1 Introduction

Let $G$ be a finite group and $p$ a prime. A subgroup $P$ containing a Sylow $p$-subgroup of $G$ is a $p$-parabolic subgroup of $G$, and $P$ is a local $p$-parabolic subgroup if in addition $O_p(P) \neq 1$.

Moreover, $G$ has characteristic $p$ if $C_G(O_p(G)) \leq O_p(G)$; and $G$ has parabolic characteristic $p$ if every local $p$-parabolic subgroup has characteristic $p$.

The standard examples for groups of parabolic characteristic $p$ are the finite simple groups of Lie type in characteristic $p$. In these examples every proper parabolic subgroup is a local $p$-parabolic subgroup, and for maximal parabolic subgroups $M$ the normal subgroup $\Omega_1 Z(O_p(M))$, considered as a $GF(p)M$-module, has a remarkably restricted structure. In this paper we try to understand this phenomena in arbitrary finite groups.

What kind of properties of the module $\Omega_1 Z(O_p(M))$ should one aim at in general? A possible answer arose during our detailed study of the $p$-local structure of groups of local characteristic $p$ in [MSS], where a group has local characteristic $p$ if each of its $p$-local subgroup has characteristic $p$.

**Definition 1.1** Let $A$ be an elementary abelian $p$-group and $V$ a finite dimensional $GF(p)A$-module. Then $A$ is

(a) quadratic on $V$ if $[V, A, A] = 0$,

(b) nearly quadratic on $V$ if $[V, A, A, A] = 0$ and

$$[V, A] + C_V(A) = [v, A] + C_V(A) \text{ for every } v \in V \setminus [V, A] + C_V(A),$$

(c) an offender on $V$ if $|V/C_V(A)| \leq |A/C_A(V)|$,

(d) a $2F$-offender on $V$ if $|V/C_V(A)| \leq |A/C_A(V)|^2$,

(e) non-trivial on $V$ if $[V, A] \neq 0$.

A $p$-subgroup $Y$ of $G$ is called $p$-reduced (for $G$) if $Y$ is elementary abelian and normal in $G$, and $O_p(G/C_G(Y)) = 1$. The largest $p$-reduced subgroup of $G$ is denoted by $Y_G$; for the existence of $Y_G$ see 2.2(a).

Let $M$ be a subgroup of $G$. Then $M$ is $F$-stable (in $G$) if none of the elementary abelian $p$-subgroups of $N_G(Y_M)/C_G(Y_M)$ are non-trivial offenders on $Y_M$. Similarly, $M$ is $2F$-stable (in $G$) if none of the elementary abelian $p$-subgroups of $N_G(Y_M)/C_G(Y_M)$ are non-trivial nearly quadratic $2F$-offenders on $Y_M$. 


Modules admitting non-trivial $2F$-offenders have been investigated by Guralnick, Lawther and Malle in [GLM],[GM1],[GM2], and [L]. They have classified all pairs $(V,G)$, where $V$ is an irreducible $GF(p)G$-module and $G$ is a known finite almost quasisimple group containing a non-trivial $2F$-offender on $V$.

Their result is a major generalization of earlier results, where $G$ was assumed to contain a non-trivial offender.

For stating our results we need some further definitions.

**Definition 1.2** By $S(X)$ we denote the subgroups of $G$ containing $X$. Let $S$ be Sylow $p$-subgroup of $G$.

\[ B(S) := C_S(\Omega_1 Z J(S)), \]

\[ C^*(G,S) := \langle C_G(\Omega_1 Z(S)), N_G(C) \mid 1 \neq C \text{ char } B(S) \rangle, \]

and

\[ C^{**}(G,S) = \langle N_G(J(S)), C_G(\Omega_1 Z(S)) \rangle. \]

A factorization family for $S(S)$ is a subset $\mathcal{F}(S) \subseteq S(S)$ with the following two properties:

(i) For every $H \in S(S)$ there exists $M \in \mathcal{F}(S)$ with $H \subseteq C_G(Y_H)M$ and $Y_H \leq Y_M$.

(ii) If $H \in S(S)$ and $M \in \mathcal{F}(S)$ with $M \subseteq C_G(Y_M)H$ and $Y_M \leq Y_H$, then $Y_M = Y_H$ and $H \leq M$.

Property (i) implies

\[ H/C_H(Y_H) \cong HC_G(Y_H)/C_G(Y_H) \cong (HC_G(Y_H) \cap M)C_G(Y_H)/C_G(Y_H), \]

so the action of $H$ on $Y_H$ is isomorphic to the action of $HC_G(Y_H) \cap M$ on the submodule $Y_H$ of $Y_M$. In particular, it suffices to identify $M/C_M(Y_M)$ and its action on $Y_M$ to identify $H/C_H(Y_H)$ and $Y_H$.

Property (ii) is the crucial one for applications since it has strong consequences. For example, if $G$ is of parabolic characteristic $p$ and $S \leq H \leq M \in \mathcal{F}(S)$ such that $M = HC_M(Y_M)$, then $M$ is the unique maximal $p$-local subgroup of $G$ containing $H$ (see 3.5).

Of course, it is not clear a priori that factorization families exist. The existence (and uniqueness) will be established in Theorem 3.4.

**Theorem 1.3** Let $G$ be a finite group and $S \in \text{Syl}_p(G)$. There exists a unique factorization family $\mathcal{F}(S)$ for $S(S)$ in $G$. Moreover, at most one member of $\mathcal{F}(S)$ is $F$-stable, and

\[ \Omega_1 Z(S) \leq Y_M \text{ and } M = N_G(Y_M) \text{ for every } M \in \mathcal{F}(S); \]

in particular, the elements of $\mathcal{F}(S)$ are $p$-local subgroups of $G$ if $S \neq 1$.

In the following results $\mathcal{F}(S)$ is always a factorization family for $S(S)$. Recall that a finite group $H$ is $p$-constrained if $H/O_p(H)$ is of characteristic $p$.

**Theorem 1.4** Let $G$ be a finite group and $S \in \text{Syl}_p(G)$, and let $1 \neq C \text{ char } B(S)$ and $M := N_G(C)$. Suppose that there exists $N \in \mathcal{F}(S)$ that is $F$-stable.

(a) If $C = B(S)$, then $Y_N = Y_M$ and $N = C_G(Y_M)M = N_G(Y_M)$. 

2
(b) If $Y_N \leq O_p(M)$, then $Y_M = Y_N$ and $M \leq N$.

(c) If $M$ is $p$-constrained, then $M = O'_p(M)(M \cap N)$.

**Theorem 1.5** Let $G$ be a finite group and $S \in \text{Syl}_p(G)$, and let $M \in S(S)$ such that $\Omega_1 Z(S) \leq M$ or $M = N_G(C)$ for some $1 \neq C \text{ char } B(S)$. Suppose that there exists $N \in F(S)$ that is $2F$-stable.

(a) If $Y_N \leq O_p(M)$, then $M \leq N$.

(b) If $M$ is $p$-constrained, then $M = O'_p(M)(M \cap N)$.

(c) The following hold for any $p$-constrained $H \in S(B(S))$ with $H \nsubseteq O'_p(H)N$ (where $\overline{H} = H/O'_p(H)$):

(a) $\overline{Y_N} \leq O_p(\overline{H})$.

(b) $C_{O_p(\overline{H})}(\overline{Y_N}) \leq \overline{H}$.

(c) $Y_{\overline{H}}$ is not $F$-stable in $\overline{H}$.

(d) $C^{**}(\overline{H}, T) \leq \overline{H} \cap N < \overline{H}$, where $B(S) \leq T \in \text{Syl}_p(H)$.

For groups of parabolic characteristic $p$ more can be said about the members of the factorization family $F(S)$.

**Theorem 1.6** Let $G$ be a finite group of parabolic characteristic $p$ and $1 \neq S \in \text{Syl}_p(G)$. Then the members of $F(S)$ are maximal $p$-local subgroups of $G$. Moreover, if $N \in F(S)$ is $2F$-stable and $H \in S(B(S))$ with $B(S) \leq T \in \text{Syl}_p(H)$, then $C^*(H, T) \leq N$.

**Corollary 1.7** Let $G$ be a finite group of parabolic characteristic $p$ and $S \in \text{Syl}_p(G)$. If $S$ is contained in at least two maximal $p$-local subgroups of $G$, then there exists $M \in F(S)$ such that $M$ is not $2F$-stable.

Let $G$ and $N$ be as in 1.6, and let $H$ be a $p$-local subgroup containing $S$ such that $H \nsubseteq N$. Then by 1.6 $C^*(H, S)$ is a proper subgroup of $H$. In this case the structure of $H$ can be described precisely using the Local $C(G, T)$-Theorem proved in [BHS].

The proof of the above theorems relies heavily on two elementary results from [PPS] and [Ste], the $L$-Lemma and the $qrc$-Lemma. The authors found it remarkable that these results allow to study finite groups in this context without any $K$-group assumption.

In fact, using the $L$-Lemma another result is proved, which is interesting in its own right and which can be used to improve the $qrc$-Lemma.

**Theorem 1.8** Let $G$ be a finite group, $S \in \text{Syl}_p(G)$, and $V$ be a finite dimensional faithful $GF(p)G$-module. Suppose that $O_p(G) = 1$ and $S$ is contained in a unique maximal subgroup of $G$. Then $|A| = |V/C_V(A)|$ for every offender $A$ of $G$ on $V$. 

3
2 Elementary Properties

In this section \( G \) is a finite group, \( p \) is a prime, and \( S \in \text{Syl}_p(G) \).

**Notation 2.1** Let \( X \) be a \( p \)-subgroup of \( G \). A subgroup \( P \) of \( G \) is \( X \)-minimal if \( X \) is contained in a unique maximal subgroup of \( P \) and \( X \not\subseteq O_p(P) \).

**Lemma 2.2** Let \( L \) be a subgroup of \( G \) and \( P \) be a \( p \)-parabolic subgroup of \( L \).

(a) There exists a unique largest \( p \)-reduced subgroup \( Y_L \) of \( L \).

(b) If \( Y \) is a \( p \)-reduced subgroup of \( P \) with \( Y \leq O_p(L) \), then \( \langle Y^L \rangle \) is \( p \)-reduced for \( L \) and so \( Y \leq Y_L \).

(c) If \( L \) is of characteristic \( p \), then \( Y_P \leq Y_L \).

**Proof:** (a): Let \( A \) and \( B \) be \( p \)-reduced subgroups of \( L \). It suffices to show that also \( AB \) is \( p \)-reduced. Then \( Y_L \) is the product of all \( p \)-reduced subgroups of \( L \).

Since \( A \) is \( p \)-reduced, \( B \leq O_p(L) \leq C_L(A) \) and so \( AB \) is elementary abelian. Let \( D \) be the inverse image of \( O_p(L/C_L(AB)) \). Since \( C_L(AB) \leq C_L(A) \), \( D\langle C_L(A)/C_L(AB) \rangle \leq O_p(L/C_L(AB)) \) and so \( D \leq C_L(A) \). By symmetry, \( D \leq C_L(B) \) and thus \( D \leq C_L(A) \cap C_L(B) = C_L(AB) \).

(b): Since \( P \) is a \( p \)-parabolic subgroup of \( L \), \( O_p(L) \leq P \). Hence \([Y,O_p(L)] = 1 \) since \( Y \) is \( p \)-reduced in \( P \). By assumption \( Y \leq O_p(L) \) and so \( Y \leq Y_L \).

(c): As in (b), \([Y_P,O_p(L)] = 1 \). Since \( L \) is characteristic \( p \), \( Y_P \leq O_p(L) \). So (b) implies \( Y_P \leq Y_L \).

\( \square \)

**Lemma 2.3** Let \( X \leq S \leq P \leq G \). Suppose that \( P \) is \( X \)-minimal and \( N \trianglelefteq P \). Then either \( O_p(P) \leq N \) and \( P = XN \), or \( S \cap N \leq O_p(P) \). In particular, \( P = XO^p(P) = \langle X^P \rangle \).

**Proof:** Observe that \( P = NN_P(S \cap N) \). As \( P \) is \( X \)-minimal, either \( NX = P \) or \( N_P(S \cap N) = P \), and in the second case \( S \cap N \leq O_p(P) \).

Since \( X \nsubseteq O_p(P) \), \( S \cap XO^p(P) \nsubseteq O_p(P) \) and so \( P = XO^p(P) \). A similar argument gives \( P = \langle X^P \rangle \).

**Lemma 2.4** Let \( A \) be an \( F \)-stable elementary abelian \( p \)-subgroup of \( G \), and let \( Q \) be a \( p \)-subgroup of \( G \) with \( A \leq Q \). Then the following hold:

(a) \( A \leq Z(J(Q)) \).

(b) \( \langle A^{N_G(Q)} \rangle \) is elementary abelian.

**Proof:** (a): Let \( B \in A(Q) \). Then \( B \) acts on \( A \), and \(|B| \geq |C_B(A)A| \) by the maximality of \( B \). Also \( C_B(A) \cap A \leq A \cap B = C_B(A) \) and so \( C_B(A) \cap A = A \cap B \). Hence

\[ |C_B(A)||A||C_B(A)^{-1} | = |C_B(A)||A||A \cap B|^{-1} = |C_B(A)A| \leq |B|, \]

and \(|A/C_B(A)| \leq |B/C_B(A)| \) follows. The \( F \)-stability of \( A \) gives \(|A,B| = 1 \) and (a) holds.

(b): This is a direct consequence of (a) since \( Z(J(Q)) \leq N_G(Q) \).

\( \square \)
Lemma 2.5 Let $Q$ be a normal $p$-subgroup of $G$ with $C_G(Q) \leq Q$ and $Y$ be an abelian $p$-subgroup of $G$. If $C_Q(Y) \unlhd G$ and $Q$ normalizes $Y$, then $Y \leq O_p(G)$.

Proof: Observe that
\[
\langle Q,Y \rangle \leq Q \cap Y \leq C_Q(Y).
\]
Since $C_Q(Y) \leq G$ this shows that $\langle Y^G \rangle$ centralizes $Q/C_Q(Y)$ and $C_Q(Y)$. Hence $O_p((Y^G))$ centralizes $Q$ and since $C_G(Q) \leq Q$, $O_p((Y^G)) = 1$ and $\langle Y^G \rangle$ is a $p$-group. Thus $Y \leq O_p(G)$. \hfill \Box

Lemma 2.6 Let $A$ be a finite elementary abelian $p$-group and $V$ a finite dimensional $GF(p)A$-module. Suppose that $A$ is quadratic on $V$ and $[v,A] = [V,A]$ for every $v \in V \setminus C_V(A)$. Then $A$ is a quadratic offender on every $A$-submodule of $V$.

Proof: Since every $A$-submodule of $V$ satisfies the same hypothesis it suffices to show that $A$ is an offender on $V$. Without loss, $|V,A| \neq 1$. Choose $W \leq [V,A]$ with $|V,A|/|W| = p$ and put $\overline{V} = V/W$. Let $U$ be the inverse image of $C_{\overline{V}}(A)$ in $V$. Then $[U,A] \leq W$ and so $[V,A] \nleq [U,A]$. Thus $U \leq C_V(A)$ and $C_{\overline{V}}(A) = C_V(A)$; in particular, $|V/C_V(A)| = |\overline{V}/C_{\overline{V}}(A)|$. Note that $\overline{V}$ satisfies the hypothesis, so replacing $V$ by $\overline{V}$ we may assume that $|[V,A]| = p$. Let $B < A$ with $|A/B| = p$. Since $[V,B]$ is at most 1-dimensional, $B$ in place of $A$ also satisfies the hypothesis of the lemma. Hence by induction on $|A|$, $|V/C_V(B)| \leq |B|$.

Let $a \in A \setminus B$. Since $|[V,a]| = p$, $|V/C_V(a)| \leq p$ and so also $|C_V(B)/C_V(B) \cap C_V(a)| \leq p$. But $C_V(A) = C_V(B) \cap C_V(a)$ and so
\[
|V/C_V(A)| \leq |V/C_V(B)|p \leq |B|p = |A|.
\]
\hfill \Box

3 A Partial Ordering

In this section $G$ is a finite group, $p$ is a prime, and $S \in Syl_p(G)$.

Notation 3.1 Let $A$ and $B$ be subgroups of $G$. The relation $\ll$ on the subgroups of $G$ is defined by
\[
A \ll B : \iff A \subseteq C_G(Y_A)B \text{ and } Y_A \leq Y_B.
\]
Furthermore, we define
\[
A^\dagger := C_G(Y_A)A \text{ and } S^\dagger := \{L \leq G \mid L = L^\dagger \}.
\]

Lemma 3.2 Let $L$ and $M$ be subgroups of $G$.
\[(a) \ Y_L \leq Y_{L^\dagger}, \ L \ll L^\dagger, \text{ and } (L^\dagger)^\dagger = L^\dagger.
\[(b) \ S^\dagger = \{L \leq G \mid C_G(Y_L) \leq L \}.
\[(c) \ll \text{ is reflexive and transitive.}
\[(d) \ L \subseteq C_G(Y_L)M \text{ if and only if } L \leq C_G(Y_L)N_M(Y_L).
\[(e) \text{ Suppose that } L \subseteq C_G(Y_L)M \text{ and } L \cap M \text{ is a } p\text{-parabolic subgroup of } L \text{ and } M. \text{ Then } Y_L \text{ is } p\text{-reduced for } N_M(Y_L) \text{ and } L \ll N_M(Y_L).
\]
(f) If \( L = L^\dagger \), then \( L \ll L \) if and only if \( Y_L \leq Y_M \) and \( L = C_G(Y_L)(L \cap M) \).

(g) Restricted to \( S^\dagger \), \( \ll \) is a partial ordering.

**Proof:**
(a): Clearly \( Y_L \) is a \( p \)-reduced subgroup of \( L^\dagger \), so \( Y_L \leq Y_L^\dagger \). Thus \( C_G(Y_L^\dagger) \leq C_G(Y_L) \leq L^\dagger \) and \( L^\dagger = (L^\dagger)^\dagger \).

(b): This is an immediate consequence of the definition of \( L^\dagger \).

(c): Obviously \( \ll \) is reflexive. If \( A, B, C \leq G \) with \( A \ll B \) and \( B \ll C \), then \( Y_A \leq Y_B \leq Y_C \) and so \( Y_A \leq Y_C \). Also \( C_G(Y_A) \leq C_G(Y_B) \) and hence
\[
A \subseteq C_G(Y_A)B \subseteq C_G(Y_A)C_G(Y_B)C = C_G(Y_A)C.
\]

Thus \( A \ll C \) and \( \ll \) is transitive.

(d): If \( L \subseteq C_G(Y_L)M \) then \( L \leq N_G(Y_L) \cap C_G(Y_L)M = C_G(Y_L)N_M(Y_L) \). The other direction is obvious.

(e): Since \( L \cap M \) is a \( p \)-parabolic subgroup of \( L \),
\[
Y_L \leq O_p(L) \leq L \cap M \leq N_M(Y_L),
\]

so \( Y_L \) is an elementary abelian normal subgroup of \( N_M(Y_L) \). Since \( L \cap M \) is a \( p \)-parabolic subgroup of \( M \), \( C_G(Y_L)(L \cap M) \) and thus also \( C_G(Y_L)L \) are \( p \)-parabolic subgroups of \( C_G(Y_L)N_M(Y_L) \).

As \( Y_L \) is a \( p \)-reduced subgroup of \( C_G(Y_L)L \), 2.2(b) shows that \( Y_L = (Y_L^{C_G(Y_L)N_M(Y_L)}) \) is \( p \)-reduced for \( C_G(Y_L)N_M(Y_L) \). Hence \( Y_L \) is also a \( p \)-reduced subgroup of \( N_M(Y_L) \). Thus \( Y_L \leq Y_{N_M(Y_L)} \) and so \( L \ll N_M(Y_L) \).

(f): Since \( L \in S^\dagger \) we have \( C_G(Y_L) \leq L \) and so \( L \subseteq C_G(Y_L)M \) implies \( L = C_G(Y_L)(L \cap M) \). Now (f) is obvious.

(g): Let \( L, M \in S^\dagger \) with \( L \ll M \) and \( M \ll L \). Since \( Y_L \leq Y_M \leq Y_L \), we have \( Y_L = Y_M \). By (f) \( L = C_G(Y_L)(L \cap M) \) and \( M = C_G(Y_M)(M \cap L) \). Hence \( Y_M = Y_L \) gives \( L = M \). So the restriction of \( \ll \) to \( S^\dagger \) is anti-symmetric. Now (g) follows (c). □

**Notation 3.3** Put \( S^\dagger(S) := \{ L \in S^\dagger \mid S \leq L \} \). According to 3.2(g) \( \ll \) restricted to \( S^\dagger(S) \) is a partial ordering on \( S^\dagger(S) \). We denote the set of maximal elements of \( S^\dagger(S) \) with respect to \( \ll \) by \( F(S) \).

**Theorem 3.4** \( F(S) \) is the unique factorization family for \( S(S) \).

**Proof:** Let \( G \) be a factorization family for \( S(S) \) and let \( M \in G \). Clearly \( M \leq M^\dagger \) and by 3.2(a), \( Y_M \leq Y_M^\dagger \). So Condition (ii) of 1.2 gives \( M = M^\dagger \). Thus \( M \in S^\dagger(S) \) and \( G \subseteq S^\dagger(S) \).

Now let \( G \) be any subset of \( S^\dagger(S) \). Then Condition (i) of 1.2 is fulfilled for \( G \) if and only if for each \( L \in S(S) \) there exists \( M \in G \) with \( L \ll M \). Since \( L \ll L^\dagger \) and \( \ll \) is transitive by 3.2, we conclude that \( G \) fulfills (i) if and only if \( G \) contains all the maximal elements of \( S^\dagger(S) \) with respect to \( \ll \). And Condition (ii) holds if and only if all elements of \( G \) are maximal with respect to \( \ll \) in \( S^\dagger(S) \). Thus \( F(S) \) is the unique factorization family for \( S(S) \). □

**Lemma 3.5** Let \( M \in F(S) \) and \( H \in S(S) \) with \( M = C_M(Y_M)(M \cap H) \). If \( H \) is \( p \)-constrained, then \( H = O_p(H)(H \cap M) \). In particular, if \( G \) is of parabolic characteristic \( p \) and \( S \leq L \leq M \) with \( M = C_M(Y_M)L \), then \( M \) is the unique maximal \( p \)-local subgroup of \( G \) containing \( L \).
Proof: Put $\overline{T} = H/O_{p'}(H)$. Since $M = C_M(Y_M)(H \cap M)$, $Y_M$ is $p$-reduced for $H \cap M$ and $\overline{Y_M}$ is a $p$-reduced subgroup of $H/\overline{M}$. So by 2.2(c), $\overline{Y_M} \leq Y_{\overline{T}}$. Let $Y \leq S$ with $Y = Y_{\overline{T}}$ and $K := N_H(Y)$. Then by the Frattini argument, $H = O_{p'}(H)K$. It follows that $Y$ is a $p$-reduced subgroup of $K$, so $Y_M \leq Y \leq Y_K$.

As $YO_{p'}(H) \cap M = Y(O_{p'}(H) \cap M)$, we also get, using the Frattini argument one more time,

$$H \cap M = (O_{p'}(H) \cap M)(K \cap M) = O_{p'}(M \cap H)(K \cap M).$$

Thus $M = C_M(Y_M)(H \cap M) \leq C_G(Y_M)K$ since $O_{p'}(M \cap H)$ centralizes $Y_M$. Now 1.2(ii) implies that $K \leq M$ and so $H = O_{p'}(H)(H \cap M)$. Hence the first statement holds.

To prove the second statement, let $H$ be a $p$-local subgroup containing $L$. Then $M = C_M(Y_M)L$ implies $M = C_M(Y_M)(H \cap M)$. On the other hand $H$ is of characteristic $p$ since $G$ has parabolic characteristic $p$, so $H$ is $p$-constrained and $O_{p'}(H) = 1$. Hence by the first statement $H = O_{p'}(H)(H \cap M) \leq M$.  

□

Lemma 3.6 Let $M \in \mathcal{F}(S)$, $S_0 := C_S(Y_M)$ and $M_0 := N_M(S_0)$.

(a) $M = C_M(Y_M)M_0$, $S_0 = O_p(M_0)$ and $C_S(S_0) \leq S_0$.

(b) $\Omega_1 Z(S) \leq Y_M = Y_{M_0} = \Omega_1 Z(S_0)$.

Proof: (a): The Frattini argument gives $M = C_M(Y_M)M_0$. Hence $O_p(M_0) = S_0$, since $Y_M$ is $p$-reduced. Clearly $Y_M \leq \Omega_1 Z(S_0)$, and so $C_S(S_0) \leq C_S(\Omega_1 Z(S_0)) \leq C_S(Y_M) \leq S_0$.

(b): Let $S_0 \leq S_1 \leq S$ with

$$S_1 C_{M_0}(\Omega_1 Z(S_0))/C_{M_0}(\Omega_1 Z(S_0)) = O_p(M_0/C_{M_0}(\Omega_1 Z(S_0))).$$

Then $S_1 C_M(Y_M)/C_M(Y_M)$ is a normalized by $M_0 C_M(Y_M) = M$. Since $O_p(M/C_M(Y_M)) = 1$ we get $S_1 \leq C_M(Y_M)$, so $S_1 = S_0 = O_p(M_0)$ by (a), and $\Omega_1 Z(S_0)$ is $p$-reduced for $M_0$. Together with 3.2(a) this gives

$$Y_M \leq \Omega_1 Z(S_0) \leq Y_{M_0} \leq Y_{M_1}. $$

In particular $M \ll M_1$, and the maximality of $M$ yields $Y_M = Y_{M_1}$. Now (b) follows, since also $\Omega_1 Z(S) \leq \Omega_1 Z(S_0)$.

Proof of Theorems 1.3 and 1.4:

By 3.4 $\mathcal{F}(S)$ is the unique factorization family for $S(S)$. Let $M \in \mathcal{F}(S)$. By 3.6(b) $\Omega_1 Z(S) \leq Y_M$ and by 3.2(e) $M \ll N_G(Y_M)$. Hence the maximality of $M$ gives $M = N_G(Y_M)$.

Assume that there exists $N \in \mathcal{F}(S)$ that is $F$-stable, i.e. $Y := Y_N$ is $F$-stable in $N_G(Y_N) = N$. Then by 2.4 $B(S) \leq C_G(Y)$ and

$$N = C_G(Y) N_N(B(S)) \subseteq C_G(Y)L, \quad \text{where } L := N_G(B(S));$$
in particular $Y \leq \Omega_1 Z(B(S)) \leq O_p(N_G(B(S)))$. Now 2.2(b) implies that $Y \leq Y_L$ and so by 1.2(ii) $Y = Y_L$. It follows that $N = N_G(Y_L)$. In particular, $N$ is the unique $F$-stable member of $\mathcal{F}(S)$. This finishes the proof of 1.3 and also shows 1.4(a).
Now let \( 1 \neq C \text{char} B(S) \) and put \( M := N_G(C) \). Then \( N_N(B(S)) \leq M \) and thus also \( N = C_G(Y)(M \cap N) \). Suppose that \( Y_N \leq O_P(M) \). Then as above 2.2(b) implies that \( Y_N \leq Y_M \), and by 1.2(ii) \( Y_N = Y_M \) and \( M \leq N \). So 1.4(b) holds.

Suppose next that \( M \) is \( p \)-constrained. From \( N = C_G(Y)(N \cap M) \) and 3.5 we get that \( M = O_{p'}(M)(M \cap N) \). Hence 1.4(c) holds. \( \square \)

4 The L-Lemma and the qrc-Lemma

In this chapter we will work with the following hypothesis.

**Hypothesis 4.1** Let \( P \) be a finite group of characteristic \( p \), \( T \in \text{Syl}_p(P) \), \( Y \trianglelefteq T \), and \( R := C_T(Y) \). Suppose that \( P \) is \( RO_p(P) \)-minimal with \( M \) being the unique maximal subgroup of \( P \) containing \( RO_p(P) \).

**Notation 4.2** Let \( X \) be a finite group and \( V \) a finite dimensional \( GF(p)X \)-module. By \( c(V, X) \) we denote the number of non-central chief factors of \( X \) in \( V \) (in a given chief series). We define \( q(V, X) := 0 \) if every quadratically acting subgroup of \( X \) already centralizes \( V \), and

\[
q(V, X) := \min\{\log_{|A/C_A(V)|}|V/C_A(V)||A \leq X, [V, A] = 1 \neq [V, A]\}
\]

otherwise. Moreover, \( r(V, X) := 0 \) if \( V \) does not possess non-central \( X \)-chief-factors, and

\[
r(V, X) := \min\{q(C, X) \mid C \text{ non-central } X\text{-chief-factor on } V\}
\]

otherwise.

**Lemma 4.3 (L-Lemma)** Assume Hypothesis 4.1. Let \( A \) be a subgroup of \( T \) such that \( A \not\leq O_p(P) \). Then there exists a subgroup \( L \leq P \) with \( AO_p(L) \leq L \) satisfying:

(a) \( AO_p(L) \) is contained in a unique maximal subgroup \( L_0 \) of \( L \), and \( L_0 = L \cap M^g \) for some \( g \in P \).

(b) \( L = \langle A, A^x \rangle O_p(L) \) for every \( x \in L \setminus L_0 \).

(c) \( L \) is not contained in any \( P \)-conjugate of \( M \).

**Proof:** See [PPS]. \( \square \)

The next lemma is very similar to [Ste, 3.3].

**Lemma 4.4** Assume Hypothesis 4.1. Suppose \( V := \langle Y^P \rangle \) is elementary abelian, \( C_{O_p(P)}(Y) \not\leq P \) and \( c(V, P) = 1 \). Then \( [O_p(P), O^p(P)] \) is non-trivial quadratic offender on \( Y \).

**Proof:** Since \( P \) is \( RO_p(P) \)-minimal, we get from 2.3 that \( P = RO^p(P)O_p(P) \). Put

\[
Q := [O_p(P), O^p(P)], \quad W := [V, O^p(P)] \quad \text{and} \quad D := C_V(O^p(P)).
\]

Since \( c(V, P) = 1 \), \( W/W \cap D \) is chief-factor for \( P \) on \( V \). Hence \( [W, O_p(P)] \leq D \). Note that \( P = TO^p(P) \) normalizes \( YW \) and so \( V = YW \). Thus \( [Y, Q]D \leq P \). Observe that \( R \) centralizes
$[Y, Q]/D$ is. Since $O^p(P) \leq \langle R^p \rangle$ we conclude that $[Y, Q, O^p(P)] \leq D$. Hence $O^p(P)$ centralizes $[Y, Q]$. So $P = TO^p(P)$ normalizes $[Y, Q]$ and $[V, Q] = [Y, Q] \leq D$. It follows that

$$[V, O^p(P), O_p(P)] = [W, O_p(P)] \leq D \quad \text{and} \quad [O_p(P), O^p(P), V] = [Q, V] \leq D.$$ 

Hence the Three Subgroup Lemma implies $[V, O_p(P), O^p(P)] \leq D$ and so

(*) $$[V, O_p(P)] \leq D.$$ 

Pick $x \in Y \setminus D$. Since $R$ centralizes $x$ we conclude from (*) that $P = RO^p(P)O_p(P)$ normalizes $\langle x^{O^p(P)} \rangle D$ and so $W \leq \langle x^{O^p(P)} \rangle D$. Put $X := [x, Q]$. Since $X \leq D$ it follows that

$$[W, Q] \leq [\langle x^{O^p(P)} \rangle D, Q] = [x, Q] = X.$$ 

As $[V, Q, O^p(P)] = 1 \leq X$ and $[V, O^p(P), Q] = [W, Q] \leq X$, the Three Subgroup Lemma implies $[Q, O^p(P), V] \leq X$. Since $[Q, O^p(V)] = Q$ we get $[V, Q] = X$. In particular,

$$[y, Q] = X \text{ for every } y \in Y \setminus C_Y(Q).$$

Now 2.6 shows that $Q$ is a quadratic offender on $Y$.

If $Q$ acts trivially on $Y$, then $Q \leq C_{O_p(P)}(Y)$ and so $C_{O_p(P)}(Y) \leq TO^p(P) = P$, a contradiction. \qed

**Lemma 4.5** Let $L$ be a finite group acting on a $p$-group $E$, and let $A$ and $B$ be $p$-subgroups of $L$ and $X$ and $Z$ subgroups of $E$. Suppose that

(i) $B \not\subseteq O_p(L),$

(ii) $[E, A] \leq X \leq C_E(A)$ and $[E, B] \leq Z \leq C_E(B),$

(iii) $L$ is $AO_p(L)$-minimal and $[E, O^p(L)] \neq 1,$

(iv) $X$ is normalized by $E$ and $O_p(L)$, $X$ is abelian, and $E = \langle X^L \rangle.$

Then

(a) $C_B(E) \leq B \cap O_p(L),$

(b) $E = X^gZ = X^gC_E(B)$ for some $g \in L,$

(c) $Z = [E, B]C_Z(E) = [E, b]C_Z(E)$ for all $b \in B \setminus O_p(L),$

(d) $[B, E, E, E] = 1,$

(e) $|Z/C_Z(E)| \leq |ZD/D| \leq |E/C_E(B)|,$

(f) $|BO_p(L)/O_p(L)| \leq |E/C_E(B)|.$

**Proof:** By (iii) there exists a unique maximal subgroup $L_0$ of $L$ containing $AO_p(L)$, and by 2.3 $(\bigcap_{g \in L} L_0^g)/O_p(L)$ is a $p'$-group.

Pick $b \in B \setminus O_p(L)$. Then there exists $g \in L$ with $b \notin L_0^g$. Put $H := \langle A^g, b \rangle$. Then $L = HO_p(L)$ since $H \not\leq L_0^g$. Furthermore, we put $D := \bigcap_{g \in L} X^g$. 

9
Since \( Z \in C(E) \) by (ii), (b) holds.

(c): Since \( X^g \cap Z \subseteq C(E) \subseteq D \), we get \( X^g \cap Z = D \cap Z \) and so by (b) and (*)

\[
|E/X| = |E/X^g| = |Z/X^g|/|X/Z| = |Z/Z \cap X^g| = |Z/Z \cap D|.
\]

Moreover, using (*)

\[
Z = (X^g \cap Z)[E,b] \leq (Z \cap D)[E,b] \leq C(Z)[E,b] \leq C(Z)[E,B] \leq Z,
\]

and so (c) holds.

(d): Since \( L \) is \( A_0(L) \)-minimal, \( A \notin O_p(L) \) and so (b) can be applied with \( A \) and \( A^g \) in place of \( A \) and \( B \). Then \( E = X^g X^t \) for some \( t \in L \); in particular \( [X^t, X] \leq X \cap X^t \leq D \leq Z(E) \). Thus \( E' \leq D \leq Z(E) \) and \( [B, E, E, E] \leq [E', E] = 1 \). So (d) holds.

(e): By (ii) and (b)

\[
C(E) \leq C(E) = C(E) = DZ.
\]

Hence

\[
|E/C(E)| \geq |E/D| = |X^g ZD/ZD| = |X^g/D|.
\]

On the other hand, by (b) \( |E/X| = |Z/Z \cap X^g| = |Z/Z \cap D| \), while the same result applied to \( A \) in place of \( B \) gives \( |E/X| = |X/D| = |X^g/D| \). Since \( D \leq Z(E) \) this gives

\[
|E/C(E)| \geq |Z/Z \cap D| \geq |Z/C(Z(E))|.
\]

(f): Let \( x \in X^g \setminus D \) and suppose that \( [x, b] \in D \). Then \( (x)D \) is normalized by \( \langle X^g, b \rangle = H \) and so \( x \in D \), a contradiction. This shows that \( [x, c] \notin D \) for every \( c \in B \setminus O_p(L) \). Since \( B \) acts quadratically on the abelian group \( E/D \) we conclude

\[
|[x, b]/D/D| = |[x, c]/D \mid c \in B) | \geq |BO_p(L)/O_p(L)|.
\]

Note that by (ii), \( |x, b| \) \( D \leq ZD \) and so (f) now follows from (e).

\( \square \)

**Theorem 4.6** Assume Hypothesis 4.1. Let \( V \) be a finite dimensional \( GF(p) \)P-module such that \( [V, O_p(P)] = 0 \) and \( |V, O_p(P)| \neq 0 \). Then \( q(V, P) = 0 \) or \( q(V, P) \geq 1 \).

**Proof:** Let \( A \leq T \) be a quadratic on \( V \) with \( [V, A] \neq 0 \). We need to show that \( |V/C_V(A)| \geq |A/C_A(V)\). The proof is by induction on \( |A| \).

Let \( Y \) be a non-central \( P \)-chief factor in \( V \). By 2.3 \( C_T(Y) \leq O_p(P) \leq C_T(V) \). It follows that

\[
|Y/C_V(A)| \leq |V/C_V(A)| \quad \text{and} \quad |A/C_A(Y)| = |A/C_A(V)|
\]

for every \( A \leq T \). Hence we may assume that

1° \( V \) is a non-trivial simple \( P \)-module.
We now apply 4.3. Then there exists \( A \leq L \) such that \( L \) has the properties given in 4.3. In particular, there exists \( g \in P \) such that \( A \leq T^g \cap L \in Syl_p(L) \), and \( L \cap M^g \) is the unique maximal subgroup of \( L \) containing \( AO_p(L) \). Put \( U := (C_V(T^g))^L \).

2° \[ [U, O^p(L)] \neq 0 \text{ and } [U, A] \neq 0. \]

By (1°) \( C_V(T^g) \) is not \( P \)-invariant, so \( N_P(C_V(T^g)) \leq M^g \). Since \( L \not\leq M^g \), we get that \([U, O^p(L)] \neq 0 \) and thus also \([U, A] \neq 0. \]

3° Put \( D := C_A(U) \). Then \([A/D] \leq [U/C_U(A)] \).

Observe that by the definition of \( U \), \([U, O^p(L)] \neq 0 \). Thus, for \( E := U, B := A \), and \( X := Z := C_U(A), L \) satisfies the hypothesis of 4.5. By 4.5(f)

\[ |A/D| = |A/C_A(U)| \leq |A/A \cap O_p(L)| \leq |U/C_U(A)|. \]

So (3°) holds.

4° \[ |D/C_D(V)| \leq |V/C_V(D)|. \]

Since \([U, A] \neq 0 \), \( D < A \) and (4°) follows by induction on \(|A| \).

Using (3°) and (4°) we compute

\[ |A/C_A(V)| = |A/D||D/C_D(V)| \leq |U/C_U(A)||V/C_V(D)| \leq |C_V(D)/C_V(A)||V/C_V(D)| = |V/C_V(A)|. \]

□

The next lemma is a variation of [Ste, 3.2].

**Lemma 4.7** (grc-Lemma) Assume Hypothesis 4.1. Let \( V := \langle Y^P \rangle \). Suppose that

(i) \( Y \leq \Omega_1 Z(J(O_p(P))) \),

(ii) \( C_{O_p(P)}(Y) \not\leq P \),

(iii) \( J(R) \not\leq O_p(P) \).

Then \( V \leq \Omega_1 Z(J(O_p(P))) \), \( V \neq Y \), \( N_L(Y) \leq M \), \([V, O^p(P)] \neq 1 \), \( C_T(V) \leq O_p(P) \) and there exists \( A \in \mathcal{A}(R) \) with

\[ [V, A, A] = 1 \neq [V, A] \text{ and } A \not\leq O_p(P). \]

Moreover, one of the following holds, where \( q := q(Y, O_p(P)), r := r(V, P) \) and \( c := c(V, P) \):

(a) \( 0 \neq q \leq 1 \).

(b) \( 2 \leq c, 1 \leq r, \) and \((q - 1)(rc - 1) \leq 1 \). In particular, \( 0 \neq q \leq 2 \).
Proof: By (i) $V \leq ZJ(O_p(P))$. If $V = Y$, then $C_{O_p(P)}(Y) \subseteq P$, a contradiction to (ii). Hence $V \neq Y$ and $RO_p(P) \leq T \leq N_P(Y) < P$. Since $P$ is $RO_p(P)$-minimal we conclude that $N_P(Y) \leq M$. So $[V,O^p(P)] \neq 1$. Hence 2.3 gives $C_T(V) \leq O_p(P)$. In particular, $(P,Y)$ satisfies Hypothesis III of [Ste].

As $C_T(V) \leq O_p(P)$, (iii) shows that there exists $A \in \mathcal{A}(R)$ such that $[V,A] \neq 1$. By the Timmesfeld Replacement Theorem [KS] we may assume that $[V,A,A] = 1$. Moreover, (i) implies that $A \notin O_p(P)$. Assume that (b) holds. Then also $| \langle A \rangle \cdot \mathcal{O} | \leq | A/C_A(U) |$ and thus $A$ is minimal. Hence 4.7(iii) holds for $A$. By 4.3(a) there exists $J$ and (c) holds.

Proof: From (ii) $[V,J(T)] = 1$ and so $J(T) \leq R$ and $J[R] = J(T)$. So the assumptions of 4.7 are fulfilled. In particular, $V$ is elementary abelian and there exists $A \in \mathcal{A}(T)$ with $[V,A,A] = 1 \neq [V,A]$ and $A \notin O_p(P)$.

Hence we are allowed to apply the $L$-Lemma 4.3. This gives a subgroup $L$ having the properties (a) - (c) given in 4.3. By 4.3(c) $L$ is not a $p$-group, and so $L$ is $AO_p(L)$-minimal. This is (a).

According to 4.3(a) there exists $g \in P$ such that $AO_p(L) \leq T^g \cap L \leq Syl_p(L)$, and $L \cap M^g$ is the unique maximal subgroup containing $AO_p(L)$. Hence replacing $A$ by $A^g$ and $L$ by $L^g$ we may assume that (b) holds.

Clearly $Y \notin L$ since $L \not\leq M$ but $N_P(Y) \leq M$. Since $V$ is abelian, $V_0$ is abelian and (c) holds.

From $A \notin O_p(P)$ and (ii) we get that $J(C_T \cap L(Y)) \subseteq O_p(L)$ and $L$ is $C_T \cap L(Y)O_p(L)$-minimal. Hence 4.7(iii) holds for $L$ and $Y$. Assume that $C_{O_p(L)}(Y) \subseteq L$. Then also $C_{O_p(L)}(Y) \subseteq L$ and thus $P = \langle T, L \rangle \leq N_P(C_{O_p(L)}(Y))$. This contradicts (i). Hence also 4.7(ii) holds for $L$ and $Y$, and (d) follows.

Lemma 4.8 Assume hypothesis 4.1. Suppose that

(i) $C_{O_p(P)}(Y) \not\subseteq P$,

(ii) $Y \leq Z(J(O_p(P))) \cap Z(J(T))$,

(iii) $J(T) \not\subseteq O_p(P)$.

Then there exist subgroups $A \in \mathcal{A}(T)$ and $L$ of $P$ such that the following hold:

(a) $L$ is $AO_p(L)$-minimal.

(b) $O_p(P)A \leq T \cap L \subseteq Syl_p(L)$, and $M \cap L$ is the unique maximal subgroup of $L$ containing $AO_p(L)$.

(c) $Y \not\subseteq L$, and $V_0 := \langle Y^L \rangle$ is abelian.

(d) If $Y \leq Z(J(O_p(L)))$, then $L$ and $Y$ satisfies the hypothesis of 4.7 with $L$ in place of $P$.  


5 F-stability

In this section we explore the following hypothesis:

**Hypothesis 5.1** Let $p$ be a prime and $H$ a finite group. Suppose that $Y$ is an elementary abelian $p$-subgroup of $H$ such that for $T \in \text{Syl}_p(N_H(Y))$ and $R := C_T(Y)$ the following hold:

(i) $Y \leq N_H(J(R))$.

(ii) $Y$ is $F$-stable in $H$.

(iii) Either $Y \leq O_p(H)$ or $H$ is of characteristic $p$.

This hypothesis is motivated by the following observation:

**Lemma 5.2** Let $G$ be a finite group, $S \in \text{Syl}_p(G)$ and $J(S) \leq H \leq G$, and let $\mathcal{F}(S)$ be a factorization family for $S(S)$. Suppose that $N \in \mathcal{F}(S)$ is $F$-stable.

(a) If $Y_N \leq O_p(H)$, then $Y := Y_H$ and $H$ satisfy Hypothesis 5.1.

(b) If $H$ is $p$-constrained and $\overline{H} := H/O_p(H)$, then $\overline{Y_N}$ and $\overline{H}$ satisfy Hypothesis 5.1 in place of $Y$ and $H$.

**Proof:** Let $T \in \text{Syl}_p(H)$ with $J(S) \leq T$. Put $Y := Y_N$ and $R := C_T(Y)$. Since $Y \leq S$ and $Y$ is $F$-stable, 2.4(a) implies that $Y \leq \Omega_1 Z(J(S)) \leq H$ and $J(S) = J(T) = J(R)$. Observe that $Y \leq O_p(N_G(J(S)))$ and so by 1.4(b), $N_G(J(S)) \leq N$. In particular, $T \leq N_G(Y)$ and so $T \in \text{Syl}_p(N_H(Y))$. Now (a) follows.

Assume that $H$ is $p$-constrained. Then $\overline{H} = H/O_p(H)$ is of characteristic $p$. By the Frattini-argument, $N_{\overline{H}}(\overline{Y}) = N_H(Y)$ and $N_{\overline{H}}(\overline{J(R)}) = N_H(J(R))$. Moreover since $Y$ is $F$-stable in $G$, $\overline{Y}$ is $F$-stable in $\overline{H}$. Thus Hypothesis 5.1 holds for $\overline{Y}$ and $\overline{H}$. \hfill $\square$

**Lemma 5.3** Assume Hypothesis 5.1. Then $Y \leq T$, $Y \leq Z(J(T))$, $J(R) = J(T)$, $N_H(T) \leq N_H(Y)$ and $T \in \text{Syl}_p(H)$.

**Proof:** Clearly $Y \leq T$. Thus by 2.4(a), $[Y, J(T)] = 1$. So $J(T) \leq R$ and $J(T) = J(R)$. Therefore $N_H(T) \leq N_H(J(R))$ and so by Hypothesis 5.1(i) $N_H(T) \leq N_H(Y)$. Hence $T \in \text{Syl}_p(H)$. \hfill $\square$

**Theorem 5.4** Assume Hypothesis 5.1 and suppose $C_{O_p(H)}(Y) \not\in H$. Then $Y$ is not $2F$-stable in $H$.

**Proof:** If any subgroup of $H$ satisfies the conclusion of 5.4 with respect to $Y$, then also $H$ does. Thus we may assume:

1° No proper subgroup of $H$ satisfies the hypothesis of 5.4 with respect to $Y$.

Put

$$H_0 = N_H(C_{O_p(H)}(Y)).$$

From 5.3 we conclude

2° $N_H(T) \leq N_H(Y) \leq H_0$, $J(R) = J(T)$, $Y \leq Z(J(R))$ and $T \in \text{Syl}_p(H_0)$.\hfill 13
Next we show:

3° Let \( J(R)O_p(H) \leq \tilde{H} < H \). Then \( C_{O_p(\tilde{H})}(Y) \leq \tilde{H} \) and \( \tilde{H} \leq H_0 \).

By \((2')\) there exists \( \tilde{R} \in \text{Syl}_p(C_{\tilde{H}}(Y)) \) with \( J(R) \leq \tilde{R} \) and \( J(R) = J(T) = J(\tilde{R}) \), and so \( Y \leq N_{\tilde{H}}(J(\tilde{R})) \). Since \( O_p(H) \leq O_p(\tilde{H}) \), \( Y \leq O_p(\tilde{H}) \) or \( \tilde{H} \) is of characteristic \( p \). Moreover \( Y \) is \( F \)-stable in \( \tilde{H} \), and so \( Y \) and \( \tilde{H} \) satisfy Hypothesis 5.1. Now \((1')\) shows that \( C_{O_p(\tilde{H})}(Y) \leq \tilde{H} \).

Hence \( C_{O_p(H)}(Y) = O_p(H) \cap C_{O_p(\tilde{H})}(Y) \) is normal in \( \tilde{H} \) and \( \tilde{H} \leq H_0 \).

4° \( H_0 \) is the unique maximal subgroup of \( H \) containing \( J(R)O_p(H) \), and \( H \) is \( J(R)O_p(H) \)-minimal.

The first statement follows from \((3')\). If \( J(R) \leq O_p(H) \), then by 5.3, \( J(R) = J(O_p(H)) \leq H \), and so by Hypothesis 5.1(i), \( H \leq N_{\tilde{H}}(Y) \leq H_0 \), a contradiction. Hence \( J(R)O_p(H) \nleq O_p(H) \), and \( H \) is \( J(R)O_p(H) \)-minimal.

5° Put \( W := (Y^{H_0}) \). Then \( W \) is elementary abelian.

We will first show that \( Y \leq O_p(H_0) \). If \( Y \leq O_p(H) \), this is obvious. Otherwise \( H \) is of characteristic \( p \) and by 5.3, \( O_p(H) \) normalizes \( Y \). So by 2.5 and \((2')\) \( Y \leq O_p(H_0) \leq T \). Now \((5')\) follows from Hypothesis 5.1(ii) and 2.4(b).

Let \( W \) be the set of all \( p \)-subgroups \( D \) of \( H \) satisfying:

(a) \( WO_p(H) \leq N_H(D) \nleq H_0 \).

(b) \( D = J(D) \leq H_0 \).

Clearly \( 1 \in W \) and so \( W \neq \emptyset \). Pick \( D \in W \) such that first \( \lvert A \rvert \) is maximal for \( A \in A(D) \) and then \( \lvert D \rvert \) is maximal. Put \( N := N_H(D) \) and \( T_0 := DO_p(H) \) and let \( T_1 \in \text{Syl}_p(N \cap H_0) \). Since \( T_0 \leq O_p(N \cap H_0) \), \( T_0 \leq T_1 \). As \( W \) is \( H_0 \)-invariant and by \((2')\) \( T \in \text{Syl}_p(H_0) \), there exists \( g \in H_0 \) with \( W^g = W \) and \( T_1^g \leq T \); in particular \( D^g \in W \). Thus, after replacing \( D \) by \( D^g \) we may assume that \( T_1 \leq T \).

6° \( Y \leq Z(J(T_1)) \), and if \( Y \leq T_0 \) then \( Y \leq Z(J(T_0)) \).

Since \( T_1 \leq T \), \( T_1 \) normalizes \( Y \). So \((6')\) follows from 2.4(a).

7° Let \( U \) be a \( p \)-subgroup of \( H_0 \) containing \( D \). Suppose that \( W \leq N_H(U) \) and \( N_H(U) \nleq H_0 \). Then \( J(U) = D \), and if \( Y \leq U \) then \( Y \leq Z(D) \).

Observe that \( W \leq N_H(U) \leq N_H(UO_p(H)) \). Hence \( N_H(UO_p(H)) \nleq H_0 \) and \( J(UO_p(H)) \in W \). Since \( D \leq U \leq UO_p(H) \), the maximal choice of \( D \) gives \( D = J(UO_p(H)) = J(U) \).

Suppose that \( Y \leq U \), then \( J(U) = D \leq T_1 \leq N_H(Y) \) and so by 2.4(a), \( Y \leq Z(J(U)) = Z(D) \).

8° \( J(T) \neq D \) and \( J(T_1) \neq D \).

Suppose \( J(T) = D \). Then by 5.3 and Hypothesis 5.1(i), \( N \leq N_G(J(T)) \leq N_G(Y) \) and so \( N \leq N_{J(T)}(C_{O_p(H)}(Y)) \leq H_0 \), contrary to the choice of \( D \).

Suppose \( J(T_1) = D \). Then \( N_T(T_1) \leq N_H(J(T_1)) = N \). So \( N_T(T_1) \leq T_1 \), \( T = T_1 \) and \( J(T) = J(T_1) = D \), a contradiction.
9° Let $U$ be a $p$-subgroup of $H_0$ containing $WD$. Suppose that $J(U) \neq D$ or $Y \not\subset Z(D)$. Then $U$ is not contained in any $H$-conjugate of $H_0$ other than $H_0$.

Let $g \in H$ with $U \leq H_0^g$ and $U \leq T_2 \in Syl_p(H_0 \cap H_0^g)$. If $J(T_2) = D$, then also $D = J(U)$ and thus $Y \not\subset Z(D)$. Thus either $J(T_2) \neq D$ or $Y \not\subset Z(D)$. So (7°) gives $N_H(T_2) \leq H_0$. This implies $N_{H_0}(T_2) \leq H_0 \cap H_0^g$ and so $T_2 \in Syl_p(H_0^g)$. By (4°), $H_0^g$ is the unique maximal subgroup of $H$ containing $T_2$. Since $T_2 \leq H_0$ we get $H_0 = H_0^g$.

10° $T_1 \in Syl_p(N)$, $J(T_1) \not\subset O_p(N)$, and $WJ(T_1)D$ is not contained in any other $H$-conjugate of $H_0$.

By (8°) and (7°) $N_H(J(T_1)D) \leq H_0$, so $N_Y(T_1) \leq N \cap H_0$ and $T_1 \in Syl_p(N)$. If $J(T_1) \subset O_p(N)$, then $J(T_1) = J(O_p(N))$ and $N \leq N_H(J(T_1)D) \leq H_0$, a contradiction.

Put $U := WJ(T_1)D$. By (8°), $J(U) \neq D$ and so the last statement in (10°) follows from (9°).

11° There exists a $WJ(T_1)T_0$-minimal subgroup $H_1 \leq N$ such that $H_1 \cap H_0$ is a maximal subgroup of $H_1$ and $J(O_p(H_1)) = D$.

By definition of $W$, $N \not\subset H_0$. Choose $WJ(T_1)T_0 \leq H_1 \leq N$ such that $H_1$ is minimal with $H_1 \not\subset H_0$. Since $H_1 \not\subset H_0$, $N_H(O_p(H_1)) \not\subset H_0$. Also $WO_p(H)$ normalizes $O_p(H_1)$ and $D \leq O_p(H_1)$.

So by (7°) $J(O_p(H_1)) = D$. Since $J(T_1) \neq D$ by (8°) we conclude $J(T_1) \not\subset O_p(H_1)$. Hence also $WJ(T_1)T_0 \not\subset O_p(H_1)$, and $H_1$ is $WJ(T_1)T_0$-minimal.

In the following let $H_1$ be as in (11°). Pick $WJ(T_1)T_0 \leq T_3 \in Syl_p(H_1 \cap H_0)$. Then $H_1$ is $T_3$-minimal and so $T_3 \in Syl_p(H_1)$. Since $T_3 \leq N \cap H_0$ and $T_1 \in Syl_p(H_0 \cap N)$, there exists $g \in N \cap H_0$ with $J(T_1) \leq T_3 \leq T_3^g$. Hence $g$ normalizes $J(T_1)$, $D$ and $W$, and thus also $WJ(T_1)T_0$. So replacing $H_1$ by $H_1^g$ and $T_3$ by $T_3^g$ we may assume that $T_3 \leq T_1 \leq T$.

Case 1 The case $Y \not\subset O_p(H_1)$.

12° $Y$ and $H_1$ satisfy the hypotheses of 4.8.

Since $T_3 \leq T_1$, $Y \leq T_3$. By (5°) and (6°) $Y \leq Z(WJ(T_1))$, so $H_1$ satisfies Hypothesis 4.1. Hence (8°) and (11°) give Hypothesis 4.8(iii), while 2.4(a) gives Hypothesis 4.8(ii).

Assume that $CO_p(H_1)(Y) \leq H_1$. As $O_p(H) \leq O_p(H_1)$, also $CO_p(H)(Y) \leq H_1$, which contradicts $H_1 \not\subset H_0$. Hence also Hypothesis 4.8(i) holds.

According to (12°) we are allowed to apply 4.8 to $Y$ and $H_1$. Let $L$ and $V$ be with the properties given there. Since $Y \leq O_p(L) \leq T_1$, we get from 2.4(a) that $Y \leq Z(J(O_p(L)))$. Thus, by 4.8(d) $L$ and $Y$ satisfy the hypothesis of 4.7.

Since $Y$ is $2F$-stable we are in case 4.7(b), so $0 \neq q(Y, O_p(H_1)) \leq 2$. Thus there exists non-trivial quadratic $2F$-offender on $Y$ and the lemma is proved in (Case 1).

Case 2 The case $Y \not\subset O_p(H_1)$.

By our assumption on $H$, in this case $H$ has characteristic $p$. Hence also $H_1$ has characteristic $p$ since $O_p(H) \leq H_1$. We now apply the $L$-Lemma 4.3 with $W$ and $H_1$ in place of $A$ and $P$. Then there exists $WO_p(H_1) \leq L$ such that

(i) $L$ is $WO_p(L)$-minimal and
(ii) there exists $g \in H_1$ such that $L_0 := H_0^g \cap L$ is the unique maximal subgroup of $L$ containing $WO_p(L)$.

$13^\circ$ \quad $L_0 = L \cap H_0, \; Y \not\subseteq Z(D)$ and $J(O_p(L)) = D$.

Let $g$ as in (ii). Then $WD \leq H_0 \cap H_0^g$ and $Y \not\subseteq Z(D)$ since $D \leq O_p(H_1)$. Hence (9$^\circ$) implies $H_0 = H_0^g$; in particular $L_0 = L \cap H_0$ and $L \not\subseteq H_0$. Now (7$^\circ$) also gives $J(O_p(L)) = D$.

According to (13$^\circ$) we may assume, after conjugation by a suitable element of $H_0 \cap H_1$, that

$14^\circ$ \quad $WO_p(L) \leq T \cap L \in Syl_p(L_0)$. In particular $O_p(L) \leq T$ and $O_p(L)$ normalizes $Y$.

By (13$^\circ$), $Y \not\subseteq ZJ(O_p(L))$. Since $O_p(L)$ normalizes $Y$, we get from 2.4 that

$15^\circ$ \quad $Y \not\subseteq O_p(L)$.

Put

$$A := W, \quad B := Y, \quad X := O_p(L) \cap A, \quad E := (X^L), \quad Z := O_p(L) \cap B.$$

By (5$^\circ$) $A$ is abelian, and by (14$^\circ$) $O_p(L)$ normalizes $A$ and $B$. Moreover, since $O_p(H_1) \leq O_p(L)$ and $H_1$ has characteristic $p$, $[E, O^p(L)] \neq 1$. It follows that the hypotheses of 4.5 are satisfied.

By 4.5(a), $C_Y(E) \leq Y \cap O_p(L)$ and so by 4.5(c), $[b, Y]C_Y(E) = [E, Y]C_Y(E) = Y \cap O_p(L)$ for all $b \in Y \setminus O_p(L)$. Moreover, by 4.5(d) $[Y, E, E, E] = 1$ and by 4.5(e),(f) we have $|Y/C_Y(E)| \leq |E/C_E(Y)|^2$. So $E$ is a nearly quadratic 2F-offender on $Y$. Hence the lemma also holds in (Case 2).

\[\square\]

**Lemma 5.5** Assume Hypothesis 5.1. Suppose that $Y \not\subseteq H$ and $Y$ is 2F-stable. Then $\Omega_1 Z(T) \not\subseteq H$.

**Proof:** Let $T \leq P \leq H$ and $P$ be minimal with $Y \not\subseteq P$. By 5.3 $N_H(T) \leq N_H(Y)$, so $T \not\subseteq P$ and $P$ is $T$-minimal. Put

$$Q := C_{O_p(P)}(Y), \quad V_0 := \Omega_1(Z(Q)), \quad V := C_{V_0}(O_p(P)), \quad \mathcal{P} := P/C_P(V).$$

If $Z(T) \not\subseteq O_p(P)$, $Z(T) \not\subseteq P$. So we may assume $\Omega_1 Z(T) \leq O_p(P)$ and thus $\Omega_1 Z(T) = C_V(T)$. By 5.4 $Q \leq P$. Since either $Y \leq O_p(H) \leq O_p(P)$ or $P$ is of characteristic $p$, 2.5 implies $Y \leq O_p(P)$.

Thus $Y \leq V_0$. Since $Y \not\subseteq P$, we get that $[V_0, O^P(P)] \neq 1$. By 5.3, $J(R) = J(T)$ and so by Hypothesis 5.1(i), $J(R) \not\subseteq O_p(P)$. Hence 2.3 shows that $[O^p(P), J(R)] = O^p(P)$. Since $J(R)$ centralizes $Y$, $[O_p(P), J(R)] \leq O_p(P) \cap J(T) \leq Q$ and so $[O_p(P), O^p(P)] \leq Q$. The $P \times Q$-Lemma yields $[V, O^p(P)] \neq 1$.

Again 2.3 gives $C_T(V) = O_p(P)$ and $O_p(\mathcal{P}) = 1$. Moreover $J(T) \neq 1$ since $J(R) \not\subseteq O_p(P)$. Hence $\mathcal{P}$ and $V$ satisfy the hypothesis of [BHS, 5.6]. It follows that $[C_V(T), P] \neq 1$. Since $C_V(T) = \Omega_1 Z(T)$ and $P = \langle T^P \rangle$ we conclude that $\Omega_1 Z(T) \not\subseteq P$ and so also $Z(T) \not\subseteq H$.

\[\square\]
6 The Proof of Theorems 1.5 – 1.8

Recall that Theorems 1.3 and 1.4 have been proved in Section 3.

Proof of Theorem 1.5:

(a): Observe that $N = N_G(Y_N)$ by 1.3. Suppose $Y_N \leq O_p(M)$. If $M = N_G(C)$ for $1 \neq C \text{ char } B(S)$, then 1.5(a) follows from 1.4(b). If $\Omega_1 Z(S) \leq M$, then 5.2(a) shows that $Y_N$ and $M$ satisfy Hypothesis 5.1. Hence 5.5 gives $Y_N \leq M$ and so $M \leq N$.

(b): Put $\mathfrak{M} := M/O_p(M)$. Then 5.2(b) shows that $\overline{Y_N}$ and $\mathfrak{M}$ satisfy Hypothesis 5.1. Thus 5.5 gives $\overline{Y_N} \leq \mathfrak{M}$. By the Frattini-argument $M = O_p(M)N_M(Y_N) = O_p(M)(M \cap N)$.

(c): Let $B(S) \leq H \leq G$ and $H$ be $p$-constrained with $H \neq O_p(H)(H \cap N)$, and let $B(S) \leq T \in Syl_p(N_H(Y_N))$. Put $\overline{H} := H/O_p(H)$. Then again 5.2(b) shows that $\overline{Y_N}$ and $\overline{H}$ satisfy Hypothesis 5.1. Hence by 5.4, $C_{\overline{H}}(\overline{Y_N}) \leq H$ and by 2.5, $\overline{Y_N} \leq O_p(\overline{H})$. From 5.5 applied to $N_{\overline{H}}(\Omega_1 Z(T))$ we get $\overline{Y_N} \leq N_{\overline{H}}(\Omega_1 Z(T))$. Recall that $Y_N \leq \Omega_1 Z(J(S)) \leq O_p(N_G(J(S)))$. Thus by 1.4(b), $N_G(J(S)) \leq N$. Since $B(T) = B(S)$ we have $J(T) = J(S)$ and so $\overline{Y_N} \leq C^{**}(\overline{H}, T)$. By the Frattini Argument $N_{\overline{H}}(\overline{Y_N}) = N_H(Y_N) = \overline{H} \cap \overline{N}$. Hence also $C^{**}(\overline{H}, T) \leq \overline{H} \cap \overline{N}$.

Proof of Theorem 1.6:

Let $P \in \mathcal{F}(S)$. By 1.3 $P$ is a $p$-local subgroup of $G$. Let $L$ be a maximal $p$-local subgroup containing $P$. By 2.2(c) $Y_P \leq Y_L$ and so $P \triangleleft L$. Hence by 3.4 $P = L$.

Suppose that $N \in \mathcal{F}(S)$ is 2F-stable. Let $M = N_G(C)$ for $1 \neq C \text{ char } B(S)$ or $M = N_G(\Omega_1 Z(S))$. Then $S \leq M$, so $M$ has characteristic $p$ since $G$ is of parabolic characteristic $p$. Hence 1.5(a) implies $M \leq N$.

Let $H \in S(B(S))$ and $B(S) \leq T \in Syl_p(H)$. Then $B(S) = B(T)$ and so $N_H(C) \leq N$ for $1 \neq C \text{ char } B(T)$. Also $T \leq S^g$ for some $g \in N_G(B(S)) \leq N$ and $\Omega_1 Z(S^g) \leq J(S) \leq T$, so $\Omega_1 Z(S^g) \leq Z(T)$ and $C_H(\Omega_1 Z(T)) \leq C_G(\Omega_1 Z(S^g)) \leq N^g = N$. Thus $C^{*}(H, T) \leq H \cap N$.

Proof of Corollary 1.7:

By 1.6 the members of $\mathcal{F}(S)$ are maximal $p$-local subgroups. We may assume that there exists a 2F-stable $N \in \mathcal{F}(S)$.

Let $L$ be a maximal $p$-local subgroup containing $S$ with $L \not\leq N$ and choose $M \in \mathcal{F}(S)$ with $L \triangleleft M$. Then $L \leq C_G(Y_L)M$. On the other hand, by 2.2(c) $\Omega_1 Z(S) \leq Y_L$ and so by 1.6 $C_G(Y_M) \leq C_G(\Omega_1 Z(S)) \leq N$. Since $L \not\leq N$ we conclude that $M \not\leq N$ and $M \neq N$. By 1.4 $N$ is the only member of $\mathcal{F}(S)$ which is $F$-stable. Hence $M$ is not $F$-stable.

Proof of Theorem 1.8:

Let $P$ be the semi-direct product of $G$ and $V$. Then $O_p(P) = V$ and $[V, O_p(P)] \neq 1$. Let $A$ be an offender on $V$ such that $|A| |C_V(A)|$ is maximal. Because of [KS, 9.2.3] we may assume that $A$ is quadratic on $V$. Hence 4.6 implies $|A/C_A(V)| = |V|$, and 1.8 follows.
References


