \(Y_M^g \leq \tilde{M}\) and by symmetry \(Y_M \leq \tilde{M}^g\). Thus \(R := [Y_M, Y_M^g] \leq V \cap V^g\). Suppose that \(R \neq 1\). Then by 2.1(e), \([V, Y_M^g] \neq 1\). By 1.1 (applied to \(V_1 = Y_M\)) \(R\) is a fours group. Since \(R \leq V^g\) the action of \(\tilde{M}^g\) on \(V^g\) shows that there is \(1 \neq x \in R\) such that \(V \not\leq O_2(C_{\tilde{M}^g}(x))\). Note that \(x \in V\) and so \([x, Q_M^m] = 1\) for some \(m \in M\). Then \(V \leq Q^m\) and since \(Q^m\) is large, \(Q^m \leq O_2(C_G(x))\), a contradiction. So we have \(R = 1\). Hence \(Y_M Y_M^g\) is abelian and since \(Q_2\) normalizes \(Y_M Y_M^g\), \([Q, Y_M Y_M^g] \leq Y_M Y_M^g\) and \([Q, Y_M Y_M^g], Y_M Y_M^g] = 1\). This is (c).
(d) Since \( \hat{W} \) is a simple \( M_2 \)-module, \( Q_2 \leq C_{M_2}(\hat{W}) \). Let \( E := O^2(C_{M_2}(\hat{W})) \). Since \( L \not\leq E \), the minimality of \( L \) shows that \( [H \cap E, H] \leq Q_2 \). Hence \( H \cap E \leq Z(H) \) and so \( H \cap E \) has odd order and \( O_2((H \cap E)Y_M) = Y_M \). Since \( [E, H] \leq H \cap E \) we conclude that \( E \) normalizes \( (H \cap E)Y_M \) and \( Y_M \). (b) implies that \( E \leq B \). Thus \( [V, E] \leq V \cap Z(W) = Z \) since \( E \) normalizes \( E \). Let \( Y \) get that \( Y \) is a faithful and simple \( \hat{W} \)-submodule of \( \hat{W} \). Then \( \hat{W} \) is the only component of \( \hat{W} \) since \( \hat{W} \) is homogeneous and \( \hat{W} \) is a selfdual \( H \)-module and \( \hat{W} \) is the direct sum of at most two simple \( \hat{W} \)-submodules and all \( \hat{W} \)-submodules and all \( \hat{W} \)-submodules of \( \hat{W} \). Since \( \hat{W} \) is quasisimple, \( \hat{W} = F^*(\hat{H}) \), \( H \) normalizes \( \hat{W}_1 \), \( \hat{W}_1 \) is the spin-module and \( \hat{W}_1 \) is isomorphic to \( \Omega^+_{2n}(2) \), \( \hat{W}_1 \) is nonsolvable and let \( \hat{W}_1 \) be a simple \( \hat{W} \)-submodule of \( \hat{W} \) isomorphic to \( \Omega^+_{2n}(2) \), \( \hat{W}_1 \) is the unique minimal \( \hat{W} \)-submodule of \( \hat{W} \) isomorphic to \( \Omega^+_{2n}(2) \), \( \hat{W}_1 \) is nonsolvable and let \( \hat{W}_1 \) be a simple \( \hat{W} \)-submodule of \( \hat{W} \). Then \( \hat{W} \) is quasisimple, \( \hat{W} = F^*(\hat{H}) \), \( H \) normalizes \( \hat{W}_1 \), \( \hat{W}_1 \) is a selfdual \( H \)-module and \( \hat{W} = \hat{W}_1 \) or \( \hat{W} = \hat{W}_1 + \hat{W}_2 \) where \( \hat{W}_2 \) is a \( H \)-submodule of \( \hat{W} \) isomorphic to \( \hat{W}_1 \). Schur’s Lemma together with the fact that finite division ring are commutative shows that \( C_H(\hat{L}_1) \) centralizes \( U \). Since \( \hat{W} \) is a homogeneous \( H \)-module, this implies that \( C_H(\hat{L}_1) \) is abelian. Hence \( L_1 \) is the only component of \( \hat{W} \) and \( L_1 = L_1 \). Note that \( O_2(\hat{H}) \leq O_2(\hat{M}_2) = 1 \) and as \( \hat{H} / \hat{L} \) is a 2-group, \( F^*(\hat{H}) = \hat{L} \). Since \( \hat{W} \) is homogeneous and \( |\hat{W}Y_M| = |V| \leq 4 \), \( \hat{W} \) is the direct sum of at most two simple \( H \)-submodules and all parts of the lemma are proved. 

Let \( U \) be a simple \( H \)-submodule of \( \hat{W} \).

Lemma 2.6. [Xstruk] Suppose \( L \) is nonsolvable. Then one of the following holds:

1. \([i]\) \( \hat{H} \cong SL_n(2), SL_n(4), Sp_{2n}(2), Sp_{2n}(4), SU_n(2), \Omega^+_n(2) \) or \( G_2(2)' \) and \( U \) is corresponding natural module.

2. \([ii]\) \( \hat{H} \cong Sp_6(2) \), \( U \) is the spin-module and \( \overline{Y_M} \) is a short root subgroup of \( \hat{H} \).
3. [iii] $\mathcal{H} \cong \text{Sym}(n)$ or $\text{Alt}(n)$, $U$ is the natural permutation module and $\overline{Y_M}$ is generated by a 2-cycle or double 2-cycle.

4. [iv] $\mathcal{H} \cong \text{Alt}(7)$ and $U$ is a spin-module.

5. [v] $\mathcal{H} \cong 3\text{Alt}(6)$ and $U$ is the 6-dimensional module.

Proof. By 2.5 $F^*(\mathcal{H}) = \mathcal{L}$ and $F^*(\overline{\mathcal{H}})$ is quasisimple. Since $\hat{W}$ is a faithful, homogeneous $\mathcal{H}$-module, $C_{\mathcal{H}}(U) = 1$. Note that $|\overline{Y_M}| = 2$ and $\overline{Y_M} \leq Z(S \cap \mathcal{H})$. Thus Glauberman’s $Z^*$-Theorem implies that there exists $g \in L$ with $\overline{Y_M} \neq \overline{Y_M}^g$ and $[\overline{Y_M}, \overline{Y_M}^g] = 1$. By 2.3(c), $Y_M Y_M^g$ induces a quadratic fours group on $U$. Since $[U, Y_M] \leq \hat{V}$, $[U, Y_M]$ has order at most 4. Now the assertion follows with 1.2. □

Lemma 2.7. [Sln1] Suppose $\overline{L} \cong \text{Alt}(n)$, $n = 5$ or $n > 6$, then $U$ is not the natural permutation module for $\overline{L}$.

Proof. By 2.3 for any $g \in L$ with $[Y_M, Y_M^g] \leq Q_2$, we have that $Y_M Y_M^g$ induces a quadratic group on $\hat{W}$. By 2.6(3) $Y_M$ either corresponds to $12(34)$ or $12$. Since $\langle 12(34), (13)(24) \rangle$ does not act quadratically on $U$, we get that $\overline{Y_M}$ is conjugate to $\langle 12 \rangle$. Since $\overline{Y_M}$ is 2-central we get $n \neq 5$ and so $n > 6$. Note that $Q_1 L \leq LB = M_2$ and so $O_2(Q_1 L) = 1$ and $Q_1 L \cong \text{Sym}(n)$. Thus by 2.3(c), $\overline{B} \cap Q_1 L \cong C_2 \times \text{Sym}(n-2)$. Since $n - 2 > 4$ we have $O_2(\text{Sym}(n-2)) = 1$ and so $O_2(B \cap Q_1 L) = Y_M Q_2$. Hence $Q_1 \leq Y_M Q_2$ and $[Q_1, W] \leq [Y_M Q_2, W] \leq V$. By 2.4 $Q_M = Y_M$ and so $|S/Y_M Q_2| = |S/Q_M Q_2| = 2$, a contradiction to $(B \cap Q_1 L)/Q_2 Y_M \cong \text{Sym}(n-2)$. □

Lemma 2.8. [orth] Suppose $\overline{L} \cong \Omega_{2n}^+(2)$ or $S_{P_{2n}(2)}'$ and $U$ is the corresponding natural module. Then $\mathcal{H} \cong S_{P_{2n}(2)}$, $\hat{W}$ is the direct sum of two $H$-submodules isomorphic to $U$ and $Y_M$ induces a transvection on $U$.

Proof. Let $\overline{P}$ be the point stabilizer of $\overline{H}$ on the natural module with $S \cap L \leq P$. Then $\overline{Y_M} \leq O_2(\overline{P})$ and $O_2(\overline{P})$ is abelian. Hence $\langle \overline{Y_M}, \overline{P} \rangle$ is abelian and (by 2.3(c)) acts quadratically on $\hat{W}$ and on the natural module. The action of $\overline{P}$ on the natural module now shows that $\overline{H} \cong S_{P_{2n}(2)}$, $\overline{P}$ normalizes $Y_M$ and $Y_M$ induces a transvection on the natural module. □

Lemma 2.9. [Sln] $\overline{L}$ is none of $SL_n(2)$, $n \geq 3$, $SL_n(4)$, $n \geq 3$, $3 \cdot \text{Alt}(6)$ and $\text{Alt}(7)$.

Proof. Then by 2.5, $U$ is self-dual. Note that the natural modules for $SL_n(q)$, $n \geq 3$, is not selfdual, the 6-dimensional module for $3 \cdot \text{Alt}(6)$ is not selfdual and the 4-dimensional module for $\text{Alt}(7)$ is not self dual. Hence by 2.6 we conclude that $U$ is the orthogonal module for $\mathcal{H} \cong SL_4(2) \cong \Omega_6^+(2)$, but this contradicts 2.8. □

Lemma 2.10. [elem b]

(a) [a] Let $F \trianglelefteq B$ with $[V/Z, F] \neq 1$. Then $T \leq F$.

(b) [b] Suppose that $[V/Z, L \cap B] \neq 1$. Then $T \leq L \cap B$ and $M_2 = LS$. 9
Lemma 2.11. [mi] Suppose \( L \) is non-solvable. Then one of the following holds.

1. [a] \( M_1/Q_1 \cong SL_3(2) \times S_{p_{2n-2}}(2), B/O_2(B) \cong SL_2(2) \times S_{p_{2n-2}}(2), \overline{M}_2 \cong S_{p_{2n}}(2) \times SL_2(2) \) and \( \overline{W} \) is the tensor product of the corresponding natural modules.

2. [b] \( M_1/Q_1 \cong SL_3(2) \times SL_2(2), B/O_2(B) \cong SL_2(2) \times SL_2(2), \overline{M}_2 \cong SL(2) \sim SU_4(2) \) and \( \hat{W} \) is the corresponding natural module.

3. [c] \( M = M_1, B/O_2(B) \cong Sym(3), \overline{M}_2 \cong \Gamma GL_2(4) \sim (C_3 \times SL_2(4)).2 \) and \( \hat{W} \) is the corresponding natural module.

4. [d] \( M = M_1, B/O_2(B) \cong Sym(3), \overline{M}_2 \cong G_2(2) \) or \( G_2(2)' \) and \( \hat{W} \) is the corresponding natural module.

5. [e] \( M_1/Q_1 \cong SL_3(2), B/O_2(B) \cong SL_2(2) \times SL_2(2), \overline{M}_2 \cong Sp_6(2) \) and \( \hat{W} \) is the spin-module.

Proof. By 2.6-2.9 one of the following holds:

(a) [1] \( \overline{H} \cong S_{p_{2n}}(2), n \geq 4 \) and \( \hat{W} \) is the direct sum of two isomorphic natural modules and \( Y_M \) induces a transvection on these natural modules.

(b) [2] \( \overline{H} \cong SU_n(2), \hat{W} \) is a natural module and \( Y_M \) induces a \( F_4 \)-transvection on \( \hat{W} \).

(c) [3] \( \overline{H} \cong S_{p_{2n}}(4), \hat{W} \) is a natural module and \( Y_M \) induces a \( F_4 \)-transvection on \( \hat{W} \).

(d) [4] \( \overline{H} \cong G_2(2)', \hat{W} \) is the natural module and \( \overline{Y_M} \) is long root element.

(e) [5] \( \overline{H} \cong Sp_6(2), \hat{W} \) is the spin-module and \( \overline{Y_M} \) is a short root element.

Since by 2.3(b) \( \overline{H \cap B} = C_{\overline{H}}(\overline{Y_M}) \) this allows us to compute \( \overline{H \cap B} \). Also \( V/Z \cong [\hat{W}, Y_M] \) as a \( B \)-module and so this determines the action of \( H \cap B \) on \( V/Z \). Put \( D = C_{M_2}(\overline{H}) \). Note that \( D \leq N_{M_2}(\overline{Y_M}) = B \) and

\[
(*) \quad (M^o \cap B)O_2(B)/O_2(B) \text{ is a normal subgroup of } B/O_2(B) \text{ isomorphic to } SL_2(2).
\]

Suppose (a) holds. Then \( M_2 = DH \). Since \( M_2 \) acts simply on \( \hat{W} \), but \( H \) does not, we get \( \overline{D} \neq 1 \). Since \( W = \langle V^{M_2} \rangle \) we have \( [V/Z, D] \neq 1 \) and so by 2.10(a), \( T \leq D \). Now (*) implies that \( \overline{D} \cong Z(M_2) \) and so \( D \) is not abelian. Now \( C_{GL(V)}(\overline{H}) \cong SL_2(2) \) and thus \( \overline{D} \cong SL_2(2) \). Moreover, \( B \cap H/O_2(B \cap H) \cong S_{p_{2n-2}}(2) \) and we see that (1) holds in this case.
Suppose (b) holds. Then \( L \cap B/O_2(L \cap B) \cong C_3 \times SU_{2n-2}(2) \) and \( L \cap B/C_{L \cap B}(V/Z) \cong C_3 \). In particular, \( L \cap B \) acts non-trivially on \( V/Z \) and so by 2.10(b), \( M_2 = LS \). Then (*) shows that \( M_2 \neq \overline{L} \) and so \( \overline{M_2} \cong \Gamma SU_n(2) = SU_n(2)/\langle \sigma \rangle \), where \( \sigma \) induces a field automorphism of order 2. Thus \( B/O_2(B) \cong (C_3 \times SU_{n-2}(2))/\langle \sigma \rangle \) and (*) implies that \( n = 4 \) and \( B/O_2(B) \cong SL_2(2) \times SL_2(2) \). Thus (2) holds.

Suppose (c) holds. Then \( L \cap B/O_2(L \cap B) \cong Sp_{2n-2}(4) \) and \( L \cap B \) centralizes \( V/Z \). Thus \( T \not\leq L \) and since \( \text{Out}(\overline{H}) = 2 \) we get \( \overline{T} \neq 1 \). Hence by 2.2(d), \( T \leq D \). Since \( C_{GL(W)}(\overline{H}) \cong C_3 \) this gives \( \overline{T} = \overline{D} \cong C_3 \). Now (*) shows \( \overline{D} \not\subseteq Z(\overline{M_2}) \) and so \( \overline{M_2} \cong (C_3 \times Sp_{2n}(4))/\langle \sigma \rangle \), where \( \sigma \) induces a field automorphism of order 2. Thus \( B/O_2(B) \cong (C_3 \times Sp_{2n-2}(4))/\langle \sigma \rangle \) and (*) implies that \( n = 1 \) and \( B/O_2(B) \cong SL_2(2) \). Thus \( B = M \cap B \) and (3) holds.

Suppose that (d) holds. Then \( B \cap H/O_2(B \cap H) \cong SL_2(2) \) and \( B \cap L \) acts non-trivially on \( V/Z \). So 2.10(b) shows that \( M_2 = LS \) and \( T \leq L \cap B \). Therefore \( B = M \cap B \) and (4) holds.

Suppose that (e) holds. Then \( B \cap H/O_2(B \cap H) \cong SL_2(2) \) and \( B \cap L \) acts non-trivially on \( V/Z \). So 2.10(b) shows that \( T \leq L \cap B \) and \( M_2 = LS \). Since \( \text{Out}(\overline{H}) = 1 \), this gives \( \overline{M_2} = \overline{H} \), \( B/O_2(B) \cong SL_2(2) \times SL_2(2) \) and (5) holds.

\begin{proof}
Suppose first that \( C_{Q_2}(W) \neq Z \) and let \( D \leq M_2 \) be minimal with \( D \leq C_{Q_2}(W) \) and \( D \neq Z \). By 2.2, \([D, L] = 1 \) and \( D \leq Q_1 \). Since \( M_2 = (M \cap B)L \) and \( (M \cap B)/O_2(M \cap B) \cong SL_2(2) \) we get that either \([D, M_2] \leq Z \) and \([D/Z] = 2 \) or \( M_2/C_{M_2}(D/Z) \cong SL_2(2) \) and \([D/Z] = 4 \). In any case \([D, Q_1] \leq Z \) and \( \Phi(D) \leq Z \). Let \( g \in M_1 \setminus B \). Then \( Z \neq Z^g \).

We will now show that \( D \) is abelian. If \([D/Z] = 2 \) this is obvious. So suppose \([D/Z] = 4 \). Then \( C_{M \cap B}(D/Z) = O_2(M \cap B) \). Since \( W \cap B^g = C_W(Z^g) \) acts non-trivially on \( V/Z^g \), we have \( W \cap B^g \trianglelefteq O_2(M \cap B^g) \). Put \( R := [D^g, W \cap B^g] \). It follows that \( R \leq D^g \) and \( R \not\subseteq Z^g \). Since \( D^g \leq Q \leq N_G(W) \), \( R \leq W \). Thus by 2.1(b), \( \Phi(R) \leq Z \). On the other hand \( \Phi(R) \leq \Phi(D^g) \leq \Phi(W^g) = Z^g \). As \( Z \cap Z^g = 1 \), \( R \) is elementary abelian. Since \( B^g \) acts transitively on \( D^g/Z^g \) this implies that all non-trivial elements of \( D^g \) have order two.

Thus \( D \) is abelian. Note that \([D, D^g] \leq [D, Q_1] \cap [Q_1, D^g] \leq Z \cap Z^g = 1 \) and so \( E := (D^g)^\infty \) is abelian. Suppose that \([E, W] \leq V \). Since \( O^2(M) \leq \langle W^M \rangle \), we get \([E, O^2(M)] \leq V \). Since \( M_1 = O^2(M)B \) and \( B \) normalizes \( D \), \( E = (D^2)^\infty \) \( \leq DV \). Hence \( E = DV \), \([D, Q_1] \subseteq M \) and \( \Phi(D) \leq M \). Since \([D, Q_1] \leq Z \) and \( \Phi(D) \leq Z \) we conclude that \([D, Q_1] = 1 \), \( \Phi(D) = 1 \) and \( D \leq Y_M \). Thus \( D \leq Y_M \cap Q_2 = V \). Since \( B \) normalizes \( D \) and \( V \not\leq D \) this implies \( D = Z \), a contradiction.

Hence \([E, W] \leq V \) and so \( E \trianglelefteq Y_M Q_2 \) and \( \overline{Y_M} \leq \overline{EY_M} \). Since \( EY_M \) is abelian and \( W \) normalizes \( EY_M \), \( EY_M \) acts quadratically on \( \overline{W} \).

In all cases of 2.11 except (3) \( \overline{Y_M} \) is a maximal quadratic normal subgroup of \( \overline{B \cap M_2} = C_{\overline{P_1}}(\overline{Y_M}) \) on \( \overline{W} \). So \( \overline{M_2} \cong \Gamma GL_2(4) \). Note that \( S \cap H = Y_M Y^h_M Q_2 \) for some \( h \in M_2 \) and \([W, S \cap H] \leq [W, Y_M Y^h_M] = Y_M Y^h_M \). By 2.3(c), \( Y_M Y^h_M \) is elementary abelian and so also \([W, S \cap H] \) is elementary abelian. Since \( W = [W, H] \), Gashütz Theorem shows that \( Z(W)/Z = C_W(L) \leq [W/Z, S \cap H] \) and so \( Z(W) \leq [W, S \cap H] \). It follows that \( Z(W) \) is
elementary abelian. Since $H$ acts transitively on $\hat{W}^2$ this means that all non-trivial elements in $W$ are involutions. Thus $W$ is elementary abelian, a contradiction.

We have proved that $C_{Q_2}(W) = Z$. In particular, $Z(W) = Z$. Since $[W, Q_2] = Z$ we have $|Q_2/C_{Q_2}(W)| \leq |W|$ and so $Q_2 = WC_{Q_2}(W) = WZ = W$.

**Lemma 2.13.** $\mathcal{L} \not\cong G_2(2)'$ and $\mathcal{L} \not\cong SL_2(4)$.

*Proof.* Otherwise $\mathcal{L}$ acts transitively on $\hat{W}^2$. Since $Z(W) = Z$ and $V \leq W$ we conclude that all elements of $W^2$ have order two and $W$ is elementary abelian, a contradiction. □

**Lemma 2.14.** [e/v] Suppose $L$ is nonsolvable. Then

(a) [a] $M_1/Q_1 \cong SL_3(2) \times SL_2(2)$, $Q_1 = [Q_1, M_1]Y_M$, and $[Q_1, M_1]/V$ is a tensor product of natural modules.

(b) [b] $M_2/Q_2 \cong SL_2(2) \times Sp_4(2)$, $Q_2$ is extra special of order $2^9$ and $Q_2/Z$ is the tensor product of natural modules.

*Proof.* Put $E = ((W \cap Q_1)^{M_1})$. By 2.13 one of 2.11(1), (2) and (5) holds. Put $m = n - 1$ in the first case and $m = 1$ in the other two. Since $Z(W) = Z$ by 2.12 this implies that in all cases $W \cap Q_1 = [W, Q_1], B/O_2(B) \cong SL_2(2) \times Sp_2m(2)$ and $W \cap Q_1/V$ is the tensor product of natural modules for $B/O_2(B)$-module. In particular, $W \cap Q_1/V$ is a simple $B$-module. Moreover, $[E, Q_1] = V$ and $E/V$ is elementary abelian. Put $F/V = C_{E/V}(\langle W^{M_1} \rangle)$. Then by 1.3, $E/F$ is a simple $M_1$-module and so $E/F \cong E_1 \otimes E_2$ where $E_1$ is a simple $M^o$-module and $E_2$ is a simple $C_B(V)$-module. Since $[E_1, W] \otimes E_2 \cong [E, W]F/F \cong W \cap Q_1/V$ as an $B$-module we conclude that $E_2$ is natural $Sp_2m(2)$-module for $C_B(V)$ and $[E_1, W]$ is a natural $SL_2(2)$-module for $B \cap M^o$. Thus $E_1$ is a natural $SL_3(2)$-module for $M^o$ dual to $V$. In particular, $[E, T] \leq (W \cap Q_1)F$. Since $[Q_1, W] \leq Q_1 \cap W \leq E$ we have $[Q_1, O^2(M)] \leq E$. It follows that $[Q_1, T] \leq W$. Since $O_2(B) = Q_1/W$ by 2.2(d) this implies $[O_2(B), T] \leq W \leq Q_2$. Thus $T$ centralizes $O_2(B)/Q_2$. This rules out cases 2.11(2) and (5).

Hence 2.11(1) holds. The structure of $M_2$ shows that $C_B(V)$ has exactly three non-trivial composition factors on $O_2(B)$. Since $C_B(V)$ also has three non-trivial composition factors on $E/F$ we conclude that $[E, O^2(C_B(V))] \leq V$. On the other hand, $E/V = \langle (W \cap Q_1/V)^{M^o} \rangle$ and so $E/V$ as an $C_B(V)$-module is the direct sum of copies of the non-trivial simple $C_B(V)$-module $W \cap Q_1/V_1$. Thus $F = V$ and $E/W \cap Q_1$ is a natural $Sp_2m(2)$-module for $C_B(V)$. It follows that $E \cap Q_2 = W \cap Q_1$ and so $EQ_2/Q_2$ is a natural $Sp_2m(2)$-module for $C_B(V)$. Hence $n = 2$ (Indeed if $n \geq 3$ and so $m \geq 2$, the structure of $M_2/Q_2$ shows that $O_2(B)/Q_2$ as a $C_B(V)$-module is a non-split extension $Y_M$ by a natural $Sp_2m(2)$-module).

In $M_2$ we see that $|O_2(B)| = 2^{1+8+3} = 2^{12}$ and so $|Q_1| = 2^{10}$. This shows that $Q_1 = Y_ME$.

**Lemma 2.15.** [solv] $L$ is solvable.
Proof. We need to show that the situation described in 2.14 does not occur. For this let $D$ be a Sylow 3-subgroup of $B$, $D_1 = C_D(V)$ and $D_2 = D \cap (M^\circ Q_1)$. Then $D = D_1 D_2$ and $D_1 Q_1 \trianglelefteq M_1$. Put $N_1 = N_{M_1}(D_1)$. By the Frattini Argument $M_1 = N_1 Q_1$ and since $D_1$ acts fixed-point freely on $Q_1/Y_M$, $N_1 \cap Q_1 = Y_M$. Hence $N_1 \sim (2^{3+1})(SL_3(2) \times SL_2(2))$ and $|O_2(N_1/D_1)| = 2^6$. Therefore 1.1(b) implies that $|Z(N_1/D_1)| = 2$. Let $E_1$ be the inverse image of $Z(N_1/D_1)$ in $N_1$ and put $F_1 = C_{N_1}(E_1)$. Then $E_1 \cong SL_2(2)$ and so $N_1 = F_1 \times E_1$, $Y_M D_2 \leq F_1$ and $F_1/Y_M \cong SL_3(2)$. Put $N = N_B(D) = N_{N_1}(D_2) \cap B$. Then $|Y_M \cap N| = 4$ and $(F_1 \cap N )/(Y_M \cap N) \cong SL_2(2)$. Moreover, by 1.1(c) $|Y_M \cap N, F_1 \cap N| \neq 1$ and so $N/D \cong D_8 \times C_2$. Also $C_N(D_2)/D = (Y_M \cap N) E_1/D \cong C_3^2$.

We now investigate the embedding of $N$ in $M_2$. Since $D_1$ and $D_2$ are the only normal subgroups of order three in $N$ we have $D_1 \leq L$ and $D_2 Q_2 \leq M_2$. Thus $[O_2(B \cap F_1), D_2] \leq Q_2$ and so $|C_{Q_2}(E_1)| = 2^5$. Note that $\overline{\Phi(L) = O^2(C_{M_2}(D_2)) \cong Sp_4(2)}$ and $W/Z$ is a direct sum of two natural modules for $\overline{H}$. Since $[E_1, D_2] = 1$ we conclude that $\overline{E_1 \leq H}$ and the involutions in $E_1$ act as transvections on these natural modules. It follows that $\overline{E_1 \not\leq H} \cong Sp_4(2)'$. Put $N_2 = N_{M_2}(D_2)$ and $U_2 = C_{M_2}(D_2)'$. Then $N_2/D_2 \sim 2.(Sp_4(2) \times 2)$ and $U_2 Z \cong Sp_4(2)'$. Since $C_{N_2}(D_2)/D$ is elementary abelian of order $2^3$ we conclude that $U_2 Z$ contains a fours group and so $U_2 \cong Sp_4(2)'$. Thus $U_2 \cap N \cong SL_2(2)$ and $(U_2 \cap N) D/D \leq Z(N/D)$. Also $Z(D)/D \leq Z(N/D)$ and $E_1 D/D \leq Z(N/D)$. Since $\overline{E_1 \not\leq H} = U_2 Z$ this implies $|Z(N/D)| \geq 8$, a contradiction to $N/D \cong D_8 \times C_2$. 

\[ \text{Proposition 2.16.} \text{[end]} \quad Q_M = Y_M, Q \text{ is extraspecial of order 32 and } \hat{C}/Q \cong \text{Sym}(3) \times \text{Sym}(3). \]

Proof. By 2.15 we have that $L$ is solvable and so by minimality $\overline{L}$ is a $r$-group for some odd prime $r$, $M \cap B$ acts simply on $\overline{L/\Phi(L)}$, $Y_M$ inverts $\overline{L/\Phi(L)}$ and $Y_M$ centralizes $\Phi(L)$. Thus $\Phi(L) \leq Z(\langle Y_M \rangle) \cong \overline{\Phi(L)}$. By 2.2 $W = [W, L]$ and $[W/Z, Q_2] = 1$, so $C_{W/Z}(L) = 1$ and $Z(W) = Z$ by 2.2(f). Thus $W$ is an extra-special 2-group.

Suppose for a contradiction that $L$ is not abelian. Then $Z(L) = Z(\overline{H}) \neq 1$. Since $W = V_{\overline{H}}$ and $L$ acts faithfully on $\overline{W}$, we get that $Z(L)$ acts faithfully on $\overline{V/Z}$. Thus $|Z(L)| = 3$ and $L$ is an extraspecial 3-group. Let $Z(L) \leq A \leq L$ with $|A| = 9$ and put $A_1 = [A, Y_M]$. Then $A = A_1 \times Z(L)$ and $A$ is elementary abelian. Let $A_1, A_2, A_3, Z(L)$ be the subgroups of order 3 in $A$. From $C_{W/Z}(Z(L)) = 1$ we have

$$W/Z = \bigoplus_{i=1}^{3} C_{W/Z}(A_i).$$

Since $L$ acts transitively on $\{A_1, A_2, A_3\}$ we have $|W/Z| = |C_{W/Z}(A_i)|^3$. As $Z(L)$ acts non-trivially on $C_{W/Z}(A_i)$, $|C_{W/Z}(A_i)| \geq 4$. Note that $Y_M$ does not normalize $A_2$ and that $|W/Z, Y_M| = 4$. Hence $|C_{W/Z}(A_i)| = 4$ and so $|W/Z| = 2^6$. It follows that $|L| = 3^3$. Since $|Z(L), Y_M| = 1$, 2.3(b) gives $Z(L) \leq B$. Hence $|O_2(B), Z(L)| = 1$. Since $C_{Out(L)}(Z(L)) \cong SL_2(3)$ and $|C_{GL_{W/Z}}(L)| = 3 = |Z(L)|$ we get that $O_2(B)$ is isomorphic to subgroup of $SL_2(3)$ and so to a subgroup of $Q_8$. Thus $\Omega_1(O_2(B)) \leq Y_M$. Put $E = \langle (W \cap Q_M)^M \rangle$. Since
\[ \Phi(W \cap Q_M) \leq Z \leq V \] we conclude that \( E/V \) is generated by involutions. As \( V \leq Q_2 \) this gives \( E \leq \Omega_1(O_2(B)) \leq Y_M \) and \( E \leq Y_M Q_2 \). Hence by 2.4 \( Q_M = Y_M \) and so \( |S| = 2^7 = |W| \), a contradiction.

So we have shown that \( L \) is abelian. It follows that \( L \) is elementary abelian and \( Y_M \) inverts \( L \). Let \( R \) be a simple \( L \)-submodule of \( W \). Note that \( C_L(R) \) is normalized by \( LY_M = H \) and so centralizes \( (RH) \). Since \( W \) is a homogeneous \( H \)-module by 2.2(g), this gives that \( C_L(R) = 1 \) and so \( L \) is cyclic. Thus \( |W/Z| = |[W/Z, Y_M]|^2 = 4^2 = 16 \). Hence \( W \) is extra special of order \( 2^4 \) and since \( V \leq W \), \( W \cong Q_8 \circ Q_8 \). Thus \( \text{Out}(W) \cong O_4^+(2) \cong \text{SL}_2(2) \lhd C_2 \) and \( L \cong C_3 \). Since \( [T, Y_M] \leq V \leq Q_2 \), \( T \not\subseteq L \) and so \( T \overline{L} \cong C_3 \times C_3 \). Moreover, \( [W, Q_M] \leq C_W(V) = V \) and so \( [O^2(M), Q_M] \leq V \). Now 2.4 gives \( Q_M = Y_M \) and so \( |S| = 2^7 \). In particular, \( Q_M \cap Q_2 = V = Q_M \cap W \) and \( Q_1W = Q_1Q_2 = O_2(B) \). Thus \( Q_2 = W = Q \) and \( |S/Q_2| = 2^2 \). It follows that \( M_2 = T \overline{L}S \cong \text{Sym}(3) \times \text{Sym}(3) \). Since \( C_G(Q) \leq Q \) and \( \text{Out}(Q) \cong O_4^+(2) \) we have \( |N_G(Q)/M_2| \leq 2 \). Since \( S \in \text{Syl}_2(G) \) this forces \( M_2 = N_G(Q) \).

**Proof of Theorem 1:**

We are now able to prove the theorem. By 2.16 we have that \( M \) is an extension of an elementary abelian group of order 16 by \( \text{SL}_3(2) \). Let \( z \in Z^2 \). Since \( Q \) is large, \( C_G(z) \leq N_G(Q) \) and so \( N_G(Q) = C_G(z) \). Since \( Q \) is generated by involutions, there exists involutions in \( M \setminus Y_M \) and so \( M/V \not\cong \text{SL}_2(7) \). Hence \( M \) has a subgroup \( M^* \) of index two, which is an extension of \( V \) by \( \text{SL}_3(2) \).

Let \( y \in Y_M \setminus V \). 1.1(c) implies that \( C_M(y) \) is divisible by seven. Since \( C_G(z) = N_G(Q) \) is not divisible by seven, \( y \) and \( z \) are not conjugate in \( G \). Note that \( V \leq Q = [Q, B] \leq M^* \). Hence every involutions in \( M^* \) is conjugate to an involution in \( Q \). Since \( M_2/Q \cong \text{Sym}(3) \times \text{Sym}(3) \) we see that all involutions in \( Q \setminus Z(Q) \) are conjugate under \( M_2 \). Thus all involutions in \( M^* \) are conjugates of \( z \) in \( G \). This shows that \( y \) is not conjugate to any involution in \( M^* \). By Thompson’s Transfer Lemma we get that \( G \) possesses a subgroup \( G^* \) of index two. Since \( M^* \) is perfect, \( M^* = M \cap G^* \). Moreover \( O^2(M_2) \leq G^* \), \( M_2 \cap G^* = C_{G^*}(z) \), \( O^2(M_2) \cong \text{SL}_2(3) \times \text{SL}_2(3) \) and \( |(M_2 \cap G^*)/O^2(M_2)| = 2 \). Hence [Asch] shows that \( G^* \cong G_2(3) \). Since \( |\text{Out}(G_2(3))| = 2 \) we conclude that \( G \cong \text{Aut}(G_2(3)) \).

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