1. Introduction

Suppose that $p$ is a prime, $P$ is a finite group and $S \in \text{Syl}_p(P)$. Then $P$ is $p$-minimal if $S$ is not normal in $P$ and $S$ is contained in a unique maximal subgroup of $P$. Now suppose that $G$ is a finite group and $S \in \text{Syl}_p(G)$, if $S \leq H \leq G$ and $S$ is not normal in $H$, then we call $P$ a $p$-parabolic subgroup of $G$. In most cases the prime $p$ will be evident from the context in which we are working and in these cases we often simply call $P$ a parabolic subgroup of $G$. The set of maximal parabolic subgroups of $G$ (containing $S$) is denoted by $\mathcal{M}_G(S)$ and the set of $p$-minimal parabolic subgroups of $G$ (containing $S$) is denoted by $\mathcal{P}_G(S)$. Suppose that $G$ is a Lie type group defined in characteristic $p$. Then the parabolic subgroups of $G$ in the traditional sense are also $p$-parabolic subgroups in our context, though it should be noted that we do not require that our $p$-parabolic subgroups contain the Borel subgroup of $G$. If $R$ is a rank 1 parabolic subgroup in $G$, then $P = O^p(R)$ is a $p$-minimal parabolic subgroup of $G$. Moreover it is easy to see that in the Lie type groups we have $G = \langle \mathcal{P}_G(S) \rangle N_G(S)$ and it turns out that this is a property of $p$-minimal parabolic subgroups in general (see 2.1). Suppose that $P \in \mathcal{P}_G(S)$ and set

$$B = B_P = (\mathcal{P}_G(S) \setminus \{P\}, N_G(S)).$$

Notice that when $P$ is Lie type group in characteristic $p$, $B_P \in \mathcal{M}_G(S)$ for all $P \in \mathcal{P}_G(S)$. In fact minimal $p$-parabolic subgroups in the Lie type groups have two further properties. The first is that $O_p(B_P) \not\leq O_p(P)$ and the second is that $P/O_p(P)$ is a rank 1 Lie type group in characteristic $p$. We shall see that for $p$ sufficiently large ($p \geq 11$) these two properties characterize Lie type groups in characteristic $p$ among the finite simple groups.

We now make these notions more precise. We say that $P \in \mathcal{P}_G(S)$ is isolated with respect to $A$ provided $A$ is a normal $p$-subgroup of $B_P$ and $A \not\leq O_p(P)$. Notice that $A \leq O_p(B_P)$ and so we also have $O_p(B_P) \not\leq O_p(P)$. Suppose that $P$ is a $p$-minimal group, $S \in \text{Syl}_p(P)$ and $M$ is the maximal subgroup of $P$ containing $S$. Set $R = \bigcap M^P$, the core of $M$ in $P$, and $E = O^p(P) R/R$. Then, loosely approximating the structure of a rank 1 Lie type group, we say that $P$ is narrow if either $E$ is a simple group or $E$ is elementary abelian and $M$ acts primitively on $E$. Our first result, proved in Section 2 is a basic structure theorem for groups which possess a narrow, $p$-minimal, isolated parabolic subgroup.

**Theorem 1.1.** [Thm1] Suppose that $P \in \mathcal{P}_G(S)$ is narrow and isolated in $G$. Set $Y = \langle O^p(P)^G \rangle$. Then either $Y/O_p(Y)$ is quasisimple or $Y = O^p(P)$ and $G = B_P P$.

Suppose that $G$ is a $p$-minimal simple group, then so long at the unique maximal subgroup of $G$ is a $p$-local subgroup, $G$ satisfies the hypothesis of the theorem.

In keeping with the structure of the Lie type groups in characteristic $p$ (perhaps extended by field automorphisms) we wish to restrict our further attentions to those candidates for $LS$ which have $P/Q_P$ a rank 1 Lie type group in characteristic $p$ extended by automorphisms of order $p$. Thus we
let $L(p)$ consist of groups $H$ with $O_p(H) = 1$ and $O^\prime(H)$ isomorphic to $O^\prime(L)$ for some (either adjoint or universal) rank 1 Lie type group in characteristic $p$. This has the effect of including some groups which are a little bit smaller than we might expect. For example, as $O^2(2B_2(2))$ has order $5$, Dih(10) and $2B_2(2)$ are both in $H$. A similar phenomena occurs with $PSU_3(2) \cong 3^2 : Q_8$. The extra condition that we will impose is contained in the following definition.

**Definition 1.2. [p-restricted]** Suppose that $p$ is a prime and $G$ is a group and $S \in \text{Syl}_p(G)$. Then $P \in \mathcal{P}_G(S)$ is $p$-restricted in $G$ if $P$ isolated in $G$ and $P/Q_P \in L_1(p)$. If $G$ possesses a $p$-restricted $p$-minimal parabolic subgroup, then we say that $G$ is $p$-restricted.

Notice that if $P \in \mathcal{P}_G(S)$ and $P/Q_P \in L_1(p)$, then $P$ is narrow. Thus 1.1 tells us that when $O_p(G) = 1$, then $Y = \langle O^\prime(P)^G \rangle$ is either quasisimple or $Y = O^\prime(P)$. In this latter case we also have that $Y$ is quasisimple or $Y$ is soluble and $O^\prime(P)$ is isomorphic to one of $Z_3$, $Q_8$, $2^2$ or $3_2^{1+2}$ or $3^2$.

**Theorem 1.3. [Thm2]** Suppose that $p$ is a prime, $G$ is a finite group, $X = F^*(G)$ is a non-abelian simple group and $G/X$ is a $p$-group. If $G$ is $p$-restricted, then either $X$ is a Lie type group defined in characteristic $p$ or

(a) [a] $p = 2$ and $X \cong \text{Alt}(12)$ (see 4.2).

(b) [b] $X$ is a Lie type group in characteristic $r$ with $r \neq p$, $p \in \{2, 3\}$ and the possibilities for $p$, $X$, $P$ and $B_P$ are as listed in Table 1.

(c) [c] $X$ is a sporadic simple group and the possibilities for $(X, p)$ are as follows: $(\text{Mat}_{12}, 2)$, $(\text{Mat}_{12}, 3)$, $(\text{Mat}_{22}, 2)$, $(J_2, 2)$, $(J_2, 3)$, $(\text{Mat}_{23}, 2)$, $(\text{HS}, 2)$, $(J_3, 2)$, $(\text{Mat}_{24}, 2)$, $(\text{McL}, 3)$, $(\text{He}, 2)$, $(\text{Ru}, 2)$, $(\text{Suz}, 2)$, $(\text{Suz}, 3)$, $(O'N, 2)$, $(\text{Co}_3, 2)$, $(\text{Co}_3, 3)$, $(\text{Co}_2, 2)$, $(\text{Co}_2, 3)$, $(\text{Fi}_{22}, 2)$, $(\text{Fi}_{22}, 3)$, $(\text{HN}, 2)$, $(\text{HN}, 5)$, $(\text{Ly}, 5)$, $(\text{Th}, 2)$, $(\text{Th}, 3)$, $(\text{Fi}_{23}, 2)$, $(\text{Fi}_{23}, 3)$, $(\text{Co}_1, 2)$, $(\text{Co}_1, 3)$, $(\text{Co}_1, 5)$, $(\text{Fi}_{24}, 2)$, $(\text{Fi}_{24}, 3)$, $(\text{BM}, 2)$, $(\text{BM}, 3)$, $(\text{BM}, 5)$, $(\text{M}, 2)$, $(\text{M}, 3)$, $(\text{M}, 5)$, $(\text{M}, 7)$.

In particular, if $p \geq 11$, $X$ is a Lie type group in characteristic $p$ and, if $p \geq 5$, then $X$ is a Lie type group or one of the sporadic groups HN, Ly, Co1, BM or M.
<table>
<thead>
<tr>
<th>Group Condition on $r^a$</th>
<th>P</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>PGL$_2(r^a)$ 5 (mod 8)</td>
<td>Sym(4)</td>
<td>Dih(2($r^a - 1$))</td>
</tr>
<tr>
<td>PGL$_2(r^a)$ 3 (mod 8)</td>
<td>Sym(4)</td>
<td>Dih(2($r^a + 1$))</td>
</tr>
<tr>
<td>PSL$_2(7) \cong$ PSL$_3(2)$</td>
<td>Sym(4)</td>
<td>Sym(4)</td>
</tr>
<tr>
<td>PSL$_2(9) \cong$ Alt(6)</td>
<td>Sym(4)</td>
<td>Sym(4)</td>
</tr>
<tr>
<td>PSL$_2(9) \cong$ Sym(6)</td>
<td>Sym(4) x 2</td>
<td>Sym(4) x 2</td>
</tr>
<tr>
<td>PSL$_2(5) \cong$ Alt(5)</td>
<td>PSL$_2(5)$</td>
<td>Alt(4)</td>
</tr>
<tr>
<td>PGL$_2(5) \cong$ Sym(5)</td>
<td>PGL$_2(5)$</td>
<td>Sym(4)</td>
</tr>
<tr>
<td>PGL$_2(19)$</td>
<td>Dih(40)</td>
<td>Sym(4)</td>
</tr>
<tr>
<td>PSU$_3(r^a)$ 3 (mod 8)</td>
<td>$2 \cdot$ Sym(4) x 4</td>
<td>$(r^a + 1)^2 :$ Sym(3)</td>
</tr>
<tr>
<td>PSU$_3(r^a)$ : 2 3 (mod 8)</td>
<td>$2 \cdot$ Sym(4) x Q$_8$</td>
<td>$(r^a + 1)^2 : (2 \times$ Sym(3))</td>
</tr>
<tr>
<td>PSU$_3(3) \cong$ G$_2(2)'$</td>
<td>$4^2 :$ Sym(3)</td>
<td>$2 \cdot$ Sym(4) x 4</td>
</tr>
<tr>
<td>PSU$_3(3) : 2 \cong$ G$_2(2)$</td>
<td>$4^2 : (2 \times$ Sym(3))</td>
<td>$2 \cdot$ Sym(4) x Q$_8$</td>
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<tr>
<td>PSL$_2(r^a)$ : 2 5 (mod 8)</td>
<td>$2 \cdot$ Sym(4) x Q$_8$</td>
<td>$(r^a - 1)^2 : (2 \times$ Sym(3))</td>
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<tr>
<td>PSU$_4(3)$</td>
<td>$2^{2+2+2} \cdot$ Sym(3)</td>
<td>$2^{1+4} \cdot$ (Sym(3) x Sym(3)).2</td>
</tr>
<tr>
<td>PSU$_4(3).2_1$</td>
<td>$2^{2+2+2} \cdot$ Sym(3).2</td>
<td>$2^{1+4} \cdot$ (Sym(3) x Sym(3)).2</td>
</tr>
<tr>
<td>PSU$_4(3).2_2$</td>
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<td>$2^{1+4} \cdot$ (Sym(3) x Sym(3)).2</td>
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<td>$2^{1+4} \cdot$ (Sym(3) x Sym(3)).4</td>
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<tr>
<td>PSU$<em>4(3),2</em>{122}$</td>
<td>$2^{2+2+2} \cdot$ Sym(3).2$^2$</td>
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<tr>
<td>PSU$<em>4(3),2</em>{122}$</td>
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<tr>
<td>PSU$_4(3) \cong$ G$_2(2)'$</td>
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<td>$2^{1+4} \cdot$ Alt(6).2</td>
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<td>$2^{1+4} \cdot$ Alt(6).2</td>
</tr>
<tr>
<td>PSU$_6(3)$</td>
<td>$\leq 4^5 \cdot$ Sym(6)</td>
<td>$\frac{1}{2} \cdot$ GU$_2(3) \cap$ Sym(3) x PSU$_6(3)$</td>
</tr>
<tr>
<td>PSU$_6(3)$</td>
<td>$\leq 4^5 \cdot$ Sym(6) x 2</td>
<td>$\frac{1}{2} \cdot$ GU$_2(3) \cap$ Sym(3)</td>
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<tr>
<td>PSp$_4(3)$</td>
<td>$\frac{1}{2} \cdot$ O$_1(3) \cap$ Sym(7)</td>
<td>$\frac{1}{2} \cdot$ O$_1(3) \cap$ Sym(7)</td>
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<td>SL$_2(8)$ : 3</td>
<td>$7^3 : 21$</td>
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<tr>
<td>G$_2(3)$</td>
<td>4$^2 :$ Dih(12)</td>
<td>$2^{1+4} : 3^2.2$</td>
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<td>G$_2(3)$</td>
<td>4$^2 :$ Dih(12)</td>
<td>$2^{1+4} :$ Sym(3) x Sym(3)</td>
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<tr>
<td>$^3$D$_4(3)$</td>
<td>4$^2 :$ Dih(12)</td>
<td>(SL$_2(3) \times SL_2(27)) : 2$</td>
</tr>
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<td>$^3$D$_4(3)$</td>
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<td>4$^2 :$ Dih(12)</td>
<td>(SL$_2(3) \times SL_2(27)) : 2$</td>
</tr>
</tbody>
</table>

Table 1: Lie type group exceptions with $p = 2$
\[ \begin{array}{|c|c|c|c|} \hline \\[-1.5ex] 
\text{Group} & \text{Condition on } r^a & P & B \\
\hline 
\text{PGL}_3(r^a) & r^a \equiv 4, 7 \pmod{9} & 3^2 : \text{SL}_2(3) & (r^a - 1)^2 : \text{Sym}(3) \\
\text{PGU}_3(r^a) & r^a \equiv 2, 5 \pmod{9} & 3^2 : \text{SL}_2(3) & (r^a + 1)^2 : \text{Sym}(3) \\
\text{PGL}_3(7) & r^a \equiv 2, 5 \pmod{9} & 3^2 : \text{SL}_2(3) & 3^2 : \text{Sym}(3) \\
\text{PSU}_4(r^a) & r^a \equiv 2, 5 \pmod{9} & 3^{1+2} : \text{SL}_2(3) & \frac{1}{(2, r^a + 1)/2} (r^a + 1)^3 : \text{Sym}(4) \\
\text{PSU}_4(2) & & 3^3 : \text{Sym}(4) & 3^{1+2} : \text{SL}_2(3) \\
\text{PSU}_5(2) & & 3 \times 3^{1+2} : \text{SL}_2(3) & 3^4 : \text{Sym}(5) \\
\text{PSU}_6(2) & & 3^6 : \text{Alt}(6) & 3^{1+4} (Q_8 \times Q_8).3 \\
\text{PSU}_6(2) : 3 & & 3^6 : \text{Alt}(6) & 3^{1+4} (Q_8 \times Q_8).3^2 \\
\text{PSU}_7(2) \cong \text{PSL}_2(9) & & & 3^2 : 4 \\
\hline 
\text{P}\Omega_4^-(r^a) & r^a \equiv 2, 5 \pmod{9} & 3^{1+2} : \text{SL}_2(3) \times 3 & \text{O}_2(r^a) \wr \text{Sym}(4) \\
\text{P}\Omega_4^+(r^a) : \langle r \rangle & r^a \equiv 2, 5 \pmod{9} & 3^{1+2} : \text{SL}_2(3) & \text{O}_2(r^a) \wr \text{Sym}(4).3 \\
\text{E}_6(2) & & 1/3 \text{PSU}_3(2) \wr \text{Sym}(3) \cap X & 1/3 \text{PSU}_3(2) \wr \text{Sym}(3)^2 \cap X \\
\text{E}_6(2) : 3 & & 1/3 \text{PSU}_3(2) \wr \text{Sym}(3) \cap X & 1/3 \text{PSU}_3(2) \wr \text{Sym}(3)^2 \cap X \\
\text{E}_8(2) & & 3^2. (\text{PSU}_3(2)^4).3^2 : \text{GL}_2(3) & 3^2. (\text{PSU}_3(2)^4).3^2 : \text{GL}_2(3) \\
\hline 
\end{array} \]

Table 1: Lie type group exceptions with \( p = 3 \)

For our intended application of 1.3 we will know additional information about the over groups of the \( p \)-restricted minimal parabolic subgroup. Indeed we will know for such a maximal \( p \)-parabolic subgroup \( M \) that \( M/O_p(M) = \text{SL}_n(p^h) \) or \( \text{PSL}_n(p^h) \) perhaps extended by a group of field automorphisms (for some \( n \geq 2 \) and natural number \( b \)). Our next result can is designed to make 1.3 more immediately applicable in the circumstances just described.

**Corollary 1.4. [L3-restricted]** Suppose that \( p \) is a prime, \( G \) is a finite group, \( X = F^*(G) \) is a non-abelian simple group and \( G/X \) is a \( p \)-group. If \( P \in \mathcal{P}_G(S) \) is \( p \)-restricted and there is a \( P_1 \in \mathcal{P}_G(S) \) such that \( O^p(P/P_1) / O_p((P, P_1)) \cong \text{SL}_n(p^h) \) or \( \text{PSL}_n(p^h) \) for some integer \( b \), then is either \( X \) is a Lie type group defined in characteristic \( p \) or \( p = 2 \) and \( X \) is one of the following sporadic simple groups ......

Suppose that \( p \) is a prime. Then we use \( \mathcal{R}_p \) to denote the set of group \( G \) for which there is \( P \in \mathcal{P}_p(G) \) which is narrow and \( p \)-restricted. One of our goals is to determine those simple groups which are in \( \mathcal{R}_p \).

For a group \( H \), we denote the preimage of \( F(H/O_p(H)) \) by \( F_p(H) \) and the preimage of \( \Phi(H/O_p(H)) \) by \( \Phi_p(H) \). The remainder of our group theoretic notation is standard as can be found in [1].

**2. Groups with isolated \( p \)-minimal parabolic subgroups**

Recall from the introduction that for a prime \( p \), a group \( P \) is called \( p \)-minimal if for a Sylow \( p \)-subgroup \( S \) of \( P \), \( S \) is not normal in \( P \) and \( S \) is contained in a unique maximal subgroup of \( P \). For a group \( G \) and \( S \in \text{Syl}_p(G) \), we denote the set of \( p \)-minimal parabolic subgroups of \( G \) (which contain \( S \)) by \( \mathcal{P}_G(S) \). For an arbitrary subgroup \( R \) of \( G \), we set \( Q_R = O_p(R) \).

The following result is elementary to prove.

**Lemma 2.1. [gen]** \( G = \langle \mathcal{P}_G(S) \rangle N_G(S) \).

We shall also need the following general result.

**Lemma 2.2. [no subnormal]** Let \( M \) be a maximal subgroup of the finite group \( H \) and let \( N \) be a subnormal subgroup of \( H \) with \( N \leq M \). Then \( N \leq \bigcap M^H \).

**Proof.** Suppose that \( H \) is a counterexample to the statement and select \( N \) subnormal in \( H \) with \( N \leq M \) of maximal order. Let \( N = N_0 \leq N_1 \leq \ldots \leq N_k \leq N_{k+1} = H \) be a subnormal chain from
N to H. By Wielandt’s subnormal lemma, \( \langle N^M \rangle \leq H \). Because of the maximal choice of \( N \), we have \( \langle N^M \rangle = N \leq M \). Also by the maximal choice of \( N, N_1 \leq M \). Therefore, \( N \leq \langle N_1, M \rangle \). Since \( M \) is a maximal subgroup of \( H \), we have \( N \leq H \) as claimed. \( \square \)

Lemma 2.3. [basic p-minimal] Assume that \( P \) is p-minimal, \( S \in \text{Syl}_p(P) \) and \( M \) is the maximal subgroup of \( P \) containing \( S \).

(a) \[ \bigcap M^p \text{ is p-closed, that is } S \cap \bigcap M^p = Q_P. \]

(b) \[ \text{If } O_p(O^p(P)) = 1, \text{ then } \bigcap M^p \text{ is nilpotent.} \]

(c) \[ \text{If } O^p(P) \text{ is p-closed, then } P \text{ is a } \{ t, p \} \text{-group for some prime } t \neq p. \]

(d) \[ \text{If } N \text{ is a subgroup of } P \text{ with } N \leq M, \text{ then } N \cap S \leq Q_P. \]

Proof. Let \( F = \bigcap M^p \) and set \( T = S \cap F \). Plainly \( Q_P \leq T \) and, by the Frattini Argument, \( P = FN_P(T) \). Since \( N_P(T) \geq S \) and \( P \) is p-minimal, we have \( P = N_P(T) \). So \( T \leq Q_P \) and (a) holds.

Assume that \( O_p(O^p(P)) = 1 \). Then (a) implies that \( F \cap O^p(P) \) is a \( p' \)-group. Let \( T \in \text{Syl}_t(F) \) for some prime \( t \neq p \). Then \( T \leq O^p(P) \) and so \( T \leq F \cap O^p(P) \). The Frattini Argument gives \( P = N_P(T)(F \cap O^p(P)) \). Since \( p \) does not divide \( |F \cap O^p(P)| \), we infer that \( N_P(T) \) contains a Sylow \( p \)-subgroup of \( P \). Thus \( T \) is normal in \( P \) and \( F \) is nilpotent.

Without loss of generality, we now assume that \( Q_P = 1 \).

For (c) let \( t \) be a prime such that \( t \) divides \( |P/M| \). Note that \( t \neq p \). Since \( O^p(P)/Q_P \) is a \( p' \)-group, there exists an \( S \)-invariant Sylow \( t \)-subgroup \( T \) of \( O^p(P) \). Then, as \( T \not\leq M \), \( P = ST \) as \( p \) is p-minimal. Thus (c) holds. Finally (d) follows from (a) and 2.2. \( \square \)

Definition 2.4. [def:restricted] Suppose that \( G \) is a group, \( S \in \text{Syl}_p(G) \), \( P \in \mathcal{P}_G(S) \), \( B = B_P = \langle P \rangle \) and \( A \) is a normal subgroup of \( B \). If \( A \not\leq Q_P \), then we say that \( P \) is isolated in \( H \) with respect to \( A \).

Note that if \( P \) is isolated in \( H \) with respect to \( A \), then \( P \) is also isolated in \( H \) with respect to \( Q_H \). Furthermore, we note that if \( P \) is isolated in \( H \) with respect to \( A \), then certainly \( A > 1 \) and so \( Q_B > 1 \).

The next lemma is the primary structural result about groups which possess an isolated p-minimal subgroup.

**Lemma 2.5. [quasi]** Suppose that \( P \in \mathcal{P}_H(S) \) is isolated in \( H \) with respect to \( A \). Set \( Y = \langle O^p(P)^H \rangle \) and \( F = \bigcap B^H \). Then

(a) \[ \text{[a-1] Suppose that } S \leq M \leq H \text{ and } M \not\leq B, \text{ then } P \leq M, B \cap M \text{ is a maximal subgroup of } M; \text{ furthermore, } P \text{ is also isolated in } M \text{ with respect to } A; \]

(b) \[ \text{[a-2] } P \cap B \text{ is the unique maximal subgroup of } P \text{ containing } S; \]

(c) \[ \text{[a] } B \text{ is a maximal subgroup of } H; \]

(d) \[ \text{[b] } N_H(T) \leq B \text{ for all } A \leq T \leq S; \]

(e) \[ \text{[c] if } O^p(H) \text{ is p-closed, then } H = BP \text{ and } \langle A^H \rangle = \langle A^P \rangle; \]

(f) \[ \text{[d] if } R \text{ is a normal subgroup of } H \text{ and } R \not\leq B, \text{ then } Y \leq R; \]

(g) \[ \text{[e] if } R \text{ is a proper characteristic subgroup of } Y, \text{ then } Y \leq F; \]

(h) \[ \text{[e+1] } [F, Y] \leq Q_H; \]

(i) \[ \text{[f] either } Y = O^p(P) \text{ or } YQ_H/Q_H \text{ is semisimple; and} \]

(j) \[ \text{[g] if } KQ_H/Q_H \text{ is a component in } YQ_H/Q_H \text{ and } Y \neq O^p(P), \text{ then } Y = \langle K^S \rangle \text{ and } K \cap P \not\leq B. \]

Proof. Suppose that \( S \leq M \leq H \) and \( M \not\leq B \). Then, by 2.1, \( M = \langle \mathcal{P}_M(S) \rangle N_M(S) \). Since \( N_M(S) \leq B \) and \( P \) is the unique member of \( \mathcal{P}_H(S) \) which is not contained in \( B \), we have \( P \in \mathcal{P}_M(S) \).
Since $\mathcal{P}_M(S) \setminus \{P\} \subseteq \mathcal{P}_B(S)$, we have $M \cap B = \langle \mathcal{P}_M(S) \setminus \{P\} \rangle N_M(S)$. Now let $M \cap B < D \leq M$. Then also $\mathcal{P}_D(S) \subseteq \mathcal{P}_M(S)$ and so as $D > M \cap B$, we must have $P \in \mathcal{P}_D(S)$ and we conclude that $D = M$. Thus $M \cap B$ is a maximal subgroup of $M$. Finally, as $A \leq B$, $A \leq B \cap M$ and $A \not\leq Q_P$ and so $P$ is isolated in $M$ with respect to $A$. This proves (a). Parts (b) and (c) follow immediately from (a).

Suppose that $A \leq T \leq S$. Since $A \not\leq Q_P$, $P$ is not in $N_H(T)$. Taking $M = N_H(T)$, (a) gives $N_H(T) \leq B$. So (d) holds.

For the proof of the remaining assertions assume that $Q_H = 1$.

Suppose that $X := O^p(H)$ is $p$-closed. Then, as $Q_H = 1$, $X$ is a $p'$-group. Since $O^p(P) \leq X$, $O^p(P)$ is also a $p'$-group. Therefore, by 2.3(e), $P$ is a $\{t, p\}$-group for some prime $t \neq p$. Since $p$ and $|X|$ are coprime, for each prime divisor $r$ of $|X|$ there is an $S$-invariant Sylow $r$-subgroup $S_r$ of $X$. If $r \not= t$, then $P \not\leq S_rS$ and so $S_r \leq B$ by (a). Hence, by considering $[X]$, we have $H = (X \cap B)S_rS$. We now consider $S_rS$, we have that $(S_r \cap B)S_r$ is a maximal subgroup of $S_rS$ by (a) and so $S_r \cap B$ is a maximal $S$-invariant subgroup of $S_r$. Since $S_r$ is nilpotent, $N_{S_r}(S_r \cap B) > S_r \cap B$ and of course $N_{S_r}(S_r \cap B)$ is also normalized by $S$. Hence $S_r \cap B$ is normal in $S_r$. In particular, we have that $(S_r \cap B)O^p(P)$ is a subgroup of $S_r$ and, as $S_r \cap B$ is a maximal with respect to being $S$-invariant, we infer that $S_r = (S_r \cap B)O^p(P)$. Hence

$$H = (B \cap X)S_rS = (B \cap X)O^p(P)S = (B \cap X)P = BP.$$ 

Finally, we have $\langle A^H \rangle = \langle A^{BP} \rangle = \langle A^P \rangle$ and this completes the proof of (e).

Set $Y = \langle O^p(P)^H \rangle$. Suppose $R$ is a normal subgroup of $H$ which is not contained in $B$. Then, by (a), $RS \geq P$ and so $O^p(P) \leq R$. Thus (f) holds. Plainly (g) is a direct corollary of (f).

Set $F = \cap B^H$. Then as $F \leq B$, $F$ normalizes $A$. Therefore, $A \cap F \leq Q_F \leq Q_H = 1$. Therefore, as $[F, A] \leq A \cap F$, $1 = [F, A] = [F, \langle A^H \rangle]$. Since $\langle A^P \rangle \geq O^p(P)$, we have $\langle A^H \rangle \geq O^p(P)^H = Y$. Therefore, $[F, Y] = 1$ and (h) is true.

Suppose that $Y$ is not perfect. Then $Y' < Y$ hence $Y' \leq Z(Y)$ by (g) and (h). In particular, $Y$ is nilpotent and so $p$-closed. Applying part (e) to $Y$, we have that $\langle A^H \rangle = \langle A^{BY} \rangle = \langle A^Y \rangle = \langle A^P \rangle = AO^p(P)$ and so $Y = O^p(P)$. If on the other hand, $Y$ is perfect, then $Y$ is semisimple by (g) and (h). So (i) holds.

Let $K$ be a component of $Y$. Then $Y$ is not nilpotent, so $Y$ is semisimple by (j) and, in particular, $K$ is normal in $Y$. Hence, as $H = YB$, $\langle K^B \rangle = \langle K^YB \rangle = \langle K^H \rangle$ and so $K \not\leq B$. Therefore, $\langle K^S \rangle \geq O^p(P)$ by (a). If $L$ is a component of $Y$ not contained in $\langle K^S \rangle$, then $O^p(P) \leq \langle K^S \rangle \cap \langle L^S \rangle \leq Z(Y) \leq B$, a contradiction. Therefore, $K^H = K^B = K^S$ and the first part of (j) holds. Assume that $K \cap P \not\leq B \cap P$. Since $O^p(P) \leq Y$, $K \not= Y$. Furthermore, as $K \cap P \not\leq P$, 2.3(d) gives $K \cap S \leq Q_P$. Hence $Y \cap S = Y \cap Q_P \leq P$ and, in particular, $O^p(P)$ is $p$-closed. Set $R = N_Y(Y \cap S)$. The $O^p(R)$ is $p$-closed and $R \geq P$. Therefore, $R = (R \cap B)P$ and $\langle R^P \rangle = \langle A^P \rangle \leq P$ by (e). Suppose that $A \leq N_H(K)$. Then $\langle A, K \cap R \rangle \leq K \cap \langle A^H \rangle \leq K \cap P \leq B$ which normalizes $A$. It follows that $[A, K \cap R]$ is a $p$-group and so, as $K \cap R$ is $p$-closed, $[A, K \cap R] \leq Q_P \cap K$. Therefore, $[A, R] = \langle [A, K \cap R]^P \rangle \leq Q_P \cap K$. Thus $\langle A^R \rangle = \langle A^P \rangle$ is a $p$-group, a contradiction. Hence $A \not\leq N_H(K)$. Therefore, $Q_B$ properly permutes the components of $Y$. Since $[K \cap Q_B \cap B]$ is a $p$-group, we get $K \cap B \leq (K \cap S)F$ and then that $Y \cap S = O_P(Y \cap B)$. But then $Y \cap S \leq \langle O^p(P), B \cap Y \rangle = Y$ and we conclude that $H$ is $p$-closed.

Now a final application of (e) indicates that $Y \leq \langle A^H \rangle = \langle A^P \rangle \leq P$. So $Y = O^p(Y) = O^p(P)$ and thus (g) holds. □

To control the structure of $Y$ in (??) further we have to impose further conditions on the isolated $p$-minimal subgroup $P$. 

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**Note:** The above text appears to be a part of a mathematical proof, possibly from a text on group theory, involving concepts such as subgroups, normal subgroups, and $p'$-groups. The formulas and symbols suggest a deep understanding of group theory, including concepts like nilpotency and component structure. The proof involves a series of logical steps to establish certain properties and conclusions about the structure of a group, $H$. The text contains a mix of set notation, algebraic manipulation, and theorems that are typical in advanced group theory texts. The proof outlines how to handle cases where a subgroup is not nilpotent, and how to use the properties of the group to deduce further structure, including the use of Sylow theorems and the concept of $p$-closed subgroups.

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**Knowledge Integration:**

- The proof relies on several key concepts from group theory, including the properties of normal subgroups, nilpotent groups, and the role of components and Sylow subgroups. 
- It uses the notation $\langle X \rangle$ to denote the subgroup generated by $X$, and $O^p(P)$ to denote the $p$-core of a group $P$. 
- The proof's logic is built on theorems and properties that are typically covered in advanced courses on group theory.

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**Educational Relevance:**

- This text exemplifies the use of group theory in proving theorems about the structure of groups. 
- It shows how to apply and combine various theorems to deduce specific properties of subgroups. 
- Understanding such proofs is crucial for students and researchers in algebra and related fields.

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**Conclusion:**

The text presents a rigorous proof that involves advanced concepts in group theory, demonstrating how to analyze and control the structure of subgroups under certain conditions. It is an excellent example of how theoretical mathematics can be applied to prove complex results. The reader is encouraged to review and practice similar proofs to deepen their understanding of group theory.
Definition 2.6. [def:narrow] Let $P$ be a $p$-minimal group and $S \in \text{Syl}_p(P)$. Let $M$ be the maximal subgroup of $P$ containing $S$, $R = \bigcap M^p$ and $E := O^p(P)R/R$. Then $P$ is called narrow provided that either

(a) $a$ E is non-abelian and simple; or
(b) $b$ E is elementary abelian and $B$ acts primitively on $E$.

Lemma 2.7. (qs) Suppose that $P \in \mathcal{P}_G(S)$ is narrow and isolated in $G$. Then either $P/Q_P$ is soluble or $O^p(P)Q_P/Q_P$ is quasisimple.

Proof. Let $M = B \cap P$, then by 2.5(b), $M$ is the unique maximal subgroup of $P$ containing $S$. Put $R = \bigcap M^p$. Then $R$ is normal in $P$ and contained in $B$, therefore, $[Q_B, R] \leq Q_B \cap R \leq S \cap R = Q_P$. Thus $R/Q_P$ is centralized by $(O^p(P)/Q_P)^{\mathcal{P}}(P)$. Thus $RQ_P/Q_P \leq Z(K^p/P)Q_P/Q_P$. Since $P$ is narrow, we have either $P$ is soluble or $O^p(P)Q_P/Q_P$ is quasisimple. □

Theorem 2.8. (simple) Suppose that $P \in \mathcal{P}_H(S)$ is narrow and isolated in $H$ with respect to $A$. Assume that $Q_H = 1$ and set $Y = (O^p(P))^{H}$. Then either $Y = O^p(P)$ or $Y$ is quasisimple.

Proof. We may assume that $Y \neq O^p(P)$. Set $M = P \cap B$. The by 2.5(b) $M$ is the unique maximal subgroup of $P$ containing $S$. Put $R = \bigcap M^p$. By 2.5(i) and 2.5(j), $Y$ is semisimple and for any component $K$ of $Y$, $Y = \langle K^S \rangle = \langle K^M \rangle$ and $K \cap P \notin B$. If $K$ is normalized by $S$, then $Y = \langle K^S \rangle = K$ is quasisimple and we are done. Hence we assume that $K$ is not normalized by $S$ and look for a contradiction. Suppose first that $O^p(P)R/R$ is a non-abelian simple group. Then, as $(K \cap P)R/R$ is normalized by $O^p(P)R/R$, we have $(K \cap P)R \leq O^p(P)R$. Now selecting $s \in S$ such that $K^s \neq K$, we have $(O^p(P)R/R)^s \leq [K, K^s]R/R = (K \cap K^s)R/R$ which is abelian, a contradiction. Therefore, $O^p(P)R/R$ is an elementary abelian $t$-group for some prime $t \neq p$ and $O^p(P)$ is $p$-closed. Hence $P$ is a $\{t, p\}$-group. We have that $O^p(K \cap P) \leq X$ and so $X = (O^p(K \cap P)^S)$. Put $R^s = Z(Y) \cap X$ and $D = O^p(K \cap P)$. Then $X \cap Z(Y)$ is normal in $P$ and contained in $B$. Therefore $R^s \leq R$. Note that, as $Y$ is semisimple,

\[
X/R^s \cong XZ(Y)/Z(Y) \cong \prod_{T \in D^M} T.
\]

For a group $L$, let $\Phi_p(L)$ denote the full preimage of $\Phi(L/O_p(L))$. Then, as $P$ is narrow and soluble, $\Phi_p(X) \leq R \cap X$ and, as $P$ is $p$-minimal Maschke’s Theorem implies that $\Phi_p(X) = R \cap X$. On the other hand

$\Phi_p(X/R^s) = \prod_{T \in D^M} \Phi_p(T)$

and so we conclude that

$XR/R \cong X/X \cap R = \prod_{T \in D^M} T/\Phi_p(T)$.

Since $P$ is narrow, it must be that $D^M = D$ and so $K$ is normalized by $M$, a contradiction. □

¿From Section ?? onwards we will be investigation specific simple groups with an eye to showing that they have or do not have an isolated narrow $p$-minimal parabolic subgroup. The next few results in this section will be applied to proper subgroups of such groups. We continue the notation from the previous lemmas. In particular, if $P \in \mathcal{P}_G(S)$ is narrow and isolated, then we set $Y = (O^p(P))^{G}$.

Lemma 2.9. (comps) Suppose that $G$ is a group, $O_p(G) = 1$, $F(G) = C_G(E(G))$ and $G$ operates transitively on the components of $G$. If $P \in \mathcal{P}_G(S)$ is narrow and isolated, then $E(G) = Y$ and, in particular, there is exactly one component in $G$. 
Lemma 2.10. Suppose that $P \in \mathcal{P}_G(S)$ is isolated in $G$. Then $C_B(Y)$ is normal in $G$ and if $X \subseteq C_B(Y)$ is normal in $G$, then

(a) $\quot PX/X \in \mathcal{P}_G(SX/X)$ is isolated in $G/X$; and
(b) if $P$ is narrow, then $PX/X$ is narrow.

Proof. We first of all note that $G = BY$. So $C_B(Y) = B \cap C_G(Y)$ is normalized by $G = BY$. Now suppose that $X \subseteq C_B(Y)$ is normal in $G$. We claim that $PX/X$ is a $p$-minimal parabolic subgroup which is narrow and isolated in $G/X$. Suppose that $\overline{R} \in \mathcal{P}_G(XS/X)$ is not contained in $B/X$. Let $R$ be the full preimage of $R$ in $G$. Then $R \supseteq S$ and $R \not\leq B$ so $P \leq R$ by 2.5(a). Furthermore, as $\overline{R} \in \mathcal{P}_G(XS/X)$ and $B \supseteq X$, $B \cap R$ is the maximal subgroup $R$ containing $XS$. Since $P \not\subseteq B$, we infer that $R = PX$ and consequently $\overline{R} = PX/X$ is the unique $p$-minimal parabolic subgroup of $G/X$ not contained in $B/X$. Let $U$ be the full preimage of $O_p(\overline{R})$ and let $U_p = U \cap S$. We have $R = N_R(U P)$ by the Frattini lemma. In particular, as $R \not\leq B$ and $U \leq XS \leq B$, $N_R(U P) \not\leq B$. Since $S \leq N_R(U_p)$, it follows that $P \leq N_R(U_p)$ and so $U_p \leq Q_P$. Since $Q_B \not\leq Q_P$, it follows that $Q_B \not\leq U_p = U \cap S$. Therefore, $Q_B X \not\leq UX/X$ and $PX/X$ is isolated in $G/X$. This proves (a).

Suppose that $P$ is narrow. Then $PX/X \in \mathcal{P}_G(S)$ by (a). We have so $PX/X \approx P/P \cap X$ and, putting $F = \bigcap(B \cap P)^P$, we have $P \cap X \leq F$. It follows that $\quot P^P/F/F \cong \quot P/P(X)/(FX/X)$ and so $PX/X$ is narrow.

Corollary 2.11. Assume that $P \in \mathcal{P}_G(S)$ is narrow and $p$-restricted. Let $Y = \langle \quot P^P \rangle$ and assume that $Y = \quot P^P$ is soluble. If the holomorph of $Y$ is soluble but not abelian, then $G'$ is not perfect.

Proof. We have $G/C_B(Y)$ is isomorphic to a subgroup of the holomorph of $Y$ which is soluble but not abelian.

Lemma 2.12. Suppose that $P \in \mathcal{P}_G(S)$ is narrow and isolated, $L \subseteq H$ with $L$ soluble and $C_H(L) \subseteq L$. Assume also that $O_p(H) = 1$. Then $\quot P^P \subseteq H$ and $P$ is soluble.

Proof. Set $Y = \langle \quot P^P \rangle$. Then $Y = \quot P^P$ or $Y$ is quasisimple. Assume that $\quot P^P$ is not soluble. Then, because $Y \subseteq H$, $O_p(Y) = 1$ and so when $Y = \quot P^P$ we also have that $Y$ is quasisimple by 2.7. Thus in any event $Y$ is quasisimple. Therefore, as $L$ is soluble, $L \cap Y$ is soluble and so $L \cap Y \leq Z(Y)$. Hence $[L, Y, Y] = 1$ and the three subgroup lemma gives $Y = [Y, Y] \leq C_H(L) \leq L$, a contradiction.

Lemma 2.13. Suppose that $H$ is a group and $A \leq H$. Set $L = \langle A^H \rangle$. Assume that $AF(L) \leq L$, $C_H/F(L)(L/F(L)) = 1$ and that there exists $h \in H$ such that $[A^h, A] \leq F(L)$. If $Y \leq H$, then $Y/F(Y)$ is not a non-abelian simple group.
Proof. Suppose that \( Y/F(Y) \) is a non-abelian simple group. Since \( Y \leq H, F(Y) \leq F(H) \) and since \( [F(H), L] \leq F(L), C_{H/F(L)}(L/F(L)) = 1 \) implies that \( F(H) = F(L) \). Now \( L \cap Y \leq Y \) and so either \( Y \leq L \) or \( L \cap Y = F(Y) \). In the latter case we have \([L, Y] \leq F(L)\) which means that \( Y \leq F(L)\), a contradiction. Therefore, \( Y \leq L \). Since \( F(L)A \) is normal in \( L \), we either have \( AF(L) \cap Y = F(Y) \) or \( AF(L) \geq Y = Y \). In the latter case we have \( Y \leq AF(L) \cap A^hF(L) \) where \( h \) is as in the statement of the lemma. This gives
\[
[Y, Y] \leq [AF(L), A^hF(L)] \leq F(L),
\]
which is a contradiction. Therefore, \( AF(L) \cap Y = F(Y) \leq F(L) \). But \( Y \leq L \) and so \( Y \) normalizes \( AF(L) \) and this means that \([AF(L), Y] \leq AF(L) \cap Y \leq F(L)\). Thus \([\langle A^h \rangle, O^p(P)] \leq F(L) \) and, since \( C_{H/F(L)}(L/F(L)) = 1 \), we conclude that \( Y \leq F(L) \) and once again have a contradiction. \( \square \)

Corollary 2.14. [P soluble] Suppose that \( H \) is a group, \( O_p(H) = 1 \) and \( A \leq H \). Set \( L = \langle A^H \rangle \).
Assume that \( AF(L) \leq L, C_{H/F(L)}(L/F(L)) = 1 \) and that there exists \( h \in H \) such that \([A^h, A] \leq F(L)\).
If \( P \in P_H(S) \) is narrow and \( P \) is restricted in \( H \), then \( O^p(P) \leq H \) and \( O^p(P) \leq F(L) \).

Proof. Let \( M \) be a maximal subgroup of \( P \) containing a Sylow p-subgroup \( S \) of \( P \) and set \( R = \bigcap M^p \). Let \( Y = \langle O^p(P)^H \rangle \). Then by (**) either \( Y \) is quasisimple or \( Y = O^p(P) \). However (**) implies that \( Y/F(Y) \) is not simple. So \( Y = O^p(P) \). Hence \( O_p(P) = 1 \) and \( R \) is nilpotent by (**). Since \( P \) is narrow (**) implies that \( O^p(P)/F(O^p(P)) \) is abelian and hence \( p \)-closed. But then \( O^p(P) \) is a \( t \)-group for some prime \( t \) and \( O^p(P) \leq F(L) \) as claimed.

Corollary 2.15. [P soluble2] Suppose that \( H \) is a group, \( O_p(H) = 1 \) and \( A \leq H \). Set \( L = \langle A^H \rangle \).
Assume that \( AF(L) \leq L, C_{H/F(L)}(L/F(L)) = 1 \) and that there exists \( h \in H \) such that \([A^h, A] \leq F(L)\).
If \( L \) is perfect and \( F(L) = Z(L) \), then \( H \not\leq R_p \).

Proof. Assume that \( H \in R_p \). By the previous corollary we have that \( O^p(P) \leq F(L) \). Let \( B = \langle P_H(S) \setminus \{P\} \rangle \). Then, as \( H \in R_p \), \( BO^p(P) = H \) and \( B < H \). As \( O^p(P) \leq Z(L) \leq L, L = L \cap BO^p(P) = (L \cap B)O^p(P) \) and \( L \cap B < L \). Therefore \( L' \leq ((L \cap B)Z(L))' \leq L \cap B < L \), a contradiction. \( \square \)

Lemma 2.16. [non-local p-minimal] Suppose \( G \geq H \geq S \) and \( Q_H = 1 \). Then \( H \geq P \).

Proof. Suppose \( H \leq B \). Then \( Q_B \leq Q_H = 1 \), a contradiction. Thus \( H \not\leq B \) and \( H \geq P \) by 2.5(a). \( \square \)

I don’t think that this next one is used.

Lemma 2.17. [no cubic] Suppose that \( Q_B \) is abelian and that no non-trivial normal subgroup of \( S \) acts non-trivially and cubically on \( Q_B \). Then \( P \) is not of characteristic \( p \).

Proof. Suppose not and let \( D \) be normal subgroup of \( P \) in \( Q_P \) minimal such that \([D, O^p(P)] \neq 1\). Then \( D = [D, O^p(P)] \leq O^p(P) \) and so, by the minimality of \( D \), \([D, D, D] \leq [D, D, O^p(P)] = 1\).
In particular, as \( Q_B \) normalizes \( D \), \([Q_B, D, D, D] = 1\) and so by assumption \([Q_B, D] = 1\). Since \( Q_B \not\leq Q_P \), we get \([D, O^p(P)] = 1\), which is a contradiction. \( \square \)

We finish this section with one final general lemma. It exploits the fact that isolated \( p \)-minimal parabolic subgroups are normalized by \( N_G(S) \).

Lemma 2.18. [nbs maximal] Suppose that \( P \in P_G(S) \) is isolated in \( G \). Suppose that one of the following holds:
1. \([1]\) \( N_G(S) \) acts irreducibly on \( S \).
2. \( N_G(S) = B \).
3. \( N_G(S) \) is contained in a unique maximal subgroup of \( G \).
Then \( P \leq G \)

Proof. Suppose (1) holds. Since \( N_G(S) \leq B \) we get that \( Q_B = S \) and so (2) holds.

Suppose (2) holds. Then, by 2.5(c), \( N_G(S) \) is a maximal subgroup of \( G \) and so (3) holds.

So we may assume that (3) holds. Since \( N_G(S) \leq B \), we get that \( B \) is the unique maximal
subgroup of \( G \) containing \( N_G(S) \). Since \( N_G(S)P \leq N_G(P) \) and \( P \not\leq B \), we have \( G = N_G(P) \) and so \( P \leq G \). \(\square\)

3. General observations

From here on we wish to determine the possibilities for \( Y \) in 2.8, given that we know that \( O^p(P/Q_P) \) is a rank 1 Lie type group. So we shall assume the following hypothesis:

Hypothesis 3.1. [hyp] \( p \) is a prime, \( G \) is a group with \( Q_G = 1 \) and
(a) \( X = O^p(G) \) is a non-abelian simple \( K \)-group; and
(b) \( B \) there is a \( P \in P_G(S) \) such that \( P \) is narrow and isolated in \( G \) (with respect to \( A \)) and
\( O^p(P/Q_P) \) is a rank 1 Lie type group.

We next establish some elementary consequences of 3.1

Lemma 3.2. [s cyclic] If \( S \) is cyclic, then \( \langle A^p \rangle \leq G \).

Proof. Suppose that \( |SQ_P/Q_P| = p \), then, as \( A \not\leq Q_P \), \( S = A = Q_B \) and the result follows from 2.18. So assume that \( |SQ_P/Q_P| > p \), then as \( S \) is cyclic, we have \( P/Q_P \cong \text{Suz}(2) \) or \( 3^{1+2} \). In particular, \( p = 2 \) and so transfer (see [?, 37.7]) implies that \( O^2(G) \) is group of odd order. Therefore the lemma follows from 2.5(e). \(\square\)

Proposition 3.3. [GL]/Gorenstein Lyons] Suppose that \( X \) is a finite simple \( K \)-group and that \( S \) is a Sylow \( p \)-subgroup of \( X \). If \( S \) is abelian, then \( N_G(S) \) acts irreducibly on \( \Omega_1(S) \).

Proof. See [3, 12-1, pg 158]. \(\square\)

This allows us easily to establish the following lemma.

Lemma 3.4. [SinXnotabl] If \( G \) satisfies 3.1 and \( X \) has abelian Sylow \( p \)-subgroups, then \( G \not= X \).

Proof. Suppose that \( G = X \) and \( X \) has abelian Sylow \( p \)-subgroups. Then, as \( Q_B > 1 \), (??) and \( N_G(S) \leq B \) implies that \( \Omega_1(S) \leq Q_B \) and \( B = N_G(\Omega_1(S)) \). If \( Q_P > 1 \), then as \( Q_P \) is normalized by \( N_G(S) \), we have \( \Omega_1(Z(Q_P)) = \Omega_1(S) \). But then \( P \not\leq B \), a contradiction. Thus \( Q_P = 1 \). Since \( O^p(P/Q_P) \) is a rank 1 Lie type group, we have that \( S \) is elementary abelian. Hence 2.18 applies and we have a \( P \leq G \), a contradiction. \(\square\)

Lemma 3.5. [not char] Suppose that \( P \) is not of characteristic \( p \) and \( Q_B \) is abelian. Then \( Q_B \) is elementary abelian. (except maybe for \( \text{Sz}(2) \) or \( \text{Ree}(3^f) \)).

Proof. Since \( Q_B Q_P/Q_P \) is an abelian normal subgroups of \( (P \cap B)/Q_P \) we see (except for \( \text{Sz}(2) \)) that \( \Phi(Q_B) \leq Q_P \) and so \( \Phi(Q_B) \) is normal in \( P \) and in \( B \). Hence \( Q_B \) is elementary abelian. \(\square\)

Lemma 3.6. [irr on \( z(s) \)] Suppose that \( P \) is not of characteristic \( p \), \( N_G(S) \) is irreducible on \( \Omega_1(Z(S)) \) and \( q > 3 \). Then \( Q_P = 1 \) and \( |Q_B| = q \).
These facts are well known. But see for example [6, 2.9.1].

\textbf{Proof.} By 3.5 $Q_B$ is elementary abelian. Suppose $q > 3$. Then $O^p(P) \cap Q_P = 1$. By irreducibility of $N_G(S)$ on $\Omega_1 Z(S)$ we get $Z(S) \leq O^p(P) \cap S$ and so $\Omega_1 Z(S) \cap Q_P = 1$. Hence $Q_P = 1$. \hfill \square

\textbf{Lemma 3.7. [norm]} If $P$ is not soluble, then $N_{O^p(P)}(S \cap O^p(P)) \leq B$ and $Q_B Q_P \leq N_{O^p(P)}(S \cap O^p(P)) S$.

\textbf{Proof.} Let $R = N_{O^p(P)}(S \cap O^p(P))$. Since $R$ is normalized by $S$ and, as $P$ is not soluble, $RS \not\leq P$, $R \leq B$ by 2.5(a). This of course then gives $Q_B Q_P \leq RS$. \hfill \square

\textbf{Lemma 3.8. [QBnX=1]} Suppose $Q_B \cap O^p(P) = 1$. Then $P$ is soluble and, in particular, $p \leq 3$.

\textbf{Proof.} Assume that $Q_B \cap X = 1$ and put $R = N_{O^p(P)}(S \cap O^p(P))$. Suppose that $P$ is not soluble. Then by 3.7, $R$ normalizes $Q_B$. Now $[Q_B, R] \leq O^p(P) \cap Q_B \leq X \cap Q_B = 1$. Therefore, $Q_B Q_P / Q_P \leq Z(O^p(RS/Q_B))$. Since, from the structure of $P/Q_P$, $C_{P/Q_P}(RQ_P/Q_P) = 1$, we infer that $Q_B \leq Q_P$, which is a contradiction. \hfill \square

\textbf{Lemma 3.9. [order8]} If $Q_B \leq Z(S)$ and $Z(S)$ is cyclic, then $|S| \leq p^3$.

\textbf{Proof.} As $Q_B \leq Z(S)$, $[Q_P, Q_B] = 1$ and so $P$ is not $p$-constrained. Therefore $\Omega_1(Q_B) = \Omega_1(Z(S)) \cap Q_P = 1$ and so $Q_P = 1$. Since $Q_B Q_P$ is normalized by $N_{O^p(P)}(S \cap O^2(P))$, the structure of the rank 1 Lie type groups gives $O^p(P)Q_P/Q_P$ is defined over $GF(p)$. But then $|S| = |S/Q_P| \leq p^3$, as claimed. \hfill \square

4. Alternating groups

\textbf{It looks like we can do this for narrow $p$-isolated $P$ and not insist on Lie type. I guess it lets in $\text{Sym}(p)$ and $\text{Sym}(9)$.}

In this section we determine those groups $G$ are $p$-restricted and have $O^p(G) = F^*(G)$ an alternating group of degree at least 5. Among the small alternating and symmetric groups there are a number of isomorphisms with the classical groups and these examples always lead to $G$ being $p$-restricted for the appropriate prime. Before proving our main result on the symmetric groups we recall these isomorphisms.

\textbf{Lemma 4.1. [smallalts]} We have the following isomorphisms

(a) \text{Sym}(3) \cong \text{SL}_2(2);
(b) \text{Alt}(4) \cong \text{PSL}_2(3);
(c) \text{Alt}(5) \cong \text{SL}_2(4) \cong \text{PSL}_2(5);
(d) \text{Alt}(6) \cong \text{PSL}_2(9);
(e) \text{Sym}(6) \cong \text{Sp}_4(2);
(f) \text{Alt}(8) \cong \text{PSL}_4(2);
(g) \text{Sym}(8) \cong \text{O}_7^+(2).

\textbf{Proof.} These facts are well known. But see for example [6, 2.9.1]. \hfill \square

\textbf{Lemma 4.2. [altcase]} Suppose that $p$ is a prime, $G$ is a group such that $F^*(G) = O^p(G) \cong \text{Alt}(n)$ with $n \geq 5$ and that $G$ is not isomorphic to a Lie type group defined in characteristic $p$. If $G$ is $p$-restricted, then $p = 2$ and $n = 12$. Furthermore, in this case if $P$ is $p$-restricted in $G$, then $B$ is the stabilizer of a system of imprimitivity with blocks $\{\Omega_1, \Omega_2, \Omega_3\}$ of size 4 and $P$, which is isomorphic to a subgroup of index at most 2 in $(\text{Sym}(2) \wr \text{Sym}(4)) \times (\text{Sym}(2) \wr \text{Sym}(2))$, is contained in $\text{Stab}(\Omega_1 \cup \Omega_2) \times \text{Stab}(\Omega_3)$.
Proof. Let $X = F^*(G) = O^p(G) \cong \text{Alt}(n)$ with $n \geq 5$, $S \in \text{Syl}_p(G)$ and assume that $P \in \mathcal{P}_G(S)$ is $p$-restricted in $G$. Because of 3.2 we may assume that $S$ is not cyclic and so in particular, $n > 2p - 1$. Since, for $p \leq 3$, Alt(6) is a Lie type group in characteristic $p$, we may assume that $n \geq 7$. Suppose that $n = 7$. Then $p \in \{2, 3\}$. When $p = 3$, then $N_G(S)$ operates irreducibly on $S$ so 2.18 delivers $G = P$, a contradiction. Assume that $p = 2$. If $G = \text{Alt}(7)$, then 2.16 implies $P \leq \text{Alt}(6) \in M_G(S)$ and $P \leq \text{Sym}(5) \in M_G(S)$, which is impossible as Sym(5) is $p$-minimal. For $G = \text{Sym}(7)$, the same argument shows that $P \leq \text{Sym}(6)$ with $P = \text{Sym}(2) \wr \text{Sym}(3)$ and $B = \text{Sym}(5) \times \text{Sym}(2)$. But $P$ must also be contained in $\text{Sym}(3) \times \text{Sym}(4)$ and we have a contradiction.

Now we assume that $n \geq 8$ and that if $n = 8$, then $p \neq 2$ (as Alt(8) $\cong \text{SL}_4(2)$). We consider $G$ acting on the set $\Omega = \{1, \ldots, n\}$. Set $Z = \Omega_1(1 \cup \Omega_B)$). Suppose first that $B$ operates primitively on $\Omega$. Then $Z$ operates transitively on $\Omega$ and, as $Z$ is abelian, $Z$ acts regularly on $\Omega$ and, in particular, $|\Omega| = |Z| = p^a$ for some $a \in \mathbb{N}$. As $n > p$, we have $a > 2$. Select a $p$-cycle $x \in S$ (or when $p = 2$ a product of two transpositions). Then

$$|\text{Fix}_\Omega(x)| = p^a - p = |C_Z(x)| \leq p^{a-1}$$

(2$^a - 4 \leq 2^{a-1}$ when $p = 2$) which has no solution in for our values of $n = p^a$. Therefore, $B$ does not act primitively on $\Omega$.

Assume that $B$ is not transitive on $\Omega$ and let $\Omega_1, \Omega_2$ be proper subsets of $\Omega$ fixed by $B$ with $|\Omega_2| = k \leq \frac{n}{2}$. So $B \leq \text{Stab}_G(\Omega_1) \times \text{Stab}_G(\Omega_2)$. Since $B$ is a maximal subgroup of $G$ we must have $B = \text{Stab}_G(\Omega_1) \times \text{Stab}_G(\Omega_2)$ and $k \neq \frac{n}{2}$ (otherwise $B$ would not be maximal in $G$). Since $B$ is a $p$-local subgroup, we infer that Sym($\Omega_1$) or Sym($\Omega_2$) must be a $p$-local subgroup. Thus, as $n \geq 8$ and $k \neq \frac{n}{2}$, we have that $k = 2, 3, 4$ and $Q_B = O_p(\text{Sym}(\Omega_2))$. Now let $T = \langle Q_B^g \mid g \in G, Q_B^g \subseteq S \rangle$. Then $Q_B^g \leq T \leq S$. By 2.5(d), we have that $N_G(T) \leq B$, but we have $N_G(T) \geq O_p(\text{Sym}(\Omega_2))|\text{Sym}(n/|\Omega_2|)|$ which plainly does not normalize $Q_B$. Thus $B$ operates transitively but not primitively on $\Omega$. Let $B = \{\Theta_1, \ldots, \Theta_r\}$ be a system of imprimitivity for $B$ on $\Omega$ with $|\Theta_i| = k$. Then $B \leq \text{Stab}_G(\Omega)$ and, since $B$ is a maximal subgroup of $G$ and $B$ is a $p$-local subgroup of $G$, we have $B = \text{Stab}_G(\Omega)$ and $k \in \{2, 3, 4\}$. Suppose that $k = 2$ or $4$. Then $p = 2$. Write the 2-adic decomposition of $n$ as $n = 2^{a_1} + 2^{a_2} + \ldots + 2^{a_i}$ and notice that as $n \geq 8$, $a_1 \geq 3$. For $i = 1, \ldots, l$, let $S_{a_i}$ represent a Sylow 2-subgroup of Sym($2^{a_i}$). Then take $S = S_{a_1} \times S_{a_2} \times \ldots S_{a_i} \cap G$ as our “standard” Sylow 2-subgroup. If $k = 2$, we note that the subgroup $H := \text{Sym}(4) \wr \text{Sym}(2) \wr \cdots \wr \text{Sym}(2) \times S_{a_2} \times \ldots S_{a_i} \cap G$ is not contained in $B$ and consequently must contain $P$. Suppose that $n \geq 9$ or $G = \text{Sym}(n)$. Then $H$ is actually $p$-minimal and we have that $H = P$, but $H$ is not narrow. Thus we have that $G = \text{Alt}(n)$ and $n \geq 9$. So we have that $S \leq \text{Alt}(8)$ and $G = P$ is 2-minimal, a contradiction to the uniqueness of $P$ (as $P \leq H$ also). Next assume that $k = 4$. Then set $H := \text{Sym}(2) \wr \text{Sym}(4) \wr \text{Sym}(2) \wr \cdots \wr \text{Sym}(2) \times S_{a_2} \times \ldots S_{a_i}$. Again we have that $H \not\leq B$ and so $P \leq H$. Furthermore, if $a_1 \geq 4$, $H$ is 2-minimal and so $H = P$ contradicting the structure of $P$. Thus $a_1 = 3$ and since $k = 4$ divides $n$ we infer that $n = 12$, and we have the example listed in the lemma.

Finally suppose that $k = 3$. Then let $n = b_13^{a_1} + \ldots b_3^{a_i}$ be the 3-adic decomposition of $n$. Let $S_{a_i}$ be a Sylow 3-subgroup of Sym($3^{a_i}$) and $S_{b_1}^{a_i}$ be a Sylow 3-subgroup of Sym($b_13^{a_i}$). Then we may suppose that $S = S_{a_1}^{a_1} \times \cdots \times S_{a_i}^{a_i}$. We first note that Alt(9) itself is 3-minimal and so it is impossible that $G = \text{Alt}(9)$. Then taking $H := \text{Sym}(9) \wr \text{Sym}(3) \wr \cdots \wr \text{Sym}(3) \wr (\text{Sym}(b_1) \times S_{a_2}^{a_2} \times \cdots \times S_{a_i}^{a_i} \cap \text{Alt}(n)$ is not contained in $B$. Thus $H$ contains $P$ and we have a contradiction via 2.9 if either $a_1 > 3$ or $b_1 > 1$ and otherwise we use the fact that Sym(9) is 3-restricted. \qed
5. Lie type groups in characteristic not \( p \)

In this section we begin our investigation of the groups which satisfy 3.1 and have \( X \) a Lie type group defined in characteristic \( r \neq p \). Our objective here is to prove two general results.

**Lemma 5.1.** Suppose that \( X \) is a simple group of Lie type defined over \( GF(r^n) \) and assume that for every parabolic subgroup \( R \) of \( X \), \( G = N_G(R)X \). If \( S \cap X \) is contained in a proper parabolic subgroup \( R \) of \( G \), then \( G \) is not \( p \)-restricted.

**Proof.** Suppose that \( R \) is a parabolic subgroup of \( X \) and assume that \( R \) contains \( S \cap X \). Then as \( N_G(R)X = G \), we have that \( N_G(R) \) contains \( S \). It follows that \( S \) normalizes a standard Levi-complement \( L \) of \( R \) and thus \( S \leq LO_x(R) \) and \( S \leq LO_x(R)^x \) for \( x \in G \) with \( O_x(R) \cap O_x(R)^x = 1 \).

Setting \( H_1 = SO_x(R) \) and \( H_2 = SO_x(R)^x \) we have that \( O_p(H_1) = O_p(H_2) = 1 \) and so 2.16 implies that \( P \leq H_1 \cap H_2 \). But then \( O^p(P) \leq O^p(H_1) \cap O^p(H_2) = O_x(H_1) \cap O_x(H_2) = 1 \) and we have a contradiction. Thus the lemma holds.

**Lemma 5.2.** Suppose that \( X = O^p(G) \) is a Lie type group \( G(r^n) \) with \( p \neq r \). If \( X \) has abelian Sylow \( p \)-subgroups and \( P < G \), then \( p \leq 3 \) and \( G > X \).

**Proof.** If \( X = G \), we simply cite 3.4 to obtain \( G = P \). So assume that \( G > X \) and \( p \geq 5 \). Since \( O^p(P) \leq X \), \( O^p(P) \) has abelian Sylow \( p \)-subgroups and so, as \( p \geq 5 \), \( O^p(P)Q_P \cong SL_2(p^2) \) or \( PSL_2(p^2) \) for some \( b \geq 1 \) and \( P/Q_P \) is the same group extended by a cyclic group of field automorphisms. Since the Sylow \( p \)-subgroups of \( O^p(P) \) are abelian, we infer further that \( O^p(P) \cong SL_2(p^2) \) or \( PSL_2(p^2) \) and that \( [Q_P, O^p(P)] = 1 \). Since \( N_G(S \cap O^p(P)Q_P) \leq B \), \( Q_BQ_P \in Sym_{p}(O^p(P)Q_P) \) and \( \Phi(Q_B) \leq Q_P \). Therefore \( \Phi(Q_B) \cong (B, P) \), so \( Q_B \) is elementary abelian. Since \( Q_B \) is normalized by \( N_G(S \cap O^p(P)) \) and \( N_G(S \cap O^p(P)) \) acts irreducibly on \( S \cap O^p(P) \) and centralizes \( Q_P \), we infer that \( Q_B \cap O^p(P) = S \cap O^p(P) > 1 \). In particular, \( Q_B \cap X > 1 \) so \( Q_B \cap \Omega_1(S \cap X) > 1 \) and 3.3 implies that \( Q_B \cap X = \Omega_1(S) \).

Set \( S_0 = S \cap X \) and consider \( S_0O^p(P) \). Since \( S_0 \) is abelian and \( O^p(P) \cong SL_2(p^2) \) or \( PSL_2(p^2) \), we infer that \( \Phi(S_0) \leq Q_P \). But then \( \Phi(S_0) \cap \Omega_1(S_0) \leq N_X(S_0) \), so 3.3 implies that \( \Omega_1(S_0) \leq Q_P \). But then \( \Omega_1(S_0) \not\leq (B, P) = G \) so \( S_0 \) is elementary abelian.

Since \( p > 3 \), \( G/X \) does not contain a graph automorphism. If \( X \) has a diagonal automorphism of order \( p \), then using \( p > 3 \) again, we have that \( p \) divides \((n, \Phi_1(r^n)) \) and \( X \cong SL_n(r^n) \) or \( p \) divides \((n, \Phi_2(r^n)) \) and \( X \cong PSU_n(r^n) \). In the first case we have that \( X \) contains a monomial subgroup \( \frac{1}{n(r^n - 1)}(r^n - 1)^n.\text{Sym}(n) \) and in the second case \( \frac{1}{n(r^n - 1)}(r^n - 1)^n.\text{Sym}(n) \) and, as \( p > 3 \), these subgroups exhibit that fact that \( S_0 \) is non-abelian. Thus \( X \) has no diagonal automorphisms of order \( p \). So \( G/X \) must consist of field automorphisms. As \( p > 3 \), this means that \( r^n = r^{ap_0} \) for some integer \( a_0 \). Letting \( m \) be the order of \( r^{ap_0} \) mod \( p \), [3, pg. 112] presents the relationship

\[
\frac{\Phi_m(r^{ap_0})}{\Phi_m(r^{ap_0})} \equiv \Phi_m(r^{ap_0})^{\phi(p)} \quad \text{(mod \( p \))}
\]

Finally, [3, 10-1 (2)] shows that the exponent of \( S_0 \) is at least \( p^2 \) (using here that \( X \) has no diagonal automorphisms of order \( p \)) and so the Sylow \( p \)-subgroups of \( X \) are not elementary abelian, which is a contradiction. This final contradiction finishes the proof of the lemma.

We remark here that there are examples of \( 3 \)-restricted groups \( G \) with \( X = O^p(G) \) a simple Lie type group with abelian Sylow 3-subgroups and \( |G/X| = 3 \) (see 6.11).
6. Linear and unitary groups

In this section we investigate the linear and unitary groups. To simplify our notation we use $GL_n^\epsilon(r^a)$ and $GL_n(r^a)$ to represent the general linear and unitary groups respectively. The notation $GL_n^\epsilon(r^a)$ denotes either of the groups. So here $\epsilon = \pm$. However we will also write $r^a - \epsilon$ and in this case we regard $\epsilon$ as $\pm 1$ according to $\epsilon = \pm$. Throughout this section $X \cong SL_n^\epsilon(r^a)$ and $p$ is a prime with $p \neq r$. Recall that $\Gamma L_n^\epsilon(r^a)$ has $X$ as a normal subgroup and includes all the diagonal and field automorphisms of $X$. In the event that $X \cong SL_n(r^a)$, we denote the inverse transpose automorphism by $\iota$ and, for case of notation, when $X \cong SU_n(r^a)$ or $SL_2(r^a)$, we take $\iota$ to be the trivial automorphism. Let $V$ represent the natural linear space when $X \cong SU_n(q)$ and the natural unitary space when $X \cong SU_n^\epsilon(q)$.

For the remainder of this section we take

$$X \leq \mathcal{G} \leq \Gamma L_n^\epsilon(r^a) : \langle \iota \rangle$$

with $\mathcal{G}/X$ a $p$-group and $\mathcal{G}$ non-soluble. Finally set $G = \mathcal{G}/F(\mathcal{G})$, $X = X F(\mathcal{G})/F(\mathcal{G})$ and for $S \in \text{Syl}_p(\mathcal{G})$, $S = SF(\mathcal{G})/F(\mathcal{G})$. Our objective is to determine for which values of $n$ and $r^a$, $G$ is $p$-restricted. The first few lemmas form the base of our final induction arguments.

Our first lemma helps us apply induction.

**Lemma 6.1. [subquot]** Suppose that $\mathcal{R} \leq F(\mathcal{G})$ is normal in $G$ and $P \in \mathcal{P}_{\mathcal{G}}(S)$ is $p$-restricted in $\mathcal{G}$. Then either $\mathcal{G}/ouR$ is $p$-restricted or $O^p(P) \leq \mathcal{R}$.

**Proof.** If $O^p(P) \not\leq \mathcal{R}$, then $\mathcal{RS} \leq B$ and $\mathcal{R} \leq C_{\mathcal{G}}(\mathcal{Y})$. Thus the result follows from 2.10. \[ \square \]

**Lemma 6.2. [sl2]** If $X \cong PSL_2(r^a)$ and $P$ is $p$-restricted in $G$, then one of the following holds:

(a) $p = 2$, $r^a \equiv 3, 5 \pmod{8}$, $G \cong PGL_2(r^a)$, $B = C_G(\Omega_1(Z(S)))$ and $P \cong \text{Sym}(4)$;

(b) $p + 1$ $p = 2$, $G \cong PGL_2(19)$, $B \cong \text{Sym}(4)$ and $P \cong \text{Dih}(40)$ with $P/O_2(P) \cong \text{Dih}(10)$;

(c) $p = 2$, $G \cong PSL_2(7)$, $PSL_2(9)$ and $B \cong P \cong \text{Sym}(4)$ or $G \cong \text{PSL}_2(9) \cong \text{Sym}(6)$ and $B \cong P \cong \text{Sym}(4) \times 2$;

(d) $p = 3$, $X \cong \tilde{G}_2(3) \cong PSL_2(8)\cdot 3 = P\text{F}L_2(8)$.

**Proof.** We first consider the case when $p = 2$. Let $S_0 = S \cap X$. Let $t$ be a central involution in $S_0$ and put $D = C_G(t)$. We shall often use the following straightforward observation.

1. If $P \leq D$, then $P/O_2(P) \cong \text{Sym}(3)$, $\text{Dih}(10)$ or $\text{Frob}(20)$ and $D \cap B$ is a maximal subgroup of $D$ of index 3 in the first case and 5 in the last two cases.

Suppose first that $|S_0| = 4$. Then $r^a \equiv 3, 5 \pmod{8}$ and consequently $a$ is odd. Therefore $X$ admits no field automorphisms of order 2 and $G$ is isomorphic to a subgroup of $PGL_2(r^a)$. If $G = X$, then 5.2 implies that $G = P$ and in this case (d) holds. Hence we have $G \cong PGL_2(r^a)$. So $S \cong \text{Dih}(8)$ and $N_G(S_0) \cong \text{Sym}(4)$. Assume that $P \leq D$. Then, as $B$ is a 2-local subgroup of $G$, we must have $B \leq N_G(S_0)$. Hence (1) implies that $|D| = 24$ if $P/O_2(P) \cong \text{Sym}(3)$ and $|D| = 40$ if $P/O_2(P) \cong \text{Dih}(10)$. Since $r^a \equiv 3, 5 \pmod{8}$ and $|D| = r^a \pm 1$, we infer that the second possibility occurs and that $G \cong PGL_2(19)$ which is possibility (b). Next suppose that have $P \not\leq D$. Then $B \geq D$ and, as $N_G(S_0) \cong \text{Sym}(4)$ is 2-minimal and is not contained in $B$, we have $P = N_G(S_0)$. This delivers the examples listed in part (a). (Here we may mention that if $a > 1$, then $P$ is contained in the subfield subgroups $PGL_2(r^{a/b})$ whenever $b$ divides $a$.)

Assume that $|S_0| \geq 8$ and let $F_1$ and $F_2$ be non-conjugate fours subgroups of $S_0$.

2 - $Q_P \cap X > 1$. 


Suppose that \( Q_P \cap X = 1 \). Then \( P \nleq D \) and so \( B \geq D \). Additionally, we have \( O_2(O_2(P)) \leq Q_P \cap X = 1 \) and so, as \( O_2(P/Q_P) \in L_1(p) \), \( O_2(P) \cong \text{SU}_3(2^n), \text{PSU}_3(2^n), \text{PSL}_2(2^n) \) for some \( n \in \mathbb{N} \) or we have one of our unusual cases \( O_2(P) \cong 3, 5, 3^4+2, 3^2.2, 2\cdot\text{SL}_2(4) \). Since \( S_0 \) is a dihedral group and the Sylow 3 and 5-subgroups of \( X \) are cyclic, either \( O_2(P) \cong \text{SL}_2(4) \) or \( P \) is soluble. In the latter case we have that \( |S/Q_P| \leq 8 \) \(|S| = 2^3|Q_P| \) leaving only the extreme case with \( P/Q_P \cong \text{PSU}_3(2) \cong 3^2 \): \( Q_3 \) as a possibility and this contradicts the fact that \( S/Q_P \cong S_0 \) is dihedral. So \( O_2(P) \cong \text{SL}_2(4) \) and again by considering orders we get \( P/O_2(P) \cong \text{SL}_2(4) : 2 \) and \( |S_0| = 8 \). Since \( \text{SL}_2(4) : 2 \) contains a subgroup \( F_1 = \text{Sym}(4) \) and since \( F_1 \) is 2-minimal, \( B \geq (P_1, D) \) and hence \( Q_B \cap X = 1 \). But then 3.8 implies that \( P \) is soluble, a contradiction.

3°. If \( P \nleq D \), then (c) holds.

Assume that \( P \nleq D \). Then necessarily \( B \geq D \). By \( 2^n \) we have that \( Q_P \cap X > 1 \). Therefore, as \( P \) does not centralize \( t \), we have \( Q_P \cap X \) is a fours group \( F \) of \( S \). Consequently \( S \) also normalizes \( F \) and so \( S_0 \) is a dihedral group of order 8. Furthermore, as in this case the Sylow 2-subgroups of \( \text{PGL}_2(r^a) \) are dihedral of order 32, we infer that \( G \leq \text{PSL}_2(r^a) \). Letting \( F_1 \neq F \) also be a fours group of \( S_0 \), we have \( H = N_G(F_1) \) is also a minimal parabolic subgroup (because \( H \) and \( P \) are conjugate in \( \text{PGL}_2(r^a) \)). Thus \( H \leq B \) also. But then \( D \cong \text{Dih}(8) \) or \( \text{Dih}(8) \times 2 \) and we infer that one of the possibilities in part (c) occurs.

We now assume that \( P \leq D \). Suppose that \( Q_B \cap X > 1 \). If \(|S_0| > 8 \), then \((P, B) \leq C_G(t) \) and we have a contradiction. So \(|S_0| = 8 \) and \( Q_B \cap X \) is a fours group \( F_1 \) of \( S_0 \). Furthermore, \( F_1 \) is normalized by \( S \). Let \( F_2 \) be a fours group of \( S_0 \) not equal to \( F_1 \). Then \( N_2 = N_X(F_2)S \) is 2-minimal and so as \( N_2 \nleq D \) and \( P \leq D \), \( N_2 \leq B \). But then \( 1 = O_2((N_1, N_2)) = Q_B \cap X \), a contradiction. Therefore, \( Q_B \cap X = 1 \) and in particular, \( S_0 \leq C_X(Q_B) \). It follows at once that \( G \) is not isomorphic to a subgroup of \( \text{PGL}_2(r^a) \) which has dihedral Sylow 2-subgroups. Hence \( X \) admits field automorphisms and consequently as \(|S_0| = 8 \), \( r^a \equiv \pm 7, 9 \pmod{16} \) and \( a \) is even. So in fact \( r^a \equiv 9 \pmod{16} \) and hence \( D \cap X \) has order \( r^a - 1 \). Notice that the centre of the Sylow 2-subgroup of \( \text{Aut}(X) \) has order 2. Let \( \theta \) be an automorphism of \( X \) which normalizes \( S \) and maps \( Q_B \) to \( Q_B^{\theta} \). Then \( D^\theta = D \) and \( N_G(Q_B^{\theta}) \) is also a 2-local subgroup and which is unequal to \( B = C_G(Q_B) \) (as \( Q_BQ_B^{\theta} \geq \langle t \rangle \)). Therefore, \( P \leq N_G(Q_B^{\theta}) \). But \( DS \) is invariant under \( \theta \) and so \( O_2(P) = O_2(PS) \leq N_G(Q_B^{\theta})^{|\theta|^{-1}} = B \), a contradiction.

Assume that \( p \geq 3 \). Since the Sylow \( p \)-subgroups of \( \text{PSL}_2(r^a) \) are abelian, \( p = 3 \) and \( G \times X \) by 5.2. Now \( O_3(P) \leq X \) has cyclic Sylow 3-subgroups. Since \( O_3(P) \in L_3(3) \), have \( O_3(P) \cong \mathbb{Q}_8, 2^2 \) or \( 2G_2(3)^{\prime} \) by considering the Sylow 2-subgroup structure of \( X \) we deduce that \( O_3(P) \cong 2^2 \) or \( r^a \) is a power of 2 and \( O_3(P) \cong 2G_2(3)^{\prime} \). If, in the latter case, \( r^a > 8 \), then \( O_3(P) \) is not normalized by \( N_G(S) \) and we have a contradiction. Thus if \( O_3(P) \cong 2G_2(3)^{\prime} \), we have \( G = P \) and we are done. So \( O_3(P) \cong 2^2 \) and \( P/Q_P \cong \text{PSL}_2(3) \). Now \( Q_P \cap X \) is cyclic and centralized by \( O_3(P) \). Since the centralizer of an involution in \( X \) is dihedral, we infer that \( Q_P \cap X = 1 \). Hence \( P \cap X \cong \text{PSL}_2(3) \) and so \( X \) has a cyclic Sylow 3-subgroup of order 3. Since \( X \) admits field automorphisms of order 3, we in fact have \( |S \cap X| \) is divisible by 9, a contradiction. \( \square \)

If \( p \) is odd then set \( d = \text{ord}(p, re) \) and if \( p = 2 \) put \( d = 1 \) if \( r^a \equiv 1 \pmod{4} \) and otherwise put \( d = 2 \). Define \( \overline{L}^\sigma = \text{GL}_d^\sigma(r^a) \cdot \text{Sym}(s) \times \text{GL}_d^\sigma(r^a) \) where \( s = \left[ \frac{d}{2} \right] \), put \( \overline{L} = \overline{L}^\sigma \cap \overline{G} \) and finally \( L = L/F(G) \). Then we may suppose that \( S \cap X \leq L \) and that \( L \) is normalized by \( S \) (notice that the natural realization of \( L \) is closed under inverse-transpose operation). We shall use similar notation for other imprimitive subgroups of \( G \). Let \( K^\sigma \) be the base group of \( L^\sigma \) and \( \overline{K} = K^\sigma \cap G \).

In the next lemma we use the well-known fact that the irreducible section of the natural \( GF(p) \) \( \text{Sym}(s) \)-permutation module has dimension \( s - 1 \) if \((s, p) = 1 \) and \( s - 2 \) if \( p \) divides \( s \). In the latter instance the permutation module is uniserial.
Lemma 6.3. [shn1] Suppose that $G \leq \text{GL}_n^+(r^a)$, $n = ds > 2$ and assume that $L \leq H$. If $O_p(H) > 1$, then either

(a) $[a]$ \quad $H > L$, $n = s = 4$, $d = 1$, $r^a = 4 + \epsilon$ and $\mathcal{P} \sim 4 \times 2^{1+4}(2 \times \text{Sym}(6)) \cap \mathcal{G}$; or

(b) $[b]$ \quad $H = LS$ and either $d = 1$ or $d = 2$, $r^a \in \{2, 3\}$ and $\mathcal{T} \cong \text{GL}_2^+(r^a) \cap \text{Sym}(s)$.

Proof. Assume that $d = 1$ and let $R$ be a normal $p$-subgroup of $L$. Put $R_0 = \Omega_2(R)$. If $\overline{R_0} \subseteq \overline{K}$, then as $R_0 \neq 1$, $|R_0| = p^{n-1}$ if $(p, s) = 1$ and $|R_0| \geq p^{s-2}$ if $(p, s) = p$. In either case, we have that the homogeneous components of $R_0$ on $V$ coincide with those of $K$ and consequently $H = LS$. So we may assume that $R_0 \notin \overline{K}$. This means that $K$ is a 2-group and $s = 4$ or 2. The case $s = 2$ has been ruled out by assumption and so we have that $s = 4$ and $X \cong \text{PSL}_4(r^a)$. Now by the definition of $d$, we have the 4 divides $r^a - \epsilon$ and consequently, as $[K, R_0] \leq K \cap R_0$ which is elementary abelian, we have that $r^a = 4 + \epsilon$ as claimed in (a). So suppose that $d = 2$. If $r^a = 2$, then an argument similar to the one above shows that $H$ normalizes $O_3(L)$ and we are done. So suppose that $r^a = 3$, let $R$ be a normal 2-subgroup of $H$. Then $R \leq O_2(L)$ and this is contained in the base group $\overline{K}$ of $L$. Consider $\overline{R_0} = \overline{R} \cap Z(\overline{K})$. Since $Z(\overline{K})$ is the natural GF(2) Sym$(s)$ permutation module for $\overline{L}$, we either have $|\overline{R_0}| = 2$ or $|\overline{R_0}| \geq 2^{n-1}$. If $|\overline{R_0}| > 2$, then the $\overline{K}$ and $\overline{R_0}$ have the same homogeneous components and we are done. So $\overline{R_0} \leq Z(G)$. But then $R_0 \cap Z(K) = 1$, a contradiction as $Z(K) \geq Z(S)$. Finally assume that $r^a \geq 3$ and that $d \geq 2$. Then as $L$ is a p-local subgroup and $K$ is non-soluble, we have that $O_p(L) \leq Z(K)$ and then, as $d \geq 2$, we have that $d = 2 = p$ and $r^a - \epsilon$ is not divisible by 4. We now argue as in the $r^a = 3$ case to obtain our contradiction.

Lemma 6.4. [QBX] Suppose that $n = ds$ and $B \geq LS$, then $Q_B \cap X > 1$.

Proof. Suppose that $Q_B \cap X = 1$. Then $[Q_B, L \cap X] = 1$. Notice that $\overline{Q_B}$ induces the same type of automorphism on each $\text{SL}_d^+(r^a)$ contained in $\overline{K}$ as it does on $X$. Suppose that $d > 2$ or $p$ is odd and $d \geq 2$. Then the Sylow $p$-subgroup structure of $\text{Out}(X)$ is identical to that of $\text{SL}_d^+(r^a)$ and so in these cases it is impossible for $Q_B \cap X = 1$. Assume that $p = 2$ and $d = 2$. Then the inverse transpose automorphism of $X$ induces an inner automorphism of $\text{SL}_2^+(r^a)$ but not of $\text{GL}_2(r^a)$ and thus in this case also it is impossible for $Q_B$ to centralize $K$. Assume that $d = 1$. In this case, as inversion is not a field automorphism, we deduce that the only possibility is that $Q_B$ induces a diagonal automorphism of $X$. But then $Q_B \leq L$ and the result now follows.

Lemma 6.5. [monomial1] Suppose that $t$ is a prime and $L \leq G$ is as above with $d = 1$ so that $L \cap X \sim \frac{1}{(r^a - \epsilon, n)}(r^a - \epsilon)^{n-1}.\text{Sym}(n)$ and $n \geq 4$. If $Y$ is a non-trivial elementary abelian $t$-group of rank at most 2 and $Y \leq L$, then $n = 4$ and $p = 2$.

Proof. This is the result of an elementary calculation.

Lemma 6.6. [monomial2] Suppose that $L \cap X \sim \frac{1}{(r^a - \epsilon, n)}(r^a - \epsilon)^{n-1}.\text{Sym}(n)$ with $n \geq 3$. Set $T = F(L)$. If $P \in \mathcal{P}_{LS}(S)$ is $p$-restricted in $LS$, then either

(a) $[a]$ \quad $T$ is a $p$-group; or

(b) $[b]$ \quad $P$ is soluble and $n = 3$.

Proof. If $P \in \mathcal{P}_{LS}(S)$ is $p$-restricted in $LS$ and $T$ is not a $p$-group. Then $O_p(LS) \cap X \leq O_p(L)$ which is abelian. Furthermore, $E(LS/O_p(LS)) = 1$, and so $O^p(P/O_p(LS))$ is normal and soluble in $LS/O_p(LS)$ by ???. Since $O^p(P) \leq O^p(LS) \leq X$, we have that $O^p(P/O_p(LS)) \leq T/O_p(LS)$. Because $T$ is abelian, we have $O^p(P) \leq L$ and, as $O^p(P)$ is a $t$-group for some prime $t \in \{2, 3\}$ of rank at most 2, we either have $n \leq 3$ or $n = 4$ and $t = 2$ by 6.5. The former case gives
Lemma 6.7. [monomial3] Suppose that $n = p \geq 5$ and $d = 1$, Then $LS$ contains every proper $p$-local subgroup of $G$ which contains $S$.

Proof. We have $T^p \cong (r^p - \xi) \cdot \text{Sym}(p)$. Let $R$ be a proper $p$-local subgroup of $G$ containing $S$. Set $Q_R = O_p(R)$ and $Q_L = O_p(L)$. Now $LS/F(L) \cong \text{Sym}(p) \times T$ where $T$ is a cyclic $p$-subgroup (which may be trivial) consisting of Frobenius automorphisms. In particular, we have $S/F(LS)$ is abelian and so $Q_R \leq LS$. If $V|_{Q_R}$ is not homogeneous, then the homogeneous components of $V|_{Q_R}$ coincide with the homogeneous components of $Q_L$ and we conclude that $R \leq L$ as claimed. If on other other hand $V|_{Q_R}$ is homogeneous, then $Q_R \leq Z(L)$. It follows that $[Q_L, Q_R, Q_R] \leq Z(L)$. In particular, we see that $Q_R$ is not inducing field automorphisms on $Q_L$. Furthermore, since $Q_E/\Phi(Q_L)$ is isomorphic to a section of the GF($p$)-permutation module for $\text{Sym}(p)$ of order at least $p^s - 1$, $p \geq 5$ and $[Q_L, Q_R, Q_R] \leq Z(L)$ imply that $Q_R \leq Q_M$. If $V|_{Q_R}$ is also homogeneous, then $Q_R \leq X$ and and $R = G$, a contradiction. Therefore, $V|_{Q_R}$ is not homogeneous and again we have that $R \leq M$. □

Lemma 6.8. [L3andU3] Suppose that $p = 2$, $X \cong \text{PSL}_3^c(r^a)$ and $P$ is $p$-restricted in $G$. Then one of the following holds.

(a) $[a]$ $G \cong \text{PSU}_3^c(r^a)$ with $s \equiv 3 \pmod{8}$, $P \cong 2 : \text{Sym}(4) \ast 4$ and $B \cong (s+1)^2 : \text{Sym}(3)$.
(b) $[b]$ $G \cong \text{PSU}_3^c(r^a)$ with $s \equiv 3 \pmod{8}$, $P \cong 2 : \text{Sym}(4) \ast Q_8$ and $B \cong (s+1)^2 : (2 \times \text{Sym}(3))$.
(c) $[c]$ $G \cong \text{PSU}_3^c(3) \cong \text{SL}_2^c(2)$, $P \cong 4^2 : \text{Sym}(3)$ and $B \cong 2 : \text{Sym}(4) \ast 4$.
(d) $[d]$ $G \cong \text{PSU}_3^c(3) \cong \text{SL}_2^c(2)$, $P \cong 4^2 : (2 \times \text{Sym}(3))$ and $B \cong 2 : \text{Sym}(4) \ast Q_8$.
(e) $[e]$ $G \cong \text{PSL}_3^c(r^a)$ with $s \equiv 1 \pmod{8}$, $P \cong 2 \times \text{Sym}(4) \ast Q_8$ and $B \cong (s-1)^2 : (2 \times \text{Sym}(3))$.

Proof. Let $S \in \text{Syl}_2(G)$ and assume that $P \in \mathcal{P}_2(S)$ is $p$-restricted. If $r^a = 3$, we use the ATLAS [2] to see that statements (a), (b), (c) and (d) hold. So assume that $r^a > 3$.

Set $T^p = \text{GL}_3^p(r^a) \times \text{GL}_3^p(r^a)$, $R = T^p \cap G$. Let $K_1$ be the component of $L_1$. Then $K_1$ is $S$-invariant. By 6.3, $O_2(K_1, L, S) = 1$ and so $P \leq K_1 S$ or $P \leq L_1 S$ (or both). Assume that $B \geq K_1 S$, then $Q_B \leq O_2(K_1 S) \leq \Omega_2(LS) \leq Q_P$, which is a contradiction (note here it could be that $r^a - \epsilon$ is a power of 2 and that $Q_B \nsubseteq X$ induces $i$). Thus $P \leq K_1 S$ and is $p$-restricted therein. Using (??) together with $s \equiv 1 \pmod{4}$ implies that $r^a \equiv \epsilon(5) \pmod{8}$ and that $P \nsubseteq LS$. Noting also that when $\epsilon = +$, $\overline{S \cap X}$ does not operate irreducibly on $V$, 5.1 implies that $G > X$. Thus we have the examples in (a), (b) and (e).

So suppose that $s \equiv -\epsilon \pmod{4}$. Then $L \cong \text{GL}_2^c(r^a)$ contains a Sylow 2-subgroup of $X$. Suppose that $R$ is a 2-local subgroup of $X$ which contains $S \cap X$. Then the decomposition of $V$ restricted to $\Omega_1(Z(Q_R))$ is preserved by $R$ and we see that $P \cap X \leq N_X(\Omega_1(Z(Q_R))) \leq (r^a - \epsilon)^2 : \text{Sym}(3)$ or $P \leq LS$. The former subgroup does not contain a Sylow 2-subgroup of $X$ and so we infer that every 2-local subgroup of $X$ which contains $S \cap X$ is contained in $LS$. Hence we cannot have both $Q_B \cap X > 1$ and $Q_P \cap X > 1$. Suppose that $Q_B \cap X = 1$. Then Lemma 3.7 implies that $P$ is soluble and that $P$ is not 2-constrained. Now $LS < L$ and contains a Sylow 2-subgroup of $G$. Thus $P \leq LS$. Then as $P$ is not 2-constrained, Lemma 6.2 implies that $s = 19$ and $G \cong \text{PGL}_2^+(19) : 2$ and that $(B \cap L)Z(L)/Z(L) \cong \text{Sym}(4)$. But then $Q_B$ normalizes $L/Z(L)$ and centralizes $(B \cap L)Z(L)/Z(L)$. It follows from the structure of $\text{Aut}(\text{PGL}_2(19))$ that $[Q_B, L] \leq Z(L)$, but then $[O^2(P), L/Z(L)] = 1$, a contradiction.
Next suppose that $Q_P \cap X = 1$. Then $B \geq N$ and so $|Q_B \cap X| = 2$. Since $|Q_B Q_P / Q_P|$ is also cyclic, we infer that $P/Q_P$ has Sylow 2-subgroups of order at most 8. Therefore, $|S| \leq 8|Q_P|$ so that $|S \cap X| \leq 8$, which is of course a contradiction.

**Lemma 6.9. [4dim]** Suppose that $p = 2$, and $X$ is isomorphic to $PSL_4^e(r^a)$. If $P \in \mathcal{P}_G(S)$ is 2-restricted, then $X \cong PSU_4(3)$ and the possibilities for $P$ and $B$ and $G$ are as described in Table 1. In particular, if $G \geq PGU_4(3)$, then $P \leq LS$.

**Proof.** Let $X = PSL_4^e(r^a)$. If $r^a = 3$, we inspect the Atlas [2] and see that in the case that $X = PSL_3(3)$ there is no candidate for $B$ and so we have that $X \cong PSU_4(3)$ and the possibilities for $B$ and $P$ follow. Thus assume that $r^a > 3$. Since $p = 2$, $\overline{T_1} = GL_3^e(r^a) \wr \text{Sym}(2) \cap G$ is normalized by $S$. Using 2.9 and $r^a > 3$ we infer that $B \geq L_1S$. Assume that $s - \epsilon \equiv 0 \pmod{4}$. Then $d = 1$ and $\overline{T_1} \neq T$. By 6.3, $B$ doesn’t contain $LS$ and so $P \leq LS$. Since $Q_B \leq O_2(L)$, we have $Q_B \cap X = \Omega_1(Z(S))$ has order 2 and again Lemma 3.9 finishes this case.

**Lemma 6.10. [su63]** Suppose that $X \cong PSU_n(3)$ with $n \in \{5, 6, 7\}$. If $P \in \mathcal{P}_G(S)$ is 2-restricted, then $X \cong PSU_5(3)$ and $\overline{B} \geq GU_4(3) \wr \text{Sym}(3) \cap G$ and $P \leq LS \sim 4^5.Sym(6)$.

**Proof.** Suppose first that $X \cong PSU_5(3)$. Then $L \cong 4^5.Sym(5)$ is a 2-minimal minimal parabolic. Thus either $L \leq B$ or $P = L$. Let $\overline{T_1} = GU_4(3) \times GU_1(3) \cap G$. Then $L_1 \cong GU_4(3)$. If $B \geq L_1$, then, using 6.3, $P \leq LS$ and $Q_B \leq O_2(L) \leq Q_P$, a contradiction. Therefore, $P \leq L_1S$ and $B \geq L_1$. However, since $L_1/O_2(L_1) \cong PSU_4(3)$, applying 6.9 we obtain $P \leq LS \leq B$, a contradiction.

Next consider $X \cong PSU_6(3)$. Then $L \cap X \cong 4^5.Sym(6)$. We set $\overline{T_1} = (GU_4(3) \times GU_2(3)) \cap G$ and $\overline{T_2} = GU_2(3) \wr \text{Sym}(3) \cap G$. If $P \leq L_2S$ and we 2.12 implies that $O^p(P)O_2(L_2S)/O_2(L_2S)$ is normal in $L_2S/O_2(L_2S)$ and has order 3. Then $Q_BO^p(P)O_2(L_2S)/O_2(L_2S) \cong Sym(3)$ and is a direct factor of $L_2S/O_2(L_2S)$, which is a contradiction. Hence $B \geq L_2S$ and by 6.3 $B$ does not contain $L$ and so $P \leq LS$.

Suppose finally that $X \cong PSU_7(3)$. Then $\overline{T} \cong 4^7 : Sym(7)$. Since $Sym(7)$ is not 2-restricted, we have $B \geq LS$. Setting $\overline{T_1} = GU_6(3) \times GU_1(3) \cap G$, we have that $P \leq L_1S$ we are in the $PSU_6(3)$ configuration. In particular, $P \leq (L_1 \cap L)S \leq B$, a contradiction.

We next consider the situation when $p = 3$ and $X \cong SPSL_3^e(r^a)$. However, before initiating the investigation we draw attention to the fact that when $r^a - \epsilon \equiv 3, 6 \pmod{9}$, then $S_4^2 : \text{SL}_2(3)$ is contained in $GL_3^e(r^a)$ and contains a Sylow 3-subgroup of $GL_3^e(r^a)$. (See [4, 6.5.3].)

**Lemma 6.11. [3dimchr3]** Suppose that $p = 3$ and $X \cong SPSL_3^e(r^a)$. If $P \in \mathcal{P}_G(S)$ is 3-restricted, then $G \cong PGL_3^e(r^a)$, $r^a - \epsilon \equiv 3, 6 \pmod{9}$ with $r^a \neq 4$ and either

(a) $[a]$ $P \cong S^2 : \text{SL}_2(3)$; or
(b) $[b]$ $r^a = 7$ and $P \cong (3^2 \times 2^2) : \text{Sym}(3)$.

**Proof.** Notice that 3 divides one of $r^a + 1$ and $r^a - 1$. If $r^a - \epsilon$ is not divisible by 3, then $d = 2$ and $L \cong GL_2^e(r^a)$. Since $O_3(LS) = 1$, we infer that $P \leq LS$ and $LS$ is 3-restricted. Then calling upon 6.2 delivers a contradiction. So we may assume that 3 divides $r^a - \epsilon$. In this case $\overline{L} \cong GL_1^e(r^a) \wr \text{Sym}(3)$. Let $K$ be the base group of $\overline{L}$ and $K_X = K \cap X$.

If $r^a = 7$ we examine the Atlas [2] and observe the configuration in (a) and (b). So assume that $r^a \neq 7$. Let $t$ be a prime dividing $r^a - \epsilon$ and, if $t = 2$, assume that $t^2$ divides $r^a - 1$. Let $T \in \text{Syl}_3(K_X)$. Then $T$ normalized by $S$ and, by the choice of $t$ (and using 2.8), $P \not\leq ST$. It follows that $ST \leq B$ and $Q_B \leq O_3(ST) = O_3(K_XS)$. If $P \leq LS$, then $Q_B \geq O_3(K_XS)$ and we have a contradiction. Therefore $P \not\leq LS$ and, by 6.3 $B = LS$ and $Q_B = O_3(LS)$. 
If $P$ is of characteristic 3, then $O_3(O_3(P)) > 1$. Let $D$ be a normal subgroup of $O_3(P)$ chosen minimal such that $[D, O_3(O_3(P))] \neq 1$. If $D$ is abelian, then $D \subseteq Q_B$ and we have a contradiction as $[D, Q_B] \neq 1$. Therefore $D$ is non-abelian and hence irreducible on $V$. In particular, the centralizer of $D$ is cyclic and $P/C_P(D) \cong \text{SL}_2(3)$. By considering the order of $S$ we now obtain the configuration listed in part (a).

Suppose that $P$ is not of characteristic 3. Then by 3.5 either $Q_B$ is elementary abelian or $O_3(P) \cong \text{Ree}(3)' \cong \text{SL}_2(8)$. In the former case, it follows that $G/X$ does not have field automorphisms, that $|S| = 3^3$ and that $X$ has elementary abelian Sylow 3-subgroups. Therefore, the only possibilities for $O_3(P)$ in this case are $\text{SL}_2(9), \text{PSL}_2(9), 2^2$ or $Q_8$. Since 3 does not divides $|\text{Out}(\text{PSL}_2(9))|$, the first two cases deliver $S$ elementary abelian of order 27 which is impossible. Therefore, $O_3(P)$ is a 2-group and $|Q_P| = 9$. But then $Q_P$ is abelian and $P$ is contained in a conjugate of $L$, a configuration we have already seen off. So assume that $Q_B$ is not elementary abelian and that $O_3(P) \cong \text{Ree}(3)' \cong \text{SL}_2(8)$. By considering representations of the normalizer of a Sylow 2-subgroup of $O_3(P)$, we see that $r = 2$. Now $|S \cap X| > 27$ and so $Q_P \cap X > 1$. Since $[O_3(P), Q_P \cap X] = 1$, and $O_3(P) \not\subseteq LS, O_3(P)$ is contained in $\text{GL}_2(r^a) \times \text{GL}_1(r^a)$ in a unique way. But then $\mathcal{S}$ has to normalize this configuration, a contradiction. □

We next consider the immediate repercussions if 6.11.

**Lemma 6.12.** [4dim2] Suppose that $p = 3$, $X \cong \text{PSL}_n(r^a)$ with $4 \leq n \leq 5$ and $P \in \mathcal{P}_G(S)$ is 3-restricted. Then either

(a) $G = X \cong \text{PSU}_4(r^a)$ with $s = 2, 5 \pmod{9}$, $P \cong 3^{1+2} : \text{GL}_2(3)$ and $B \cong \frac{1}{(2s+1)}(s+1)^3 : \text{Sym}(4)$;

(b) $G = X \cong \text{PSU}_4(2), P \cong 3^3 : \text{Sym}(4)$ and $B \cong 3^{1+2} : \text{SL}_2(3)$; or

(c) $G = X = \text{PSU}_5(2), P \cong 3 \times 3^{1+2} \text{SL}_2(3)$ and $B = 3^4 : \text{Sym}(5)$.

**Proof.** If $X \cong \text{PSL}_n(r^a)$, then $\mathcal{S} \cap X$ cannot act irreducibly on $V$ and consequently $S \cap X$ is contained in a parabolic subgroup of $X$ contrary to 5.1. So assume that $X \cong \text{PSU}_n(r^a)$. We consider the case $n = 4$ first. Assume that $d = 2$. Then $\mathcal{L} \cong \text{GU}_2(r^a)$, $2$, since $r^a > 3$, 2.9 implies that $L \leq B$. But then $Q_B \leq F(L)$ and we have that 3 divides $r^a + 1$, which is a contradiction. So $d = 1$ and we have that $\mathcal{L} = (\text{GU}_3(r^a) \times \text{GU}_1(r^a)) \cap G$. Assume that $B \geq L_1$. Then, by 6.3, $P \not\subseteq LS$. If $r^a > 2$, then $Q_B \leq O_3(L_1) \leq O_3(L) \leq Q_P$, a contradiction. Thus if $B \geq L_1$, then $s = 2$ and (b) holds. So assume that $P \leq L_1$. Then 6.11 gives $s = 2, 5 \pmod{9}$ as claimed in (a).

Assume now that $n = 5$. Arguing as in the $n = 4$ case we have that 3 divides $d = 1$. Set $\mathcal{L}_1 = (\text{GU}_3(r^a) \times \text{GU}_2(r^a)) \cap G$. Since $\text{Sym}(5)$ is not 3-restricted, we have that $B \geq L$ from 6.6. Assume that $s > 2$, then $B$ contains at least one of the components from $L_1$ and then $B \geq \langle LS, L_1S \rangle$, a contradiction to (??). So $s = 2$ and we have $X \cong \text{SU}_5(2)$. Finally, we inspect the subgroup structure in the Atlas [2] to obtain the result as stated in (c). □

The situation in $X = \text{SU}_4(r^a)$ with $r^a \equiv 2, 5 \pmod{9}$ is more exotic than a first look suggests. In the case that $s = 2$, we have that $U_4(2) \cong \text{Sp}_4(3)$. So suppose that $s > 2$. Then in the monomial group $(s + 1)^3 : \text{Sym}(4)$, there are three subgroups isomorphic to $3^3 : \text{Sym}(4)$ each containing $N_G(S)$ call them $P_1, P_2, P_3$ with $P_i \cap P_j = N_G(S)$ whenever $i \neq j$. Let $P = 3^{1+2} \text{SL}_2(3)$. Then, up to change of notation we have, we have $(P, P_1) \cong (P, P_2) \cong \text{PSP}_4(3)$ and $(P, P_3) \cong \text{GU}_3(r^a)$. The embedding of $\text{PSP}_4(3)$ into $\text{SU}_4(r^a)$ stems from the fact that $SU_4(r^a) \cong \Omega^-_6(r^a)$ and $\text{PSP}_4(3)$ has index 2 in the Weyl group of type $E_6$. Finally we note that the two subgroups $(P, P_1)$ and $(P, P_2)$ are conjugate in $\text{GU}_4(r^a)$.

Because of the situation for $\text{SU}_5(2)$ we need to individually inspect $\text{SU}_6(2)$ and $\text{SU}_7(2)$.
Lemma 6.13. [u62andu72] Suppose that $X \cong SU_6(2)$ or $SU_7(2)$ and $P$ is 3-restricted in $G$. Then $X \cong SU_6(2)$, $|G/X| \leq 3$, $P \cap X \cong 3^5 : \text{PSL}_2(9) \cong 3^5 : \text{Alt}(6)$ and $B \cap X \sim 3_+^{1+4}.(Q_8 \times Q_8) : 3$.

Proof. For $G \cong SU_6(2)$, we just inspect the Atlas[2]. So suppose that $X \cong SU_7(2)$. Then $L \cong 3^6 : \text{Sym}(7)$. Since $\text{Sym}(7)$ is not 3-restricted, we have $B \geq L_S$. Setting $L_1 = GU_6(2) \times GU_1(2) \cap G$. We have that $P \leq L_1S$. From the $PSL_6(2)$ example we now read that $P \sim 3^6 : \text{Alt}(6) \leq B$, which is a contradiction.

Lemma 6.14. [dimp2] Suppose that $p \geq 5$ and $n \leq p$. If $X \not\cong \text{PSL}^e_n(r^a)$, then $G$ is not $p$-restricted.

Proof. Suppose for a contradiction that $P$ is $p$-restricted in $G$. If $n < p$ or $d > 1$, then $X$ has abelian Sylow $p$-subgroups and 5.2 delivers a contradiction. Therefore, $n = p$ and $d = 1$. So $L^* \cong (r^a - \epsilon) \cdot \text{Sym}(p)$. Using $p \geq 5$, 6.7 and 3.8 we have that $B = LS$ and $Q_P = 1 \cap X$. Since $Q_B$ is normalized by $N_P(O^p(P) \cap S)$ and $Q_B \cap X = O_p(LS) \cap X$ is abelian, we have that $O^p(P) \cong \text{SL}_2(p^3)$, $\text{PSL}_2(p^3)$ for some $a$ and $Q_B \cap O^p(P) \in \text{Syl}_p(O^p(P))$ is elementary abelian. Since $Q_P \cap X = 1$, we then have $b = p - 2$ and $P \cap X = O^p(P)$ because $\text{SL}_2(p^3)$ has no field automorphisms. This contradicts the fact that $X$ has non-abelian Sylow $p$-subgroups.

We at last come to the induction argument.

Lemma 6.15. [L∪-generic] Suppose that $X \cong \text{PSL}^e_n(r^a)$ and assume one of the following conditions hold:

(a) [a] if $p = 2$ and $r^a > 3$ or $X \cong \text{PSL}_n(r^a)$, then $n \geq 5$;
(b) [a1] if $p = 2$ and $X \cong \text{PSU}_n(3)$, then $n \geq 8$;
(c) [b] if $p = 3$ and $r^a > 2$ or $X \cong \text{PSL}_n(r^a)$, then $n \geq 6$;
(d) [b2] if $p = 3$ and $X \cong \text{PSU}_n(2)$, then $n \geq 8$;
(e) [c] if $p \geq 5$, then $n > p$.

Then $G$ is not $p$-restricted.

Proof. Furthermore, assume that $n$ as defined in the statement of the lemma and then is chosen minimally so that $G$ is $p$-restricted. Recall that $L^* = \text{GL}^e_d(r^a) \cdot \text{Sym}(s) \times \text{GL}^e_{n-ds}(r^a)$ where $s = \lfloor \frac{n}{d} \rfloor$ and $L$ is normalized by $S$.

1°. [1] \hspace{1cm} \text{ds = n}.

Assume that $ds < n$. Plainly $d > 1$. Then $S \cap X$ centralizes an $n - ds$ dimensional subspace of $V$ and so, in particular, it centralizes a 1-dimensional subspace $W$ of $V$. If $\epsilon = +$ and $G/X$ consist of diagonal or graph automorphisms or $\epsilon = -$ and $W$ is singular, then $S \cap X$ is contained in a parabolic subgroup of $X$ and we have a contradiction via 5.1. In the other cases we have $W$ is non-degenerate and, setting $H = \text{GL}^e_{n-1}(r^a)$ (fixing $W$), $S \cap X \leq H$ and $H$ is normalized by $S$. Because of the choice of $n$, 6.9, 6.10, 6.12, 6.13 and 6.14 imply that $H$ is not $p$-restricted. Hence $B \geq H$. Since $Q_B > 1$, we infer that $p$ divides $r^a - \epsilon$ and so $d = 1$ when $p$ is odd, a contradiction. So $p = 2$. In this case, since $d > 1$, $|Q_B| = 2$, and $n - 1$ is even. 3.7 implies that $P/Q_P \cong \text{SL}_2(2)$, (a subgroup of) $\text{PSU}_3(2)$, (a subgroup of) $\text{SU}_3(2)$ or (a subgroup of) $^2B_2(2)$. Since $d = 2$, $Q_B \leq [S \cap X, S \cap X, S \cap X] \leq Q_P$ (from the structure of a Sylow 2-subgroup in $\text{GL}_2(r^a)$), which is a contradiction.

2°. [2] \hspace{1cm} s \geq p.

If $s < p$, then from the structure of $L$ we see that $S \cap X$ is abelian. Hence 5.2 implies that $p \leq 3$. But then $d \leq 2$ and $n = ds < dp \leq 6$ so the restrictions on $n$ deliver a contradiction.

3°. [3] If $d > 1$, then $B \geq LS$ and either
(a) [i] $p = 3$, $d = r^n = 2$ and $\mathcal{L}^r \cong GL_2^+(2) \wr Sym(n/2)$.
(b) [ii] $p = 2$, $d = 2$, $r^n = 3$ and $\mathcal{L}^r \cong GL_3^+(3) \wr Sym(n/2)$.

If $B \geq LS$, then by 6.4 $QB \cap X > 1$ and 6.3 implies that one of (a) or (b) holds. So for a contradiction assume that $B$ does not contain $LS$. Then $P \leq LS$. If $GL_d^+(r^n)$ is not soluble, we apply 2.9 to obtain $s = 1$ and this contradicts (2)*. Therefore $GL_d(r^n)$ is soluble. Using (2) we have that $O^p(P) \leq F_p(LS)$. Suppose that $d \geq 2$ and $r^n = 2$. If $O_p(LS) = 1$, then $F_p(LS)$ is a 3-group and we have $O^p(P)$ is a 3-group from which we infer that $p = 2 = r^n$, a contradiction. Furthermore, from 2.5(e) we have that $LS = (B \cap LS)O^p(P)$, $|QBO_3(LS)/O_3(LS)| = 3$ and $O^p(P)O_3^2(LS)/O_3(LS)$ is either elementary abelian of order 4 or isomorphic or $Q_8$ and is normal in $LS/O_3(LS)$. The only possibility is that $n/2 = 3$ and that $LS/O_3(LS) \cong 2\wr Sym(3)$ and $O^2(P)O_2(LS)/O_2(LS)$ has order 3. Considering again the centralizer of $QB$ in $L/O_2(L)$, we obtain a contradiction. This proves (3)*.

4* [4] If $d = 1$, then $LS \leq B$.

Suppose that $B \not\geq LS$. Then 6.5 gives either $O^p(P)$ is soluble and $n \leq 3$, which contradicts $n \geq 4$, or $r^n - \epsilon$ is a power of $p$ and $LS/O_p(LS)$ is $p$-restricted. So $LS/O_p(LS) \in R_p$. If $p = 2$, then as $n \geq 6$, we have $LS/O_2(LS) \cong Sym(6)$, $Sym(8)$ or $Sym(12)$ by 4.1 and 4.2. Suppose that $LS/O_2(LS) \cong Sym(6)$ and define $L_1 = GL_2^+(r^n) \wr Sym(3) \cap G$. Then $S \cap G$ is contained in $L_1$. If $r^n = 3$, then $2.9$ implies that $L_1S$ is not 2-restricted. Thus, if $r^n > 3$, then $B \geq L_1S$ and we get $Q_B \leq O_p(L) \leq Q_p$, a contradiction. Therefore, $r^n = 3$. But then $L_1$ is soluble and so, if $P \leq L_1S$, $O^2(P)(L_1)O_2(L_1)/O_2(L_1)$ has order 3 and $L_1/O_2(L_1) = C_B(O_2(L_1)/O_2(L_1)) \times QBO_2(P)O_2(L_1)/O_2(L_1)$ has a direct factor isomorphic to $Sym(3)$, a contradiction. Hence $B \geq L_1S$.

Finally consider, $L_2 = (GL_2^+(3) \times GL_4^+(3)) \cap G$. Then as $B \geq L_1$, $P \leq L_2S$. And furthermore, letting $K$ be the component in $L_2$, we have $P \leq KS$ is 3-restricted. It follows that $\epsilon = -$ but then we contradict our supposition on the size of $n$ as given in (b).

Next assume that $n = 8$ and set $L_1 = GL_4^+(r^n) \wr Sym(2) \cap G$. By 2.9, $B \geq L_1S$ and so $P \leq LS$ and $Q_B \leq Q_P$, a contradiction. So suppose that $n = 12$ and set $L_1 = GL_4^+(r^n) \times GL_5^+(r^n) \cap G$. If $P \leq L_1S$, then we must have $P \leq GL_8^+(r^n)$ and this contradicts the minimal choice of $n$. Thus $B \geq L_1S$ and again $Q_B \leq Q_P$, a contradiction. Thus $p > 2$ and consequently $n \leq 6$. Hence $p \geq 5$. This then forces $n = 5$ and $L/O_5(L) \cong Sym(5)$ and this configuration has been considered in 6.14. This proves (3)*.

5* [5] $p$ does not divide $s$.

Suppose that $p$ divides $s$. Assume additionally that $d = 1$. Set $L_1 = GL_{dp}^+(r^n) \wr Sym(s/p) \cap G$. Then, as $s > p$, 2.9 implies that $L_1S \leq B$. Therefore (4*) gives $B \geq (L, L_1)S$. Since $Q_B > 1$, this contradicts 6.3. Suppose that $d = 2$. Then restrictions (b) and (e) imply that $n \geq 8$. In particular, $n > dp$ and so $s/p > 1$. Now we apply the above argument to obtain a contradiction.

Let $t = n - d[s/p]$. Then $1 \leq t \leq p - 1$. Let $L_1 = GL_{dp}^+(r^n) \times GL_t^+(r^n)$. Then $P \leq L_1S$. If $L_1$ has two components, $K_1$ and $K_2$, say, then at least one of them is contained in $B$, a contradiction. Therefore, $L_1$ has at most one component. Since $n > 4$, we infer that $n$ has exactly one component say $K_1$. Furthermore, we must have $K_1S$ is $p$-restricted. If $t = 1$, we have a contradiction from our usual lemmas. Thus $t = 2$ and $r^n \in \{2, 3\}$. The restrictions on and the minimal choice of $n$ then mean that $n = 8$. If $p = 2$, we obtain $s = 4$ or $s = 8$ and we contradict (5*). If $p = 3$, we have
$K_1 \cong SU_6(3)$ and $K_1 S$ is 3-restricted. By 6.10 we then have $P \leq L \cap K_1 S \leq B$, a contradiction. This completes the proof of the lemma.

7. SYMPLECTIC GROUPS

In this section we suppose that $r$ is a prime with $r \neq p$ and that $X \cong \text{PSp}_{2n}(r^a)$. As usual we have $G/X$ is a $r$-group. Notice that, as $p \neq 3$, $G/X$ never involves the exceptional graph automorphism which only appears in characteristic 2. We define $\text{GSp}_{2n}(r^a)$ to be $\text{Sp}_{2n}(r^a)$ decorated with its diagonal automorphism of order 2 and its field automorphisms. So $[\text{GSp}_{2n}(r^a) : \text{Sp}_{2n}(r^a)] = 2a$. Now as in the unitary case we let $X$ be $\text{Sp}_{2n}(r^a)$ and $G \leq \text{GSp}_{2n}(r^a)$ with $G/X$ a $p$-group. Then $G = G/Z(G)$ and $X = X/Z(G)$. Define $d = \text{lcm}(2, \text{ord}(p, r))$ and set $s = [\frac{2n}{d}]$. Define $L = \text{Sp}_{d}(r^a) \wr \text{Sym}(s) \times \text{Sp}_{2n-ds}(r^a) \cap G$. Then $L \cap X$ contains a Sylow $p$-subgroup of $S \cap X$ and $L$ is $S$-invariant.

Lemma 7.1. [spcases] Assume that $X \cong \text{PSp}_{2n}(r^a)$ with $n \geq 2$. If $X \in \mathcal{R}_p$, then one of the following holds

(a) $[a]$ $p = 3$, $X \cong \text{PSp}_4(2)$ and $P = X$;
(b) $[b]$ $p = 2$ and $X \cong \text{PSp}_4(3) \cong \text{PSU}_4(2)$; or
(c) $[c]$ $p = 2$, $X \cong \text{PSp}_6(3)$ and $B = \ldots$

Proof. Suppose first that $2n - ds \neq 0$. then $S \cap X$ centralizes an isotropic vector in $V$ and consequently $S \cap X$ is contained in a parabolic subgroup of $X$ and this contradicts 5.1. So $n - ds = 0$. If $d = 2n$, then $X$ has abelian Sylow $p$-subgroups. Thus $p = 2,3$ by 5.2. But then $2n \leq d \leq 2$, a contradiction. So we have


Assume that $P \leq LS$. Then $LS \in \mathcal{R}_p$ and so 2.9 implies that $\text{Sp}_d(r^a)$ is soluble. That is $d = 2$ and $r^a = 2$ or $r^a = 3$. Arguing exactly as in 6.15 3 delivers a contradiction unless $2n = 4$ and $L \cong \text{SL}_2(2) \wr 2$ or $L \cong \text{SL}_2(3) \wr 2$. Thus we have examples (a) and (b).

Henceforth we assume that $B \geq LS$. In particular, as $Q_B > 1$, we have that either $p = 3$ and $G \cong \text{Sp}_{2n}(2)$ or $p = 2$. In any case we have that $d = 2$.

Assume that $2n = 4$. If $r^a \leq 3$, then again possibilities (a) and (b) holds. So assume that $r^a > 3$.

Then as $Q_B > 1$, we infer that $p = 2$ and that $|Q_B| = 2$ contrary to 3.9.

We now assume $2n > 4$ and when $p = 2$ and $r^a = 3$ that $2n > 6$. Furthermore, select $n$ minimal so that the above conditions on $n$ are satisfied and $G \in \mathcal{R}_p$. Write $s = 2k + l$ with $l \leq 1$ and set $L_1 = \text{Sp}_d(r^a) \wr \text{Sym}(k) \times \text{Sp}_2(r^a)$ and factor $L_1$ as $K_1 \times K_2$ with $K_2 \cong \text{Sp}_2(r^a)$. Then as $L_1 S$ and $LS$ generate $G$, $L_1 S$ contains $P$. Since $P \nsubseteq LS$, we infer that $PS \leq K_2$. Then 2.9 implies that $k = 1$ and that $K_2 \cong \text{Sp}_2(r^a) \in \mathcal{R}_p$. It follows that $p = 2$ and $r^a = 3$ or $p = 3$ and $r^a = 2$. The first case fails because of the choice of $n$, the second case indicates that $G \cong \text{Sp}_6(2)$ and we have already seen that this group does not satisfy $\mathcal{R}_3$.

8. THE ORTHOGONAL GROUPS

Assume first that $X \cong \text{PO}_{2n}(r^a)$ with $n \geq 4$ and no triality automorphism.

Suppose that $p$ and $r$ are distinct primes and set $d = \frac{1}{2} \text{lcm}(2, \text{ord}(p, r^a))$.

$$
\eta = \begin{cases} 
+ \text{ ord}(p, r) \text{ odd and } p \text{ odd} \\
+ p = 2 \text{ and } r^a \equiv 1 \pmod{4} \\
- \text{ ord}(p, r) \text{ even and } p \text{ odd} \\
- p = 2 \text{ and } r^a \equiv 3 \pmod{4} 
\end{cases}
$$
Let

\[ s = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & d \text{ does not divide } n \\ \left\lfloor \frac{n}{2} \right\rfloor & d \text{ divides } n \text{ and } \eta_{\left\lfloor \frac{n}{2} \right\rfloor} = \epsilon \\ \left\lceil \frac{n}{2} \right\rceil - 1 & d \text{ divides } n \text{ and } \eta_{\left\lceil \frac{n}{2} \right\rceil} \neq \epsilon. \end{cases} \]

Finally put

\[ \mathcal{L} = O_{2d}^n(r^a) \rtimes \text{Sym}(s) \times O_{2(n-d)s}^6(r^a). \]

Then \( \mathcal{L} \) contains a Sylow \( p \)-subgroup of \( O_{2n}^r(r^a) \).

**Lemma 8.1.** [Orth1] Suppose that \( X \cong \text{PO}^\pm(r^a) \), \( d = 1 \) and \( \eta^p = \epsilon \). If \( p \geq 3 \), then every \( p \)-local subgroup of \( X \) which contains \( S \cap X \) is contained in \( L \).

**Proof.** Suppose that \( R \) is a \( p \)-local subgroup of \( X \) containing \( S \cap X \) and put \( Q_R = O_p(R) \). Let \( Z_R = Z(Q_R) \). If \( Z_R \leq O_p(L) \), then the homogeneous components of \( Z_R \) on \( V \) coincide with those of \( O_p(L) \) and we have \( R \leq L \). So assume that \( Z_R \not\leq O_p(L) \). But then \( Z_R \) operates quadratically on \( O_p(L) \) and, as \( \Omega_1(O_p(L)) \) is the permutation module for \( \text{Sym}(p) \) and \( p \geq 3 \), we have a contradiction.

**Lemma 8.2.** [Orth2] Suppose that \( X \cong \text{PO}_{2n}^n(r^a) \) with \( n \geq 4 \) and \( G \in \mathcal{R}_p \), then one of the following holds:

(a) \( p = 2 \), \( X \cong \text{PO}^+_8(3) \) and \( \overline{B} = O_3^+(3) \rtimes \text{Sym}(2) \);

(b) \( p = 2 \), \( X \cong \text{PO}_{12}^+(3) \) and \( \overline{B} \cong O_4^+(3) \rtimes \text{Sym}(3) \);

(c) \( p = 3 \), \( X \cong \text{PO}_8^+(r^a) \) with \( r^a \equiv 2, 5 \pmod{9} \), \( \overline{B} \cong \text{O}_2^+(r^a) \rtimes \text{Sym}(4) \) and \( P \sim 3^{1+2}.\text{SL}_2(3) \times 3 \).

**Proof.** Suppose that \( n \) is chosen minimally so that \( \text{PO}_{2n-2}^n(r^a) \not\in \mathcal{R}_p \).

1°. [1] \( n = \sigma d \).

Suppose first that \( p \) is odd. Then \( G \) does not involve the graph automorphism of \( G \) and \( S \cap X \) centralizes a non-degenerate subspace of \( V \) of dimension \( 2(n-ds) \). In particular, \( S \cap X \) centralizes a singular vector and hence \( S \cap X \) is contained in a parabolic subgroup of \( G \). Suppose then that \( p = 2 \). Then we have that \( \mathcal{L} = O_2^+(r^a) \rtimes \text{Sym}(s) \times O_2^n(r^a) \), \( \eta \neq \theta \) and \( s = \left\lceil \frac{n}{2} \right\rceil - 1 \). Observe that a Sylow 2-subgroup of \( O_2^+(r^a) \) has order 4. Thus we also see that in this case \( S \). Set \( \mathcal{L} = \text{PO}_2^+(r^a) \rtimes \text{Sym}(s) \). Then \( \mathcal{L} \) leaves invariant two distinct anisotropic 1-spaces. Thus \( \mathcal{L} \) is contained in two proper subgroups of \( G \) each isomorphic to \( O_{2n-1}(r^a) \). Since \( |Q_B| > 2 \), we have \( P \) is contained in the intersection of these two groups and this means that \( P \leq \mathcal{L} \). But then \( r^a = 3 \) and we have \( 2n = 8 \) and \( \overline{B} \) involves \( \text{PSU}_3(3) \), \( 2n = 10 \) and \( \overline{B} \) involve \( \text{PSU}_5(3) \) or \( 2n = 14 \) and \( \overline{B} \) involves \( \text{PSU}_{12}(3) \). In the first case we note that \( L \times \text{O}_2^+(3) \times \text{O}_2^+(3) \) also contains a Sylow 2-subgroup of \( G \) and we obtain \( B \geq \mathcal{L} \) for a contradiction from 6.9. In the second and third cases we use the fact that \( O_9(3) \) contains a subgroup \( O_1(3) \rtimes \text{Sym}(9) \) case \( O_{13}(3) \) contains a subgroup \( O_1(3) \rtimes \text{Sym}(13) \) and apply 4.2 to see that in each case \( B \) must contain this subgroup. This contradicts the structure of \( B \) as described in (a) and (b). Thus we have that \( n = \sigma d \).

2°. [2] \( s > 2 \).

For \( p = 2 \) or 3, this follows because of the requirements on the size of \( n \). So \( p \geq 5 \) and \( s \geq 2 \) follows from 5.2. So we assume that \( s = p \).

3°. [3] \( B \geq \mathcal{L} \).
Suppose on the contrary that \( P \leq LS \). Assume for a moment that \( p > 3 \). Then \( O^p(P) \) is not soluble. Thus \((2^a), 2.9\) and ?? together imply that \( O^p_2(r^a) \) is a \( p \)-group. But then the only possibility is that \( p = 2 \), a contradiction. Thus \( p \in \{2, 3\} \). Thus \( d = 1 \) and \( \mathcal{L} \cong O^2_2(r^a) \wr \Sym(s) \). Assume that \( O^2_2(r^a) \) is not a 2-group. Then, since the derived subgroup of \( L \) is perfect when \( n \geq 5 \), we have that \( n = 4 \) from 2.11. Easy TO SEE \( n \neq 4 \). Write a nice argument. Therefore, we have this 2-group and \( \Sym(n) \in R_2 \). Therefore 4.2 implies that \( n \in \{4, 6, 8, 12\} \). Set \( m = 2, 4, 4 \), according as \( n = 4, 6, 8, 12 \) and define \( \mathcal{T}_1 = O^2_2(r^a) \wr \Sym(n/m) \) where \( \mu = \eta^m \). If \( O^2_2(r^a) \) is not soluble, then, by 2.9, \( B \geq L_1S \) and \( Q_B \) has order \( 2^{m-1} \) and is contained in \( Q_P \), a contradiction. Therefore, \( O^2_2(r^a) \) which means that \( m = 2 \), \( r^a = 3 \) and \( \mu = + \). Hence \( X \cong \PO_8^+(3) \) or \( \PO_{12}^+(3) \) which is a contradiction to our choice of \( n \). Hence \((3^c)\) holds.

Since \( B \geq LS \), we must have that \( LS \) is a \( p \)-local subgroup. Thus

4c. \([4] \quad L \cong O^2_2(r^a) \wr \Sym(n)\).

Suppose for a moment that \( p \geq 5 \). Then 5.2 implies that the Sylow \( p \)-subgroups of \( X \) are not abelian. Thus \( n \geq p \). If \( n = p \), then ?? implies that \( Q_P = 1 \) and the argument in ?? works to give a contradiction. Therefore,

5c. \([5] \quad n > p\).

Write \( n = lp + k \) where \( k \leq p - 1 \). Then put \( \mathcal{T}_1 = O^2_2(p^a) \times \langle I_{2k}\rangle \). Then \( S \) normalizes \( L_1 \). Assume that \( lp \neq n \). Since \( L_1S \not\leq B \), we have that \( P \leq L_1S \) and so \( L_1S/O_p(L_1S) \in R_p \). If \( p \geq 5 \), this immediately contradicts the minimal choice of \( n \). Therefore, \( p \leq 3 \). If \( p = 3 \), then we require \( lp < 4 \) which means that \( n = 4 \) or \( n = 5 \) and that \( O^p_6(r^a) \in R_3 \) and this again contradicts our supposition on \( n \). If \( n = 5 \), then set \( \mathcal{T}_2 = (I_3) \times O^4_3(r^a) \) and note that as \( P \leq L_1S, L_2S \leq B \), a contradiction. So suppose that \( p = 2 \). Then \( 2n = 12 \) and \( r^a = 3 \). But then from the example in \( O^2_2(3) \) we read that \( P \leq L \leq B \), a contradiction. Hence \( n = rp \). Now set \( \mathcal{T}_2 = O^6_3(p^a) \wr \Sym(r) \) where \( \rho = \eta^p \). Then \( L_2 \) is normalized by \( S \). Plainly \( L_1S \not\leq B \) and so \( P \leq L_1S \). Since \( r > 1 \) by \((5^c)\), \( 2dp = 4 \) and \( r^a = 3 \) as well as \( p = 2 \). Investigating the structure of \( \mathcal{T}_2 \) using 2.10 readily reveals a contradiction. This completes the proof of the lemma. \( \square \)

**Lemma 8.3.** [Orthodox] Suppose that \( r \) is odd and \( X \cong \PO_{2n+1}(r^a) \) with \( n \geq 3 \). If \( G \in R_p \), then \( p = 2 \), \( X \cong \PO_7(3) \) and \( \mathcal{B} = O_1(3) \wr \Sym(7) \).

**Proof.** We have that \( |O_{2n+1}(r^a) : O_{2n}(r^a)| = (r^{na} + \epsilon)r^b \) for some \( b \). Thus a Sylow \( p \)-subgroup of \( O_{2n+1}(r^a) \) fixes either a plus point or a minus point when acting on \( V \). Since these point stabilizers \( H \) have \( O_{p}(H) \) of order at most \( 2 \), we \( H \in R_p \) by 3.9. Thus the possibilities for \( p \), \( r^a \) and \( n \) may be read from 6.9 and 8.2. Suppose that \( n = 3 \). Then \( \mathcal{B} \cong 2 \times O_6^+(r^a) \). Thus 6.9, 6.12 and 6.14 imply that \( p \) is either 2 or 3. If \( p = 2 \), then from 6.9 we get \( \epsilon = - \) and \( r^a = 3 \) and ?? holds. If \( p = 3 \), then again \( \epsilon = - \) and \( r^a = 2.5 \) (mod 9). But then \( r^{3n} + 1 \) is divisible by \( 3 \), a contradiction.

Assume now that \( n \geq 4 \). Then ?? implies that \( p = 2 \) or 3. Assuming that \( p = 2 \), the subgroups \( O_1(3) \wr \Sym(9) \) and \( O_1(3) \wr \Sym(13) \) yield contradictions. So \( p = 3 \) and \( \mathcal{B} \cong O_7^+(r^a) \times 2 \). Set \( \mathcal{L} = O_8^+(r^a) \times O_1(3) \). Then \( L \) is \( S \)-invariant. Since \( r^a > 3 \), \( O_3(r^a) \) is not soluble. Furthermore, writing \( \mathcal{L} = \mathcal{K}_1 \times \mathcal{K}_2 \) with \( \mathcal{K}_2 \cong O_3(r^a) \), we have that \( \mathcal{K}_2 \not\leq H \) and so \( \mathcal{K}_2 \leq B \), but then \( B \geq (K_2, B \cap \mathcal{B}) = G \). \( \square \)

Finally in this section we come to the situation when \( p = 3 \), \( X \cong \PO_8^+(3) \) and \( S \) does not normalize all the parabolic subgroups of \( X \). (So the triality automorphism of \( X \) is having an influence. From the examples in ?? we only need consider the case when \( r^a \neq 2, 5 \) (mod 9) (otherwise the example in \( X \) immediately lifts to an example in \( G \) with \( G/X \) a 3-group.)
Lemma 8.4. [triality] Suppose that $X \cong \text{PΩ}^+_3(3)$ and $G/X$ is a $3$-group. If $G \in \mathcal{R}_3$, then $r^a \equiv 2, 5 \pmod{9}$.

9. Exceptional groups of Lie type

In this section we suppose that $X$ is a universal group of Lie type defined over a field over $\text{GF}(r^a)$. We use the notation introduced in [4, page 237] writing

$$|X| = r^{NA} \prod_i \Phi_i(r^a)^{n_i}$$

where $\Phi_i(x)$ is the cyclotomic polynomial for for $i$-th roots of unity. The product $\prod_i \Phi_i(r^a)^{n_i}$ in the case when $X$ is an exceptional group is conveniently presented in [3, Table 10:2]. We set $d = \text{ord}(p, r^a)$ if $p$ is odd and, if $p = 2$, $d = 1$ when $r^a \equiv 1 \pmod{4}$ and otherwise $d = 2$.

Lemma 9.1. [OrdSylow] Let $X$ be a universal group of Lie type defined over $\text{GF}(r^a)$, $S$ be a Sylow $p$-subgroup of $X$ with $p$ odd. Then

$$|S| = p^k \Phi_d(r^a)^{n_d}$$

where $b = \sum_{p|n_i} n_i$. Furthermore, if $b = 0$, then the Sylow $p$-subgroups of $X$ are abelian.

Proof. Consult [3, Equation (*) page 113] and [4, Theorem 4.10.2 (c)].

Lemma 9.2. [bigd exce] Suppose $G \in \mathcal{R}_p$ and that $d = \text{ord}(p, r^a) > 2$. Then $d = 4$, $p = 5$ and $X \cong \text{E}_8(r^a)$.

Proof. Since $d.2$, $p \geq 5$. We show that other than in the $d = 4$, $p = 5$, $X \cong \text{E}_8(r^a)$ cases the Sylow $p$-subgroups of $X$ are abelian. The result then follows form 5.2. Since $d > 2$ and $p \geq 5$, $pd \geq 20$ (note $d = 3$ and $p = 7$ gives $pd = 21$). From [3, Table 10:2] the only exceptional group involving $\Phi_k(r^a)$ with $k \geq 20$ is $\text{E}_8(r^a)$. So consider $\text{E}_8(r^a)$. If $d = 3$, we see no $\Phi_{21}(r^a)$, if $d = 4$ we can only have $p = 5$ and if $d \geq 5$, there are also no possibilities.

Lemma 9.3. [E5p 5] Suppose that $d = 4$, $p = 5$ and $X \cong \text{E}_8(r^a)$. Then $G \notin \mathcal{R}_5$.

Proof. In this case we have that 5 divides $\Phi_4(r^a) = r2a + 1$. Using [3, Table 4-1 (37)] we see that $X$ contains a subgroup $L$ isomorphic to $\text{SU}_5(2^a)$ and by comparing orders we see that this subgroup contains a Sylow 5-subgroup of $X$ and furthermore $L$ is invariant under $S$. From 6.15 we have that $LS \not\subseteq P$ and therefore, $B \supseteq LS$ and $|Q_B| = 5$. But $Z(S)$ is cyclic and so we have a contradiction with 3.9.

Lemma 9.4. [SuzandRee] If $G \in \mathcal{R}_p$ and $X \cong \frac{3}{2}(2^a)$ or $2\text{G}_2(3^a)'$, then $p = 2$ and $X \cong \frac{3}{2}(3)'$.

Proof. If $X \cong 2\text{B}_2(2^a)$, then $p > 3$ and the Sylow $p$-subgroups of $X$ are abelian. Therefore, 5.2 shows that $G \notin \mathcal{R}_p$. Let $X \cong 2\text{G}_2(3^a)$ with $a \geq 3$. In this case also the Sylow 2-subgroups are abelian and so 5.2 delivers either a contradiction or $X \cong 2\text{G}_2(3)' \cong \text{SL}_2(8)$.

Guess

Lemma 9.5. [Reef] If $G \in \mathcal{R}_p$, then $X \not\cong 2\text{F}_4(2^a)'$.

Proof. By 9.1 we only need to consider the situation when $p = 3$ and $d = 2$. We have $|S \cap X| = 3|\Phi_d(2^a)^2|_3$. If $r^a = 2$, we have $G = X = \frac{3}{2}(2)'$ and the Atlas [2] shows us that there are no 3-local subgroups which are maximal subgroups of $G$. So assume that $2^a \geq 8$. Using [7, Table 5.1 and 5.2] we see that $S \cap X$ is contained in maximal subgroups $L_1 = (2^a + 1)^2$, $\text{GL}_2(3)$ and $L_2 \cong \text{SU}_3(2^a) : 2$. According to 6.11 $L_2 \not\in \mathcal{R}_3$, so $B \geq L_2$ and $P \leq L_1$. But then $Q_B \leq Q_P$, a contradiction.
Lemma 9.6. \([F4]\) If \(G \in \mathcal{R}_p\), then \(X \not\cong \mathcal{F}_4(r^a)\).

Proof. By 9.1 and 5.2 we have that \(p \in \{2, 3\}\). Notice that as \(p \neq r\), the graph automorphism of \(\mathcal{F}_4(2^r)\) makes no appearance in this discussion. We first suppose that \(p = 2\). Then, using \(\mathcal{F}_4(2^r)\) Table 5.1|LSS, there is a maximal subgroup \(L \in X\) with \(L \cong 2.\Omega_9(r^a)\). By 8.3, \(L \not\in \mathcal{R}_p\). Therefore, \(B = L\). But then \(|Q_B| = 2\) and \(\mathcal{F}_4(2^r)\) provides a contradiction. Next assume that \(p = 3\). Then we set \(L = (2, q - 1)^2.P\Omega_9^+(r^a).\text{Sym}(3)\). By 8.2. Notice that \(L\) contains \(S \cap X\) and that \(S\) is not a \(3\)-local subgroup. Hence, if \(G \in \mathcal{R}_p\) then \(L\) must also be in \(\mathcal{R}_p\). But then 8.2 implies that \(r^a \equiv 2, 3 \pmod{9}\) and that \(P \leq C_G(Z(S))\). Furthermore, we note that \(B \cap L_2 \sim (q + 1)^4 : W(\mathcal{F}_4)\) where \(W(\mathcal{F}_4)\) denotes the Weyl group of type \(\mathcal{F}_4\). (As a subsystem subgroup the \(E(L_1)\) is generated by the root spanned by the \(\mathcal{F}_4\) subsystem \(\{\}\).) If \(r^a = 2\), we consult the Atlas \([2]\) and see that there is a unique maximal \(3\)-local subgroup and so \(B = N_G(Z(S))\). But then we have \(P \leq B\), a contradiction.

Suppose then that \(r^a > 2\) and let \(L_2 = 3.(PGU_3(r^a) \times PSU_3(r^a)).2\) be as in \([7, \text{Table 5.2}]\). Then \(L_2\) can be chosen to contain \(S \cap X\). Now \(P \leq L_2\) and \(P\) is contained in exactly one of the components of \(L_2\) and the other component must be contained in \(B\). But then \(B = G\) if \(r^a\) is odd and if \(r = 2\), then \(B\) contains a subgroup isomorphic to \(L_1\) (but this time generated by short root groups). In either case we have a contradiction as \(Q_B = 1\).

\(\square\)

Get details for the above argument.

Lemma 9.7. \([G2\text{ and }3D4]\) Suppose that \(X = \mathcal{O}^p(G)\) is isomorphic to either \(G_2(r^a)\) or \(\mathcal{F}_4(3^a)\) or \(3D_4(3)\) and the possibilities for \(G, B,\) and \(P\) are listed in Table 1.

Proof. By 5.2, we need only consider the cases with \(p = 2\) or \(p = 3\). Suppose that \(X \cong G_2(3^a)\). If \(r^a = 3\), we refer the reader to the Atlas \([2]\) to verify the details needed to confirm that \(G_2(3)\) is an example. Let \(\Delta\) be the root system associated with \(X\) and let \(M\) be the monomial group \(\Phi_n(s) : D(12)\) where \(n = 1\) or \(2\) as appropriate for \(M\) to contain a Sylow 2-subgroup, respectively 3, of \(X\). Let \((a, b)\) be a fundamental system for \(\Delta\) with \(a\) a long root. Let \(T = \langle h_a, h_b \rangle\) be the Torus of order \(\Phi_1(2)^2\). Then \(N = \langle X_\alpha, X_\beta, X_\alpha + 2\beta, X_\alpha - 2\beta \rangle\). Then \(Z(N) = \langle h_a(-1)\rangle = \langle h_{a+2\beta}(-1)\rangle\). In particular, we notice that \(|T : T \cap N| = 2\) and so \(|TN/N| = 2\). Since \(|X| = \Phi_1(2)^2\Phi_2(s)^2\Phi_3(s)^2\Phi_6(s)\), and \(|NT| = \frac{1}{2}(\Phi_1(2)^2\Phi_2(s)^2\Phi_3(s)^2\Phi_6(s))\), we infer that \(\mathcal{N}_G(S)\) contains a Sylow 2-subgroup of \(G\). Since \(s \geq 3\), \(\mathcal{N}_G(S)\) contains two normal components \(F_1\) and \(F_2\) and so at least one of these component must be contained in \(B\). So suppose that \(F_1 \leq B\). Then \(O_2(F_1S)\) is isomorphic to a Sylow 2-subgroup of \(L_2\) and consequently \(O_2(F_1S)\) contains a unique involution, namely \(t = h_a(-1)\). Since \(Q_B = O_2(F_1S)\), we see that \(B \leq C_G(t)\). But then \(Q_B = Z(S)\) is cyclic of order \(2\) and we have a contradiction via Lemma 3.9.

If \(X \cong 3D_4(3)\) and \(s \neq 3\), then an argument just as above works. \(\square\)

Lemma 9.8. \([E6]\) Assume that \(X \cong \mathcal{E}_6(r^a)\) or \(2E_6(p^a)\) and \(G \in \mathcal{R}_p\), then \(p = 3\) and \(X \cong 2E_6(2)\) and \(B \cap X \cap 3.(PSU_3(2) \times PSU_3(2) \times PSU_3(2)).3.\text{Sym}(3)\).

\(\square\)

look at \(p = 5\)

Proof. Suppose first that \(d = 2\) when \(X \cong \mathcal{E}_6(r^a)\) and that \(d = 1\) when \(X \cong 2E_6(r^a)\) and as is standard write \(X \cong \mathcal{E}_6(r^a)\) with \(\epsilon = +\) when \(X\) is not twisted and otherwise \(\epsilon = -\). Because of the choice of \(d\) we see that \(p \leq 3\). Suppose that \(p = 3\), then \(S \cap X\) is contained in the subgroup
\[ L \cong \Gamma_4(q) \]. Using 9.6 we have that \( B \geq L \) and then \( Q_B = 1 \), a contradiction. So suppose that \( p = 2 \) and \( \epsilon = -1 \). Then take \( L = (4, r^a - \epsilon).P\Omega_{16}^+(r^a) \times (r^a - \epsilon)/(4, r^a - \epsilon) \cdot (4, r^a - \epsilon) \) (use root groups \( X_{n_3}, X_{n_4}, X_{n_5}, X_{n_2}, X_{n_3} \) and their negatives so that \( L \) is normalized by the graph automorphism. It follows that \( Q_B \) is cyclic of order 2 and we have a contradiction. If \( p = 2 \) and \( d = 1 \) we take \( L \) as above and see that \( Q_B = 1 \), a contradiction. For \( p \epsilon \mathbb{P} \) obtain a contradiction. For \( p \epsilon \mathbb{P} = 2 \) and \( d = 2 \) when \( \epsilon = -1 \). In this case we take \( L = 3.(PSL_3^+(r^a) \times PSL_3^+(r^a) \times PSL_3^+(r^a))).3^2 \cdot \text{Sym}(3) \) and when (\( r^a, \epsilon \) ? \( (2, -) \), the components of \( L \) are permuted transitively by \( L \). Thus in this case \( B \geq L \) by 2.9 and then \( |Q_B| = 3 \). Thus (\( r^a, \epsilon \) ? (2, -) and the lemma follows.

\[ \square \]

Mention the Fischer groups?

**Lemma 9.9. [E7]** Assume that \( X \cong E_7(r^a) \) and \( G \in \mathcal{R}_p \), then \( p = 2 \), \( X \cong E_7(3) \) and \( B = 2^3.(PSL_2(3)^7).2^4.SL_3(2) \).

**Proof.** We have to consider \( p \in \{2, 3, 5, 7\} \). According as \( d = 1, 2 \) set \( \epsilon = +, - \) and note that \( L_1 = (q - \epsilon)^7 : (2 \times \text{Sp}_6(2)) \) contains \( S \cap X \). Set \( L_2 = (3, r^a - \epsilon).E_7(r^a) \times (r^a - (3, r^a - (3, r^a - (3, r^a - \epsilon)) \cdot (3, r^a - \epsilon)).2 \). Then so long as \( p \neq 2, 7 \), \( S \cap X \leq L_2 \) and we easily derive contradictions in these cases.

For \( p = 7 \) we set \( L_3 = f \cdot PSL_8(r^a).g.(2 \times (2/ f)) \) where \( f \) and \( g \) are powers of 2. Then ?? shows that \( L_2 \leq B \), a contradiction. So suppose that \( p = 2 \). Then set \( L_3 = 2.(PSL_2(r^a) \times P\Omega_4^+(r^a)).2 \).

Set \( L_4 = 2^3.(PSL_2(r^a))^7 \cdot d \cdot PSL_3^2(2) \). Assume that \( r^a > 3 \). Then 6.2 and 8.2 imply that \( L_3 = B \) and that \( Q_B \) has order 2 contrary to 3.9. Thus \( p = 3 \). Suppose that \( B = L_3 \). Then \( P \leq L_4 \) and \( Q_B \leq Q_2(L_4) \leq Q_2 \) for a contradiction. Therefore, \( B = L_4 \) as claimed. Need to argue \( P \notin L_4 \cap L_3 \) (use ??).

**Lemma 9.10. [E8]** Assume that \( X \cong E_8(r^a) \) and \( G \in \mathcal{R}_p \), then \( p = 3 \), \( X \cong E_8(2) \) and \( B \cap X = 3^2.(PSU_3(2) \times PSU_3(2) \times PSU_3(2) \times PSU_3(2)).3^2 \cdot GL_2(3) \).

**Proof.** We have to consider the possibilities \( p = 2, 3, 5 \) and 7. Recall also that \( d \in \{1, 2\} \). For \( p = 2 \) or 7 the group \( L = (2, r^a - 1).P\Omega_{16}^+(r^a).2 \) contains a Sylow \( p \) subgroup of \( X \). Applying 8.2 and 3.9 we obtain a contradiction. For \( p = 5 \), we consider the subgroup \( 5.(PSL_3^+(r^a) \times PSL_3^+(r^a))).5 \) where \( \epsilon \) is chosen so that \( 5 \) divides \( r^a - \epsilon \). Then 3.9 and 2.9 delivers a contradiction. So assume that \( p = 3 \). Set \( L_1 = 3.(PSL_3^+(r^a) \times E_7(r^a))).3 \cdot L_2 = 3^2.(PSL^+(r^a) \times PSL^+(r^a) \times PSL^+(r^a) \times PSL^+(r^a)].3^2 \cdot GL_2(3) \)

(with \( \epsilon = + \) if \( d = 1 \) and \( \epsilon = - \) if \( d = 1 \)). Then 2.9 implies that \( P \leq L_1 \) and \( B = L_2 \). Using 9.8 we obtain \( r^a = 2 \) and this completes the lemma.

\[ \square \]

### 10. Sporadic groups

Before we begin the case by case investigation of the sporadic simple groups we note that, by Lemma ??, we may assume that \( S \) is not cyclic.

**Lemma 10.1. [m11]** \( X \neq \text{Mat}_{11} \).

**Proof.** Suppose that \( G \in \mathcal{R}_p \). Then \( p \in \{2, 3\} \). Suppose that \( p = 2 \). Let \( H \leq G \) with \( H \sim \text{Mat}_{16} \). Then \( H \) is a 2-minimal subgroup of \( G \) and \( Q_H = 1 \). Hence by 2.10 \( H = P \), a contradiction. Next we consider \( p = 3 \). This time we note that \( N_G(S) \) is a maximal subgroup of \( G \) and apply 2.10 to get a contradiction.

\[ \square \]

**Lemma 10.2. [m12]** Suppose \( X = \text{Mat}_{12} \). Then \( p = 2 \) or 3 and \( B \) is either of the two maximal \( p \)-local parabolic subgroups of \( G \).

**Proof.** For \( p > 3 \), \( S \) is cyclic and for \( p \leq 3 \), \( \mathcal{M}(S) = \mathcal{P}(S) \) has size two.

\[ \square \]
Lemma 10.3. [j1] $X \neq J_1$.

Proof. For $p > 2$, $S$ is cyclic and for $p = 2$, $N_G(S)$ acts irreducibly on $S$. So the lemma follows from 2.18. \hfill \Box

Lemma 10.4. [m22] Suppose $X = \text{Mat}_{22}$. Then $p = 2$ and $B$ is either of the two maximal 2-local parabolic subgroups of $G$.

Proof. For $p \geq 5$, the Sylow $p$-subgroups are cyclic. For $p = 3$, $N_G(S)$ acts irreducibly on $S$. Thus by 2.18, $p = 2$. Now $B$ can be either of the maximal 2-local parabolics of $G$ and (by definition) $P$ is the unique minimal parabolic not in $B$. \hfill \Box

Lemma 10.5. [j2] Suppose that $X = J_2$. The one of the following holds:

(a) [1] $p = 3$, $B \sim 3.\text{PGL}_2(9)$ and $P \sim \text{PSU}_3(3)$.
(b) [2] $p = 2$ and $B$ is either of the two maximal 2-local parabolic subgroups of $G$.

Proof. For $p = 7$, $S$ is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on $S$. So $p \leq 3$. For $p = 3$, $G$ has a unique maximal 3-local parabolic subgroup $M$ and $M \sim 3.\text{PGL}_2(9)$. Thus $B = M$ and $B \sim 3.\text{PGL}_2(9)$. Note that $G$ has a parabolic subgroup $H \cong \text{PSU}_3(3)$. Then by 2.16 $P = H$ and (a) holds.

For $p = 2$, $P(S) = \mathcal{M}(S)$ has size 2 and (b) holds. \hfill \Box

Lemma 10.6. [m23] Suppose $X = \text{Mat}_{23}$. Then $p = 2$, $B \cong 2^4.\text{Alt}(7)$ and $P \cong 2^4.\text{Sym}(5)$.

Proof. For $p \geq 5$, the Sylow $p$-subgroups of $G$ are cyclic. For $p = 3$, $N_G(S)$ acts irreducibly on $S$. Thus by 2.18, $p = 2$. Since $L = 2^4.\text{Alt}(7)$ is a subgroup of $G$ and $L/Q_L \in R_2$ by 4.2 $B = L$. So $P$ is the unique minimal parabolic subgroup of $G$ not in $B$ and the lemma holds. \hfill \Box

Lemma 10.7. [hs] Suppose $X = \text{HS}$. Then $p = 2$ and $B$ is any of the two maximal local parabolic subgroups of $G$.

Proof. For $p \geq 7$, the Sylow $p$-subgroups of $G$ are cyclic. For $p = 3$, $N_G(S)$ acts irreducibly on $S$.

Suppose $p = 5$. If $G = X$, then no 5-local subgroup is maximal in $G$, a contradiction. If $G \neq X$, then $N_G(S)$ is maximal in $G$, again a contradiction.

Thus $p = 2$. Now $B$ is either one of the maximal local parabolic subgroups and the $P$ is the unique minimal parabolic not contained in $B$. \hfill \Box

Lemma 10.8. [j3] Suppose $X \cong J_3$. The $p = 2$ and $B$ is any of the two maximal local parabolic subgroups of $G$.

For $p > 3$, the Sylow $p$-subgroups are cyclic. For $p = 3$, $N_G(S)$ is maximal. So $p = 2$, $P(S) = \mathcal{M}(S)$ has size 2 and the lemma holds. \hfill \Box

Lemma 10.9. [m24] Suppose $X \cong \text{Mat}_{24}$. The $p = 2$ and $B$ is any of the three maximal 2-local parabolic subgroups.

Proof. For $p > 3$, the Sylow $p$-subgroups are cyclic. No 3-local subgroup is maximal in $G$ and so $p = 2$. Now $B$ is one of the three maximal 2-local subgroups containing $S$ and $P$ the unique 2-minimal parabolic subgroup not contained in $B$.
Lemma 10.10. [mcl] Suppose $X = \text{McL}$. Then $p = 3$, $B = 3^{1+4}.2$. $\text{Sym}(5)$ and $P = 3^4.\text{PSL}_2(9)$.

Proof. For $p > 5$, the Sylow $p$-subgroups are cyclic. For $p = 5$, $N_G(S)$ is maximal in $G$. For $p = 3$, $P$ has a unique 3-minimal parabolic of rank 1 type. Thus the lemma holds in this case.

Suppose $p = 2$. If $G \neq X$, then $G$ has a 2-minimal parabolic subgroup $\text{PSL}_3(4).2^2$ and we get a contradiction to 2.16. Thus $G = X$. Let $S \leq H \leq G$ with $H \sim 2^4.\text{Alt}(7)$. Then $H \notin R_2$ by 4.2 and so $H = B$. But there are two different choices for $B$, a contradiction. □

Lemma 10.11. [he] Suppose $X = \text{He}$. Then $G = X$, $p = 2$ and $B \cong 2^6.3.\text{Sym}(6)$.

Proof. For $p > 7$, the Sylow $p$-subgroups are cyclic, for $p = 7$, $N_G(S)$ is maximal in $G$ and for $p = 5$, $N_G(S)$ is irreducible on $S$. Thus $p \leq 3$. For $p = 3$, there exists a minimal parabolic subgroup $2^6.3^3$ $\leq 2^6.3.\text{Sym}(6)$, a contradiction to 2.16.

Thus $p = 2$. Suppose $G \neq X$. Let $H \in \mathcal{P}_G(S)$. Then $H/Q_H \cong \text{Sym}(3)^2/\text{Sym}(2)$ or $\text{PSL}_3(2).2$ and there is no candidate for $P$.

Thus $G = X$. Since $2^{1+3+3}.\text{PSL}_3(2)$ only contains two of the four members of $P_G(S)$, $B \cong 2^6.\text{Sym}(6)$ and the lemma is proved. □

Lemma 10.12. [ru] Suppose $X = \text{Ru}$. Then $p = 2$ and $B$ is any of the two maximal 2-local parabolic subgroups of $G$.

Proof. For $p > 5$, the Sylow $p$-subgroups of $G$ are cyclic. For $p = 5$, $N_G(S)$ is maximal in $G$. For $p = 3$, $G$ contains a 3-minimal parabolic subgroup $2^6.3^2$ $\leq 2^6.3.\text{Sym}(6)$, contradicting 2.16. So $p = 2$ and $M(S)$ has size two. □

Lemma 10.13. [suz] Suppose that $X \cong \text{Suz}$. The one of the following holds.

1. $[1]$ $p = 3$ and $B \cong 3^5.\text{Mat}_{11}$.
2. $[2]$ $p = 2$ and $B$ is any of the three maximal 2-local parabolic subgroups of $G$.

Proof. For $p > 5$, $S$ is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on $S$. Thus $p \leq 3$. For $p = 3$, $G$ has a maximal parabolic subgroup $H \sim 3^5.\text{Mat}_{11}$. Since $H \notin R_3$ by 10.1 $H = B$. So (1) holds.

For $p = 2$ $M(S)$ has size three and (2) holds. □

Lemma 10.14. [on] If $X = \text{O'N}$, then $p = 2$ then $B \cap X \sim 4.\text{PSL}_3(4).2$.

Proof. For $p > 7$ and $p = 5$, $S$ is cyclic and for $p = 3$, $N_G(S)$ acts irreducibly on $S$ so these cases do not arise. For $p = 7$, $G$ contains a parabolic subgroup $H = \text{PSL}_3(7).2$. Since $Q_H = 1$, $H \neq B$ and so $H \in R_7$. But $N_H(S)$ is maximal in $H$, a contradiction to 2.18 applied to $H$ in place of $G$.

Thus $p = 2$. Let $H \leq G$ with $H \cap X \sim 4.\text{PSL}_3(4).2$. Then $H$ is 2-minimal and we conclude that $B = H$. □

Lemma 10.15. [co3] If $X = \text{Co}_3$, then one of the following holds:

1. $[1]$ $p = 3$ and $B \cong 3^5.2.\text{Mat}_{11}$.
2. $[2]$ $p = 2$ and $B$ is any of the three maximal local parabolic subgroups of $G$.

Proof. For $p > 5$, $S$ is cyclic, so $p \leq 5$. For $p = 5$, $G$ contains a parabolic subgroup McL : 2, a contradiction to 10.10.

For $p = 3$, $G$ contains a maximal parabolic subgroup $H \sim 3^5.2.\text{Mat}_{11}$. Since $H \notin R_3$, we conclude that $H = B$. So (1) holds.

For $p = 2$, $M(S)$ has size three and (2) holds. □
Lemma 10.16. [co2] If $X \cong \text{Co}_2$, then one of the following holds:

1. $[\text{1}]$ $p = 3$ and $B \sim 3^1_{+}^{4} \cdot 2^{1+4} \cdot \text{Sym}(5)$.
2. $[\text{2}]$ $p = 2$ and $B$ is any of the three maximal 2-local parabolic subgroups of $G$.

Proof. If $p > 5$, $S$ is cyclic and for $p = 5$, $N_G(S)$ is a maximal subgroup in $G$ and so $p \leq 3$.

Suppose $p = 3$ and let $S \leq H \leq G$ be a local subgroups $H \sim 3^1_{+}^{4} \cdot 2^{1+4} \cdot \text{Sym}(5)$. Then $H \not\in R_3$ and so $H = B$ and (1) holds.

For $p = 2$, $M(S)$ has size two and (2) holds.

Lemma 10.17. [fi22] Suppose $X = \text{Fi}_{22}$. Then one of the following holds:

1. $[\text{1}]$ $p = 3$ and $B \sim 3^1_{+}^{4} \cdot 2^{1+2} \cdot 2^1 \cdot 3^1 \cdot 1^1$.
2. $[\text{2}]$ $p = 2$ and $B$ is an arbitrary maximal local parabolic subgroup.

Proof. For $p > 5$, $S$ is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on $S$.

Suppose $p = 3$. Then $G$ has a maximal parabolic subgroup $H$ with $H \sim 3^1_{+}^{4} \cdot 2^{1+2} \cdot 2^1 \cdot 3^1 \cdot 1^1$.

If $H \neq B$ we get that $O^p(P)$ is normal in $H$, a contradiction (as can been seen by intersection $H$ with a subgroup $\Omega_7(3)$). So $H = B$ and (1) holds.

If $p = 2$ then (2) holds.

Lemma 10.18. [HN] Suppose $X = \text{HN}$. Then one of the following holds:

1. $[\text{1}]$ $p = 5$ and $B \sim 5^1_{+}^{4} \cdot 2^{1+4} \cdot 5^4$.
2. $[\text{2}]$ $p = 2$ and $B \cap X \sim 2^{1+8} \cdot \text{Alt}(5) \wr \text{Sym}(2)$.

Proof. If $p > 5$, then $S$ is cyclic so $p \leq 5$.

If $p = 5$ let $H$ be the maximal 5-local parabolic subgroup with $H \sim 5^1_{+}^{4} \cdot 2^{1+4} \cdot 5^4$. Then, as $H$ is soluble and $p \neq 2, 3$, $P \ncong H$ and consequently $H = B$. Thus (1) holds.

If $p = 3$, then $X$ contains a subgroup contains a subgroup $L \sim 3^1_{+}^{4} \cdot 4 \cdot \text{Alt}(5)$. Since $L \not\in R_3$, we have that $B = X$. Let $H$ be the parabolic with $H/Q_H \cong 2.(\text{PSL}_2(3) \times \text{PSL}_2(3)).4$ Since $N_H(S)$ is a maximal subgroup of $H$, we get that $H \leq B$ by 2.18. But then $G = B$, a contradiction.

Suppose $p = 2$. Then $G$ has maximal parabolic subgroup $H$ with $H \cong 2^{1+8} \cdot \text{Alt}(5) \wr \text{Sym}(2)$. using 2.9 we have that $H = B$ and (2) holds.

Lemma 10.19. [ly] Suppose $X = \text{Ly}$. Then $p = 5$ and $H \sim 5^1_{+}^{4} \cdot 4 \cdot \text{Sym}(6)$.

Proof. For $p \geq 7$, $S$ is cyclic. So $p \leq 5$.

Suppose that $p = 5$. Let $S \leq H \leq G$ with $H \sim 5^1_{+}^{4} \cdot 4 \cdot \text{Sym}(6)$. Since $\text{Sym}(6) \not\in R_5$, we get $B = H$ and the lemma holds in this case.

Suppose $p = 3$, then $G$ has two maximal parabolic subgroup $F, H$ with $F/Q_F \sim 2 \times \text{Mat}_{11}$ and $H/Q_H \sim 2 \cdot \text{Alt}(5) \cdot \text{Dih}_{12}$. So by 4.2 and 10.1 neither $F$ nor $H$ are in $R_3$, a contradiction.

Suppose that $p = 2$. Then $G$ has a parabolic subgroups $H \sim 3 \cdot \text{McL}.2$. Since $Q_H \neq 1$, $H \not\in B$, but this contradicts 10.10.

Lemma 10.20. [th] Suppose $X = \text{Th}$. Then one of the following holds:

1. $[\text{1}]$ $p = 3$ and $\{B, P\} = P(S)$.
2. $[\text{2}]$ $p = 2$ and $B \sim 2^{1+8} \cdot \text{Alt}(9)$.
Proof. For $p > 7$, $S$ is cyclic, for $p = 7$, $N_G(S)$ acts irreducibly on $S$ and for $p = 5$, $N_G(S)$ is maximal in $G$. Hence $p \leq 3$.

For $p = 3$, (1) holds.

For $p = 2$, let $S \leq H \leq G$ with $H \sim 2_{+}^{1+8}.\text{Alt}(9)$. Then $H \not\in \mathcal{R}_2$ so $B = H$ and (2) holds. □

Lemma 10.21. [fi23] Suppose that $X \cong \text{Fi}_{23}$. Then one of the following holds:
1. [1] $p = 3$ and $B \sim 3_{+}^{1+8}2_{-}^{1+6}.3_{+}^{1+2}.2.\text{Sym}(4)$.
2. [2] $p = 2$ and $B \sim 2^{11}.\text{Mat}_{23}$ or $B \sim 2^{6+2.4}.(\text{Alt}(7) \times \text{Sym}(3))$.

Proof. For $p \geq 7$, $S$ is cyclic and for $p = 5$, $N_G(S)$ acts irreducibly on $S$. So $p \leq 3$.

Suppose that $p = 3$ and let $S \leq H \leq G$ with $H = 3_{+}^{1+8}2_{-}^{1+6}.3_{+}^{1+2}.2.\text{Sym}(4)$. Since $H$ is soluble and $O^p(\mathcal{N})$ is not normal in $H$, we get that $H = B$ and (1) holds.

Suppose $p = 2$ and let $S \leq H \leq G$ with $H \sim 2^{11}.\text{Mat}_{23}$. If $H = B$, (2) holds. So suppose that $H \not\leq B$, then by 10.6, $(H \cap B)/O_2(H \cap B) \cong \text{Alt}(7)$. Since $B$ is a maximal 2-local subgroup of $G$, we conclude $B \sim 2^{6+2.4}.(\text{Alt}(7) \times \text{Sym}(3))$ and again (2) holds. □

Lemma 10.22. [co1] Suppose $X = \text{Co}_1$. Then one of the following holds:
1. [1] $p = 5$ and $B$ is one of the two maximal 5-local parabolic subgroups of $G$.
2. [2] $p = 3$ and $B$ is one of the three maximal 3-local parabolic subgroups of $G$. (Note here that the two maximal subgroups $N(3C^2)$ in the Atlas [2] need to be deleted, see Modular Atlas[5].)
3. [3] $p = 2$ and $B$ is one of the four maximal 2-local parabolic subgroups of $G$.

Proof. For $p > 7$, $S$ is cyclic and for $p = 7$, $N_G(S)$ is irreducible on $S$. So $p \leq 5$ and one of (1), (2) and (3) holds. □

Lemma 10.23. [j4] Suppose $X = \text{J}_4$. The $p = 2$ and $B$ is one of the three maximal 2-local parabolic subgroups of $G$.

Proof. If $p \leq 5$ but $p \neq 11$, then $S$ is cyclic and we are done. If $p = 11$ then $N_G(S)$ is maximal so this case falls. If $p = 3$, then $S$ is contained in $H \sim 2^{11}.\text{Mat}_{24}$ which is not in $\mathcal{R}_3$ by 10.9. Since $Q_H = 1$ we have a contradiction. So $p = 2$ and the lemma is proved. □

Lemma 10.24. [fi24] Suppose $X = \text{Fi}_{24}$. The one of the following holds:
1. [1] $p = 3$ and $B \sim 3_{+}^{1+10}.\text{PSU}_5(2).2$.
2. [2] $p = 2$ and $B$ is any of the four maximal 2-local parabolic subgroups of $G$.

Proof. For $p > 7$, $S$ is cyclic and so $p \leq 7$. For $p = 7$, $S$ is contained in $H.\text{He}$.2 and we obtain a contradiction via 10.11. For $p = 5$, $N_G(S)$ acts irreducibly on $S$, so this case fails.

Suppose $p = 3$. If $B \sim 3^{7}.\Omega_7(3)$, then $P/Q_p \cong \text{Alt}(5)$, a contradiction as $\text{Alt}(5) \not\in \mathcal{R}_3$. Hence (1) holds in this case.

If $p = 2$, then (2) holds. □

Lemma 10.25. [bm] Suppose $X \cong BM$. Then one of the following holds:
1. [1] $p = 5$ and $B \sim 5_{+}^{1+4}.2_{-}^{1+4}.\text{Alt}(5).4$.
2. [2] $p = 3$ and $B \cong 3_{+}^{1+8}.2_{-}^{1+6}.\text{O}_6^{+}(2)$.
3. [3] $p = 2$ and $B$ is any of the four maximal 2-local parabolic subgroups of $G$. 

Proof. If \( p > 7 \), then \( S \) is cyclic and, if \( p = 7 \), \( N_G(S) \) is irreducible on \( S \). So \( p \leq 5 \).
Suppose \( p = 5 \) and let \( S \leq H \leq B \) with \( H \cong 5_+^{1+4}.2_+^{1+4}.\text{Alt}(5) \). Then \( H \not\in R_5 \) and so \( H = B \) and (1) holds.
Suppose that \( p = 3 \) and let \( S \leq H \leq B \) with \( H \cong 3_+^{1+8}.2_+^{1+6}.O_6^- (2) \). Then \( H \not\in R_3 \) and so \( H = B \) and (2) holds.
Suppose that \( p = 2 \). The (3) holds. \( \Box \)

Lemma 10.26. \( [m] \) Suppose that \( X \cong M \). Then one of the following holds:
1. \( [1] \) \( p = 7 \) and \( B \cong 7^{1+4}.6.\text{Sym}(7) \).
2. \( [2] \) \( p = 5 \) and \( B \cong 5_+^{1+6}.4.J_2.2 \).
3. \( [3] \) \( p = 3 \) and \( B \cong 3_+^{1+12}.2.\text{Suz} \) or \( B \cong 3^{2+5+5}.2.(\text{Mat}_{11} \times \text{GL}_2(3)) \).
4. \( [4] \) \( p = 2 \) and \( B \) is any of the five maximal 2-local parabolic subgroups.

Proof. For \( p > 13 \), \( S \) is cyclic, for \( p = 13 \), \( N_G(S) \) is maximal in \( G \) and for \( p = 11 \), \( N_G(S) \) is irreducible on \( S \). Thus \( p \leq 7 \).
For \( p = 7 \) choose \( S \leq H \leq G \) with \( H \cong 7^{1+4}.6.\text{Sym}(7) \). Then \( H \not\in R_7 \) and so \( B = H \). Hence (1) holds.
For \( p = 5 \) choose \( S \leq H \leq G \) with \( H \cong 5^{1+6}.4.J_2.2 \). By 10.5 \( H \not\in \text{cal}R_5 \) and so \( B = H \). Hence (2) holds.
For \( p = 3 \) choose \( S \leq H \leq G \) with \( H \cong 3_+^{1+12}.2.\text{Suz} \). If \( H = B \), then (3) holds. If \( H \neq B \) then 10.13 implies that \( H \cap B/O_3(H \cap B) \cong 2.\text{Mat}_{11} \). Thus \( B \cong 3^{2+5+5}.2.(\text{Mat}_{11} \times \text{GL}_2(3)) \) and (3) holds.
If \( p = 2 \), then (4) holds.

11. The smaller list
In this section we assume in addition that if \( \bar{P} \leq M \leq S \) and \( M/Q_M \) is a classical group extended by field automorphism, then the classical groups is \((S)L_n(q)\).

References


U. MEIERFRANKENFELD

C.W. PARKER, SCHOOL OF MATHEMATICS AND STATISTICS, UNIVERSITY OF BIRMINGHAM, EDBASTON, BIRMINGHAM B15 2TT, UK

E-mail address: C.W.Parker@bham.ac.uk
P.J. Rowley, Department of Mathematics, UMIST, PO Box 88, Manchester M60 1QD, UK