Ascending subgroups of irreducible finitary linear group

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1 Introduction

Let K a field and V a vector space over K. Let $FGL_K(V)$ be the finitary linear group of V over K, namely $FGL_K(V) = \{g \in GL_K(V) |$ [V,g] has finite K-dimension $\}$. Subgroups of $FGL_K(V)$ are called finitary groups. Recently a good amount of work has been done towards a classification of the locally finite, finitary groups (see [1, 2, 7]). On the otherhand very little is known without the assumption of locally finiteness. This paper is meant as a contribution to the general theory of finitary groups.

Throughout this paper G is a subgroup of $FGL_K(V)$. Suppose that G is irreducible and H an ascending subgroup of G, that is there exist a well ordered set I and subgroups $H_i, i \in I$, of G including H and G such that $H_i \lhd H_{i+1}$ and if i is a limit ordinal in I, then $H_i = \cup_{j<i} H_j$. Then our main theorem (7.6) asserts that H acts completely reducibly on V. The most important step in the proof of this is (6.2), which provides a component type subgroup in G in the case where G acts irreducibly on V and has a reducible ascending subgroup. As a consequence (7.4) we can prove that ascending subgroups of primitive, infinite dimensional, finitary groups are primitive. In (4.5) we prove a Jordan-Hölder Theorem for finitary modules of finitary groups.

2 Preliminaries

Definition 2.1 *Let H be a group and V a KH-module.*

1. *A series for H on V is a set Γ of submodules of H in V such that*
   (a) *Γ is totally ordered by inclusion,*
   (b) *Γ is complete, i.e. if $\Lambda \subset \Gamma$, then $\cup\Lambda \in \Gamma$ and $\cap\Lambda \in \Gamma$*
   (c) *$0 \in \Gamma$ and $V \in \Gamma$.*
2. *Let Γ be a series for H on V. For S in Γ let $S_- = \cup\{T \in \Gamma | T < S\}$. If $S \neq S_-$ then (S_-, S) is called a jump of Γ and $\overline{S} = S/S_-$ is called a factor of $\mathcal{S}$*
3. *A series for H on V is called a composition series if all its factors are irreducible KH-modules.*
4. *A class $\mathcal{F}$ of KH-modules is called closed if every non-zero section of H of an element of $\mathcal{F}$ is in $\mathcal{F}$.*

5. *Let $\mathcal{F}$ be a closed class of KH-modules. Then series for H on a KH-module is called a $\mathcal{F}$-series provided that all its factors are in $\mathcal{F}$.*

6. *A series for H on V is called finite dimensional if all its factors are finite dimensional: i.e if it is a $\mathcal{F}$-series, where $\mathcal{F}$ is the class of finite dimensional KH-modules.*

Lemma 2.2 *Let $H \leq GL_K(V)$ and $\mathcal{F}$ a closed class of KH-modules. Then H has a unique minimal submodule $m_{V,\mathcal{F}}(H)$ in V such that H has a $\mathcal{F}$-series on $V/m_V(H)$.*

Proof: Let$\mathcal{W}$ be the set of a H-submodules F in V such that H has a $\mathcal{F}$-series on V/F. We need to show that $\cap\mathcal{W}$ is in $\mathcal{W}$ and for this we may assume that $\cap\mathcal{W} = \mathrm{O}$.Thus the goal is to show that V has an $\mathcal{F}$-series for H.

For any series Γ of H on V,define

$$m(\Gamma) = \{S \in \Gamma | 0 \neq \overline{S} \notin \Gamma\}$$

and

$$M(\Gamma) = \{S \in \Gamma | m(\gamma) \leq S\}$$

On the set $\mathcal{C}$ of H-composition series on V define a partial order $\leq$ by

$$\Gamma \leq \Lambda \text{ if } M(\Gamma) \subseteq M(\Gamma)$$

We will show that

(*) $\mathcal{C}$ is linearly ordered.

Suppose for the moment that (*) is true. Then by Zorn's lemma $\mathcal{C}$ has a maximal element Γ. We claim that Γ is a $\mathcal{F}$ series. Otherwise $m(\|Gamma) \neq 0$. Since by assumption $\cap\mathcal{W} = 0$, there exists F in $\mathcal{W}$ with $m(\Gamma) \nleq F$. But then $m(\Gamma)/(m(\Gamma \cap F \cong m(\Gamma) + F)/F$ has a $\mathcal{F}$-composition series. It is now easy to see that there exists a compostion series Γ^* of H on V with $m(\Gamma^*) \leq m(\Gamma) \cap F$ and $M(\Gamma) \subset M(\Gamma^*)$, a contradiction to the maximality of Γ.

It remains to prove (*). We remark that $\Gamma \leq \Lambda$ implies that $M(\Gamma) = \{T \in \Lambda | m(\Gamma) \leq T\}$. Indeed, let $T \in \Lambda$ with $m(\Gamma) \leq T$. Since $M(\Gamma) \subseteq M(\Lambda) \subseteq T, X \leq T$ or $T \leq X$ for any $X \in M(\Gamma)$. Put $T_- = \cup\{X \in M(\Gamma) | X \leq T\}$ and $T_+ = \cap\{X \in M(\Gamma) | T \leq X\}$. Then $T_- \leq T \leq T_+$. Since $\Gamma$is a composition series, T_+/T_- is irreducible as H-module. Hence T is equal to T_+ or T_- and so $T \in M(\Gamma)$.

Let D be any chain in $\mathcal{C}$ and put

$$m = \cap\{m(d) | d \in D\}$$

and

$$M = \cup\{M(d) | d \in D\}.$$

Let N be any composition series for H on m and $\Gamma = M \cup N$. We wish to show that Γ is a composition series for H on V. Clearly Γ is totally ordered. To show that *Gamma* is complete, let Λ be any subset of Γ.

If $N \cap \Lambda \neq \emptyset$, then$\cap\Lambda \in N$. If $\Lambda \subseteq M(d)$ for some d in D, then $\Lambda \in M(d)$. If $\Lambda \subseteq M$, but $\Lambda \not\leq M(d)$ for every d in D, then for every d in D there exists T in Λ with $T \not\leq M(d)$. Pick e in D with $T \in M(e)$. Then $d \leq e$. By the above remark, $m(d) \not\leq T$ and so $T \leq m(d)$. Thus $m \leq \cap\Lambda \leq \cap\{m(d)|d \in D\} = m$. Hence in all cases $\cap\Lambda \in \Gamma$.

If $\Lambda \subseteq N$, then $\cup\Lambda \in N$. If $\Lambda \cap M(d) \neq \emptyset$for some d in D, then $\cup\Lambda \in M(d)$. Hence also $\cup\Lambda \in \Gamma$ and Γ is complete. To finish the proof that S is a composition series we have to show that $\overline{S}$ is irreducible for all S in Γ. If $S \in N$ this is obvious. If $S \notin N$ there exists d in D with $m(d) < S$, since otherwise $S \leq m(d)$ for all d in D and so $S \leq m$ and $S \in N$,a contradiction. Thus S and S_{-} are in M(d) and so $\overline{S}$ is a factor of d. In particular, $\overline{S}$ is irreducible and $S \in \mathcal{F} \cup \{0\}$

We have proven that Γ is a composition series and that $S \in \mathcal{F} \cup \{0\}$ for all $S \in \Gamma$ with $S \not\leq m$. It follows that $m(\Gamma) \leq m$ and $M \subseteq M(\Gamma)$. Thus
*Gamma*is an upper bound for D and (*) is proved.

Lemma 2.3 *Let Y be a group, $R \leq Y$ and s in R with $[R, R^s] = 1$.Then $R' \leq [R, s]$.*

Proof: Let r,t be in R. Then

$$[r,t] = r^{-1}s^{-1}rt = (r^{-1}t^{-1}(rt)^s)(rt)^{-1}rt = r^{-1}t^{-1}r^st^s[rt,s]^{-1}$$

=

$$= r^{-1}r^st^{-1}t^s[rt,s]^{-1} = [r,s][t,s][tr,s]^{-1} \in [R,s]$$

Lemma 2.4 *Let Y be a group and R and S subgroups of Y. Suppose that the following three conditions hold:*

(a) $R \cap R^s = 1$ and $[R, R^s] = 1$ for all $s \in S \setminus N_S(R)$.

(b) $S \cap S^r = 1$ and $[S, S^r] = 1$ for all $r \in R \setminus N_R(S)$.

(c) R does not normalize S and S does not normalize R.

Then one of the following holds:

(i) $R' = S' = 1, N_R(S) = N_S(R) = 1$ and any two non-trivial elements in $R \cup S$ have the same order.

(ii) If $\{T,U\} = \{R,S\}$ and $X = N_T(U)$, then $|T/X| = 2$, X is abelian, $C_X(T) = 1$ and T inverts X.

Proof: Put $A = \cap\{N_R(S^r)|r \in R\}$ and $B = \cap\{N_S(R^s)|s \in S\}$. Pick $r \in R \setminus N_R(S)$ and $s \in S \setminus N_S(R)$. Note that $[(R \cap S)^s, R] \leq [R^s, R] = 1$ and so $(R \cap S)^s = (R \cap S)^{sr} \leq S \cap S^r = 1$. Hence

(1) $R \cap S = 1$.

(2) [R,S] normalizes R and S, $R\prime S\prime \leq [R,S], R' \leq A$ and $S' \leq B$.

For the proof of (2), note that since S is normal in $< S^R >$, [R,S] normalizes S. Furthermore, by (2.3), $R' \leq [R,S]$ and so R' normalizes S and $R' \leq A$.

(3) $A' = B' = [A,B] = 1$.

By (2.3), $A' \leq [A,S]$. But A normalizes S and so $A' \leq R \cap S = 1$. Now also $[A,B] \leq R \cap S = 1$, and (3) is established.

(4) $C_A(R) = 1$ and $C_B(S) = 1$.

Let $t \in S$. If $R = R^t$ then $C_A(R)^t \leq Z(R)^t = Z(R)$. If $R \neq R^t$, then $[R, R^t] = 1$. Thus in any case, $[R, C_A(R)^t] = 1$. Hence $[C_A(R), S] \leq C_S(R) \leq S \cap S^r = 1$ and $C_A(R) \leq C_R(S) \leq R \cap R^s = 1$.

(5) If $R' \neq 1$, then (ii) holds.

Suppose that $R' \neq 1$. Assume also that $R^S \neq \{R, R^s\}$ and pick t in S with $R^t \not\leq \{R, R^s\}$. Let $u, v \in R$. Then $[[t,u],[s,v]] = [u^{-t}u, v^{-s}v] = [u,v]$. If $u \in R'$, then by (2) u normalizes S and so $[t,u] \in S$. Now [s,v] normalizes S and so $[u,v] \in R \cap S = 1$. Thus $[R', R] = 1$ and (2) and (4) imply that $R' = 1$. Hence $R^S = \{R, R^s\}$ and so $|S/B| = 2$. Note that $R \cong (R \times R^s)/R^s = [R,s]R^s/R^s$. It follows that [R,S] is not abelian and so S is not abelian. Therefore $|R/A| = 2$. By (4) R inverts A, and S inverts B. Thus (ii) holds.

We may assume from now on that R and S are abelian. By (4), A = B =1, and so $N_R(S) = N_S(R) = 1$. Moreover, $< R^S >$ is abelian and so [S,R,R] = 1 and [R,S,S] =1. Let k be a positive integer. Then $[r^k, s] = [r,s]^k = [r, s^k]$. So $r^k \neq 1$ implies $[r^k, s] \neq 1$ and $s^k \neq 1$. Hence $|r| = |[r,s]| = |s|$ and (i) holds.

Lemma 2.5 *(a) Let Ω be a set with at least 5 elements. Then Alt(Ω) has no proper ascending subgroups.*

(b) Let Ω be a finite set, H a primitive subgroup of Sym(Ω) and N a non-trivial subnormal subgroup of H. Then N acts fixed-point freely on Ω.

Proof:(a) This is well known and easy to proof, see for example [8].

(b) We prove (b) by induction on $|H|$. Suppose $1 \neq$ N is subnormal in H and has a fixed-point ω in Ω.

Let $N \lhd \lhd L \lhd H$ with L $\neq$ H. Then L acts transitively and, by induction, imprimitively on Ω. Let R be the largest subgroup of L which acts trivally on all maximal systems of imprimitivity for L on Ω. Then H normalizes R and R acts intransitively on Ω. So R = 1. On the other hand let Π be any maximal system of imprimitivity for L on Ω. Pick P in Π with $\omega \in P$. Then N normalizes P. Since L acts primitively on Π we conclude by induction that N acts trivially on Π. So $N \leq R$ and N = 1; this contradiction completes the proof.

Lemma 2.6 *Let Ω be an infinite set, L a transitive subgroup of $FSym(\Omega)(= \{g \in Sym(\Omega)|$ supp(g) is finite$\})$ and H an ascending subgroup of L. Then either H is transitive on Ω or Ω is the disjoint union of H invariant sets of imprimitivity for L on Ω.*

Proof: Let $L_i, i \in I$, be an ascending normal series from H to L with $L_0 = H$. Assume first that H as an orbit of infinite length on Ω, then clearly L_1 normalizes that orbit and in easy induction proof shows that L normalizes that orbit. Hence H is transitive, a and the Lemma follows in this case.

Assume next that every orbit of H on Ω is finite and that L has a maximal system Π of imprimitivity on Ω. Put $D = \{d \in L | P^d = P forall P \in \Pi\}$. Then by [9], Satz 9.4, L/D = FSym(P) or L/D = Alt(P). Since clearly every orbit for H on P is finite, $Alt(P) \leq HD/D$. Thus by 2.5(a), $H \leq D$ and the lemma holds also in this case.

Assume finally that every orbit of H on Ω is finite and L has no maximal system of imprimitivity. By [5],(2.2), there exists a chain $\Omega_1, \Omega_2, \ldots$ of sets of imprimitivity for L on Ω such that $\cup_i \Omega_i = \Omega$ and such that no set of imprimitivity of L on Ω lies between Ω_i and Ω_{i+1}. Put

$$M_0 = \{\Omega_i^g | i \leq 1, g \in L, \Omega_i^g \text{ is H invariant}\}$$

and let M be the set of mimimal elements in M_0. We claim that Ω is the disjoint union of the elements in M. Let $X, Y \in M$ with $X \cap Y \neq \emptyset$. Without loss $|X| \leq |Y|$. Hence there exists $g \in L$ with $X \subseteq Y^g$. Thus $Y \cap Y^g \neq \emptyset$and so $Y = Y^g$. By minimality of Y, X = Y.

Suppose $\cup M {\neq} \Omega$, Then there exists an orbit O for H on Ω with $O \not\subseteq \cap M$. Since O is finite, there exists i with $O \subseteq \Omega_i$. Then H normalizes Ω_i and $\Omega_i \in M_0$. Let $X \in M_0$ be of mimimal order with respect to $X \not\subseteq \cup M$. Pick $Y \in M$ with $Y \subseteq X$ and let $i \leq 1$ maximal with respect to $Y \subseteq \Omega_i^g \subset X$ for some $g \in L$. Then H normalizes Ω_i^g. By choice of g, $N_L(X)$ acts primitively on $\{\Omega_i^{gk} | k \in N_L(X)\}$ and so by 2.5(b) H acts trivially on $\{\Omega_i^{gk} | k \in N_L(X)\} \subseteq M_0$. The minimal choice of $|X|$ implies $X = \cup \{\Omega_i^{gk} | k \in N_L(X)\} \leq \cup M$, a contradiction.

Hence $\Omega = \cup M$ and the Lemma holds also in this last case.

3 The submodules $\overline{U}_V(G)$ and $\underline{U}_V(G)$

Definition 3.1 *Let H be a group, $h \in H$ and W a KH-module.*

1. *h acts unipotently on W if $W(h-1)^n$ =0 for some non-negative integer n.*
2. *H acts unipotently on W if each of its elements act unipotently.*
3. $\overline{U}_W(H) =< \{X|$ *X a KH-submodule in W such that H acts unipotently on W/X*$\}$¿.
4. $\underline{U}_W(H) = \sum\{X|$ *X a KH-submodule in W such that H acts unipotently on X*$\}$.
5. *W is a perfect KH-module if $W = \overline{U}_W(H)$: i.e if no non-zero factor module of W is a unipotent KH-module.*

Lemma 3.2 *(a) G acts unipotently on V if and only if V has a series with respect to G all whose factors are central.*

(b) The subgroup of G generated by all the ascending unipotent subgroups of G acts unipotently on V.

(c) If G acts irreducibly on V, no non-trivial normal subgroup of G acts unipotently on V.

Proof: see Theorem B in [4].

Lemma 3.3 *(a) G acts unipotently on $V/\overline{U}_V(G)$.*

(b) G acts unipotently on $\underline{U}_V(G)$.

(c) Let Y be a KG-submodule in V. Then $\overline{U}_Y(G) \leq \overline{U}_V(G)$. Moreover, if $\overline{U}_V(G) \leq Y$, then $\overline{U}_Y(G) = \overline{U}_V(G)$.

(d) Suppose that $G =< L_i | i \in I >$ with L_i asc G and $L_i^G \subseteq \{L_j | j \in I\}$, for all i in I. Then

$$\overline{U}_V(G) = \sum_{i \in I} \overline{U}_V(L_i) \text{ and } \underline{U}_V(G) = \cap_{i \in I} \underline{U}_V(L_i)$$

(e) Let S be a set of KG-submodules in V. Then

$$\overline{U}_{\sum_S}(G) = \sum \{\overline{U}_s(G) | s \in S\} \text{ and } \underline{U}_{\cap S}(G) = \underline{U}_{\cap \{U_s(G) | s \in S\}}(G)$$

.

Proof: We first prove that

(*) Let x be a unipotent element in $FGL_K(W)$. Then $W(x-1)^{deg(x)+1} = 0$.

Indeed let d be minimal with $W(x-1)^d = 0$. Then

$$W(x-1) > W(x-1)^2 > ... > W(x-1)^{d-1} > 0.$$

So $deg(x) = dim W(x-1) \leq d-1$, and (*) is proved.

Put $\mathcal{Z} = \{X |$ X a G-submodule such that G acts unipotently on V/X$\}$and let $g \in G$. By (*), $V(g-1)^{deg(g)+1} \leq X$ for all X in $\mathcal{Z}$. Thus $V(g-1)^{deg(g)+1} \leq \overline{U}_V(G)$ and so $\overline{U}_V(G) \in \mathcal{Z}$. This proves (a)

By (*) $\underline{U}_V(G)(g-1)^{deg(g)+1} = 0$ for all g in G. So (b) holds.

Let Y be a KG-submodule in V. Then G acts unipotently on $Y + \overline{U}_V(G)/\overline{U}_V(G)$ and so also on $Y/Y \cap \overline{U}_V(G)$. Thus $\overline{U}_Y(G) \leq Y \cap \overline{U}_V(G) \leq \overline{U}_V(G)$.

If $\overline{U}_V(G) \leq Y$, then G acts unipotently on V/Y and $Y/\overline{U}_Y(G)$. Hence G acts unipotently on $V/\overline{U}_Y(G)$ and therefore $\overline{U}_V(G) \leq \overline{U}_Y(G)$. Thus (c) holds.

To prove (d), note that G normalizes $\sum\{\overline{U}_V(L_i) | i[I\}$ and that L_i acts unipotently on $V \sum\{\overline{U}_V(L_i) | i \in I\}$. By part (b) of (3.2), G acts unipotently on $V \sum\{\overline{U}_V(L_i) | i \in I\}$. Thus $\overline{U}_V(G) = \sum\{\overline{U}_V(L_i) | i \in I\}$. A similar argument shows that $\underline{U}_V(G) = \cap\{\underline{U}_V(L_i) | i \in I\}$.

(e) This is readily verified and we omit the proof.

We remark that (2.2) provides an alternative proof for part (a) of 3.3. Indeed, let $\mathcal{F}$ be the class of one dimensional central KG-modules and X a KG-submodule in V. Then by part (a) of (3.2), G acts unipotently on V/X if and only if G has an $\mathcal{F}$ series on V/X.Thus $m_{V,\mathcal{F}}(G) = \overline{U}_V(G)$.

Proposition 3.4 *If G is locally nilpotent and V is a perfect KG-module, then every KG-submodule of V is perfect.*

Proof: Assume first that G is nilpotent and V is finite dimensional. If $\overline{U}_X(G) \neq 0$, then by induction on the dimension of V, $X/\overline{U}_X(G)$ is a perfect submodule of $V/\overline{U}_X(G)$. So X is perfect. If $\overline{U}_X(G) = 0$, G acts unipotently on X. Since X is finite dimensional, part (a) of (3.2) implies that $C_X(G) \neq 0$. Let $1 \neq z \in Z(G))$. Since $C_V(z) \neq 0$,V $\neq$ [V,z]. Moreover, $[V, z] \cong V/C_V(z)$ and so [V,z] is perfect. It follows by induction that $X \cap [V, z]$ and X+[V,z]/[V,z] are perfect. Hence X is perfect.

In the general case let $\mathcal{F}$ be the set of finitely generated subgroups of G. Note that G acts unipotently on $V/\cup\{\overline{U}_V(F)|F \in \mathcal{F}\}$ and so $V = \cup\{\overline{U}_V(F)|F \in \mathcal{F}\}$. This implies $U = \cup\{X \cap \bar{U}_V(F)|F \in \mathcal{F}\}$. Moreover, $\overline{U}_V(F) \leq [V, F]$ and so $\overline{U}_V(F)$ is finite dimensional. Since F is nilpotent, the preceeding paragraph implies that $X \cap \overline{U}_V(F)$ is a perfect module for every $F \in \mathcal{F}$. Thus X is a perfect KG-module and the lemma is proved.

4 A Jordan-Hölder Theorem for Finitary Modules

In this section we will prove that the non-central factors in a composition-series for G on V are independent from the choice of the composition series. This statement is not true for the central factors. The concept introduced in the following definition is designed to isolate the non-central factors of a series.

Definition 4.1

1. *A u-series for G on V with respect to a normal subgroup H of G is a set S of KG-submodules in V such that*
   (a) *S is totally ordered by inclusion,*
   (b) *if $T \subseteq S$, then $\cup T \in S$ and $\overline{U}_{\cap T}(H) \in S$,*
   (c) *$0 \in S$ and $\overline{U}_V(H) \in S$,*
   (d) *all elements in S are perfect KH-modules.*
2. *Let S be a u-series for G on V with respect to H. For X in S let*
   $$X_- = \cup\{T \in S|T < S\}, X_+/X_- = \underline{U}_{X/X_-}(H) \text{ and } \overline{X} = X/X_+$$
   If $X \neq X_-$, then (X_-, X) is called a jump of S and $\overline{X}$ is called a factor of X.
3. *A u-composition series for G on V with respect to H is a u-series for G on V with respect to H all of whose factors are irreducible as KG-modules.*
4. *A u-series (u-composition series) for G on V is a u-series (u-composition series)for G on V with respect to G.*

Lemma 4.2 *Let H be a normal subgroup of G and for any KH-module X put $U(X) = \overline{U}_X(H)$*

(a) Let S be a series for G.

(a1) Put

$$T = \{U(X)|X \in S\}$$

Then T is a u-series for G on V with respect to H.

(a2) Let $Y \in T$ and put

$$X = \cap\{Z \in S|U(Z) = X\}.$$

Then $Y = U(X)$, $Y_- = U(X_-)$ and

$$\overline{Y} \cong U(\overline{X})/(\underline{U}_{U(\overline{X})}(H)$$

as a KG-module. In particular, if S is a composition series for G on V, then $\bar{(}Y) \cong \bar{(}X)$, T is a u-composition series for G on V with respect to H and the factors of T are the factors of S not centralized by H.

(b) Let Y be a KG-submodule in V and S a u-series for G on V with respect to H.

(b1) Let

$$T = \{U(X \cap Y)|X \in S\}.$$

Then T is a u-series for G on X with respext to H.

(b2) Let $Z \in T$ and put

$$X = U(\cap\{W \in S|U(X \cap W) = Z\}).$$

Then $\overline{Z}$ is isomorphic to a submodule of $\overline{X}$. In particular, if S is a u-composition series for G on V with respect to H, then $\overline{Z} \cong \overline{X}$ and T is a u-composition series for G on X with respect to H.

Proof: (a) By part (c) of (3.3), T is totally ordered and U(W) = W for all W in T. Let R be a subset of T and put $Q = \{W \in S|U(W) \in R\}$. By part (e) of (3.3)

$$\cup R = \cup\{U(W)|W \in Q\} = U(\cup Q) \in T$$

and

$$U(\cap R) = U(\cap\{U(W)|W \in Q\}) = U(\cap Q) \in T.$$

Hence T is a u-series. Pick Y in T and define X as above. Then clearly U(X) = Y. Let W be in S. It follows from the definition of W that $W < X$ if and only if and only if $U(W) < Y$. Therefore $Y_- = U(X_-)$ and the remaining assertions in (a2) follow easily.

(b) Clearly T is totally ordered and U(W) = W for all W in T. Let R be a subset of T and put $Q = \{W \in S | U(Y \cap W) \in T\}$. By part(e) of (3.3),

$$\cup R = \cup \{U(Y \cap W) | W \in Q\} = U(Y \cap \cup Q) \in T$$

and

$$U(\cap R) = U(\cap \{U(Y \cap W) | W \in Q\}) = U(Y \cap \cap Q) = U(Y \cap U(\cap Q)) \in .T$$

Hence T is a u-series for G on Y with respect to H. Pick Z in T and define X as above. Then clearly $U(Y \cap X) = Z$. Let W be in S. It follows from the definition of X that $W < X$ if and only if $U(Y \cap W) < Z$. Therefore $Z_- = U(Y \cap X_-)$. Note that

$$U(Y \cap X_-) \leq U(Y \cap U(X_+)) \leq U(Y \cap X_-) = Z_-.$$

So $Y \cap X_+ \leq Z_+$. On the other hand $U(Z_+) \leq Z_- \leq X_-$ and $Z_+ \leq X_+$. Thus

$$Z \cap X_+ = Y \cap X_+ = Z_+ \text{ and } \bar(Z) = Z/Z_+ = Z/(Zcap X_+) \cong Z + X_+/X_+$$

. Now $Z + X_+/X_+$ is a submodule of $\overline{X}$ and this finishes the proof of 4.1

Lemma 4.3 *Let T and B be KG-submodules of V such that $B \leq T$, T/B is irreducible and [T/B, G] $\neq$ 0. Then there exists a minimal G-supplement to B in T, i.e. a KG -submodule C in T such that $T = B + C$ and every proper KG-submodule in C is contained in B.*

Proof: Pick h in G with [T/B, h] $\neq$ 0 and put $H =< h^G >$. Let U be any G-submodule in V and let S be a composition series for G on U. Since deg(h) is finite, h and H centralize all but finitely many of the factors of S. Let d(S) be the number of factors of S not centralized by H, d(U) = min { d(S) | S a KG composition series on U} and d = min { d(U) | U a KG-submodule of T with T = U+B}. Replacing T by an appropiate supplement,there is no loss in assuming that d(T) = d and that $T = \overline{U}_T(H)$. Let S be a composition series for G on T with d(S) = d. Let C be a supplement to B in T.

Suppose that C $\neq$ T. Put $D = \{\overline{U}_X(H) | X \in S\}$. Then $|D| = d + 1$. For $Y \in D$, let $\tilde{Y} = \overline{U}_{Y \cap C}(H)$. Put $E = \{\tilde{Y} | Y \in D\}$. Then by (4.1), D and E are u-composition series for G with respect to H on T and C, respectively. Thus

$$d + 1 \leq d(C) + 1 \leq |E| \leq |D| = d + 1,$$

and so $|E| = |D|$. Pick Y in D with $Y_- \leq K$ and $Y \not\leq K$. Since $|E| = |D|, \tilde{Y_-} \neq \tilde{Y}$, and so $\tilde{Y} \not\leq Y_+$. Since Y/Y_+ is an irreducible KG-module, we conclude that $Y = Y_+ + \tilde{Y}$. Hence H acts unipotently on $Y/Y + Y_-$. Since Y is a perfect KH-module, $Y = Y + Y_- \leq C$, a contradiction to the choice of Y. Thus C = T and every proper submodule of T is contained in B.

Definition 4.4 *Let R and S be sets of KG-modules.Then R and S are KG-isomorphic if there exists a bijection $\alpha : R \to S$ such that for all $MinR$, M and $\alpha(M)$ are isomorphic KG-modules.*

Theorem 4.5 *Let R and S be composition series for G on V. Then the sets of non-central factors of R and S are KG-isomorphic.*

Proof: Let T and B be G-submodules in V with $B \leq T$, T/B irreducible and [T/B, G] $\neq 0$. We claim that T/B is isomorphic to a factor of S. By (4.3) we may assume that every proper G-submodule in T is contained in B. Put $Y = \cap\{X \in S | T \leq X\}$. Let $R \in S$ with R ¡ Y. Then $T \not\leq R, T \cap R$ is a proper submodule of T and so $T \cap R \leq B$. Thus $T \cap Y_- \leq B$. Suppose that $B \not\leq Y_-$. Then $Y = B + Y_-$ and so $T = B + (T \cap Y_-) \leq B$, a contradiction. Thus $B \leq Y_-$ and $T \cap Y_- = B$. It follows that

$$T/B = T/T \cap Y_- \cong T + Y_-/Y_- = Y/Y_-.$$

Let X be a non-central factor of S. By the preceeding paragraph X isomorphic to a factor of R. Pick g in G with [X, g] $\neq 0$. Since g has finite degree, g centralizes all but finitely many factors of R and S. In particular only finitely many factors of R and S are isomorphic to X. Let $(B_i, T_i), 1 \leq i \leq n$, and $(C_i, U_i), 1 \leq i \leq m$, be the jumps of R and S, respectively, with factors isomorphic to X.

To complete the proof of the theorem it is enough to show that n = m. We assume without loss that $T_i \leq B_{i+1}, 1 \leq i < n, U_j \leq C_{j+1}, 1 \leq j < m$, and $n \leq m$.

Note that $S^* = \{T_n \cap D | D \in S\}$ is a composition series for T_n whose set of factors is isomorphic to a subset of the set of factors of S. Suppose that the number of factors of S^* isomorphic to X is smaller than m. Then there exists $1 \leq i \leq m$ with $T_n \cap U_i = T_n \cap C_i$. It follows that $U_i + T_n/T_n \cong U_i/U_i \cap T_n \cong U_i/C_i \cap T_n$ and so $U_i + T_n/T_n$ has a factor-module isomorphic to X. On the other hand $\{Y/T_n | Y \in R, T_n \leq R\}$ is a composition series for G on V/T_n none of whose factors is isomorphic to X, a contradiction to the first paragraph of the proof.

Thus S^* has m factors isomorphic to X. Replacing S by S^* we therefore may assume that $V = T_n$. Similarly we may assume that $V = U_m$. Put $Q = \{B_n \cap D | D \in S\}$; then Q is a composition series for B_n. By induction on n, we conclude that Q has exactly n-1 factors isomorphic to X. Suppose that $m \geq n$; then there exists $1 \leq i < j \leq m$ with $B_n \cap U_i = B_n \cap C_i$ and $B_n \cap U_j = B_n \cap C_j$. Suppose that $U_i \leq B_n$; then $U_i \leq C_i$, a contradiction. Thus $V = U_i + B_n$ and so $U_j = U_i + (B_n \cap U_j) = U_i + (B_n \cap C_j) \leq C_j$, again a contradiction.. Thus n =m and the proof is completed.

Remark 4.6 *(a) (4.5) can be rephrased as:*

Let R and S be u-composition series for G on V. Then the sets of factors of R and S are KG-isomorphic.

(b) By (4.5), if G has a finite dimensional series on V, then every composition series for G on V is finite dimensional.

Example 4.7 *The number of central factors in two distinct composition series for G can be different as the following example shows:*

Suppose that K has a element k with k $\neq$ 0,1. Let V be the vectorspace over K with basis $v_0, v_1, v_2, \ldots$. Define $x_i, i \geq 1$, in $FGL_K(V)$ by

$$v_j^{x_i} = v_j, 0{\neq}j{\neq}i,\ v_i^{x_i} = kv - i,\ v_0^{x_i} = v_0 + v_i$$

Put $V_i =< v_j | j > i >$ and $U_i = V_i + < (1-k)v_0 + \sum_{1 \leq j \leq i} v_j >$. Then it is readily verified that

$$V > V_0 > V_1 > V_2 > \ldots > 0\ mboxandV > U_1 > U_2 > \ldots > 0$$

are composition series for $< x_i | i \geq 1 >$ on V. Moreover, V/V_0 is a central factor in the first series, while the second one has no central factors.

Using (4.5) we are now able to prove a dual version of (2.2) for finitary modules. We remark that the following lemma is false for non-finitary modules.

Lemma 4.8 *Let $\mathcal{F}$ be a closed class of KG modules which includes all 1-dimensional central KG-modules. Then there exists a unique KG-submodule $M_{V,\mathcal{F}}(G)$ in V maximal with respect to G having a $\mathcal{F}$-series on $M_{V,\mathcal{F}}(G)$.*

Proof: Let S be the set of all KG-submodules U in V such that G has an $\mathcal{F}$-series on U. Let U be in S. Then by the properties of $\mathcal{F}$, G has a $\mathcal{F}$ -composition series on U and by 4.5 every composition series for G on U is an $\mathcal{F}$-series. Order S by inclusion and let R be a chain in S. Put$N = \cup R$ and extend R to a composition series C for G on N. Then, for all U in R, $\{X \in C | X \leq U\}$ is a composition series for G on U and hence a $\mathcal{F}$-series for G on U. It follows that C is a $\mathcal{F}$-series and so N is in S. By Zorn's Lemma S has a maximal element M. Let U be in S; then G has a $\mathcal{F}$-series on $U + M/M \cong U/U \cap M$ and on M. Thus U + M is in S and $U \leq M$.

5 Complete Reducibilty of Normal Subgroups of Irreducible Finitary Groups

Theorem 5.1 *If G acts irreducibly on V and H is a normal subgroup of G, then H acts completely reducibly on V.*

Proof: Suppose the Theorem is false. If H has an irreducible submodule I in V, then $V = \sum_{g \in G} I^g$ and V is completely reducible. So H has no irreducible submodule in V and in particular, H has no non-zero finite dimensional submodule in V. Pick $1{\neq}r \in H$ and put $R =< r^H >$. We will first prove that

(1) [V,R]+Y/Y is finite dimensional for any non-zero H-submodule Y in V.

Since G is irreducible on V, $V=\sum_{g\in G}Y^g$. Also since [V,r] is finite dimensional there exists $g_1,\dots,g_s$ in G with [V,r] Z, where $Z=\sum_{1\leq i\leq s}Y^{g_i}$. Now Z is a H submodule and so $[V,r^h]=[V,r]^h\leq Z$ for all h in H. Thus $[V,R]\leq Z$. Note that $C_Y(g_i)\leq Y\cap Y^{g_i}$ and hence $Y/C_Y(g_i)$ is finite dimensional for all $1\leq i\leq s$. Therefore $Y^{g_i}+Y/Y, Z+Y/Y$ and $[V,R]+Y/Y$ are finite dimensional, proving (1).

Since H has no non-zero irreducible submodule in V we can find a descending chain

$$[V,R]>Y_1>Y_2>....Y_n>...$$

of H submodules Y_i in [V,R]. Let $Y=\cap_{i\geq 1}Y_i$. Then [V,R]/Y is infinite dimensional and (1) implies that Y = 0.(in fact all non-zero KH-submodules of [V,R] have finite codimension in [V,R]). Hence there exists n with $[V,r]\cap Y_n=0$. It follows that $[Y_n,r]=0$. Thus $[Y_n,R]=0$ and $C_V(R)\neq 0$. (1) implies that $[V,R]+C_V(R)/C_V(R)$ is finite dimensional. In particular,

(2) $\overline{U}_V(R)+\underline{U}_V(R)/\underline{U}_V(R)$ is finite dimensional.

For $N\leq G$ let

$$U(N)=\overline{U}_V(N)\cap\underline{U}_V(N) \text{ and } d(N)=dim_K\overline{U}_V(N)/U(N).$$

Put $d=min\{d(N)|1\neq N\trianglelefteq H\}$ and pick $1\neq L^*\trianglelefteq H$ with $d(L^*)=d$. Put

$$L=<N|1\neq N\trianglelefteq H,\overline{U}_V(N)=\overline{U}_V(L^*),U(N))=U(L^*)>.$$

By (3.3), $\overline{U}_V(L)=\overline{U}_V(L^*)$ and $U(L))=U(L^*)$. We prove next that:

(3) For all g in G, $L=L^g$ or$L\cap L^g=1$.

Indeed, let g be in G with $L\cap L^g\neq 1$. Then $\overline{U}_V(L\cap L^g)\leq\overline{U}_V(L)$ and $U(L)\cap\overline{U}_V(L\cap L^g)\leq U(L\cap L^g)$. By the minimal choice of d, $d(L\cap L^g)\leq d(L)$. On the other hand

$$d(L\cap L^g)=dim_K\overline{U}_V(L\cap L^g)/U(L\cap L^g)\leq dim_K\overline{U}_V(L\cap L^g)/U(L)\cap\overline{U}_V(L\cap L^g)\leq$$

$$\leq dim_K\overline{U}_V(L\cap L^g)+U(L)/U(L)\leq dim_K\overline{U}_V(L)/U(L)=d(L).$$

Hence equality holds at each place and we conclude that $U(L)\cap\overline{U}_V(L\cap L^g)=U(L\cap L^g)$ and $\overline{U}_V(L\cap L^g)+U(L)=\overline{U}_V(L)$. In particular, L acts unipotently on $\overline{U}_V(L)/\overline{U}_V(L\cap L^g)$ and so $\overline{U}_V(L)=\overline{U}_V(L\cap L^g)$. Thus $U(L)=U(L\cap L^g)$. By symmetry, $\overline{U}_V(L^g)=\overline{U}_V(L\cap L^g)$ and $U(L^g)=U(L\cap L^g)$. Therefore $\overline{U}_V(L)=\overline{U}_V(L^g)$ and $U(L)=U(L^g)$. By definition of L, $L^g\leq L$ and $L\leq L^g$. Thus $L=L^g$ and (3) is established.

If g is an element of G with $L\neq L^g$, then by (3) $L\cap L^g=1$. Since L and L^g are normal subgroups of H we get $[L,L^g]=1$. Put $L^\perp=<L^g|g\in G\setminus N_G(L)>$. Then $[L,L^\perp]=1$. Put $Y=[V,L^\perp]\cap\overline{U}_V(L)$. Since $\overline{U}_V(L)/U(L)$ is finite dimensional, there exist $l_1,\dots,l_t$ in $L^\perp$ with $Y\leq U(L)+Z$, where $Z=\sum_{1\leq i\leq t}[V,l_i]$. In particular, L acts unipotently on Y+Z/Z and so $\overline{U}_V(L) \leq Z$. Since Z is finite dimensional, $\overline{U}_Y(L)$ is a finite dimensional H submodule in V and so $\overline{Y}_V(L)) = 0$. Thus

$$\overline{U}_{[V,L^\perp]}(L) \leq \overline{U}_{[V,L^perp]\cap\overline{U}_V(L)}(L) \leq \overline{U}_Y(L) = 0.$$

It follows that $[V, L^\perp] \leq \underline{U}_V(L)$ and in particular, $\overline{U}_V(L^) \leq \underline{U}_V(L)$. So $\overline{U}_V(L^g) \leq \underline{U}_V(L)$ for all $g \in G \setminus N_G(L)$ and $\overline{U}_V(L) \leq \underline{U}_V(L^g)$ for all $g \in G \setminus N_G(L)$. Hence $\underline{U}_V(L) \leq \underline{U}_V(< L^G >)$. Note that G normalizes $\underline{U}_V(< L^G >)$. Since G acts irreducibly on V, part (c) of (3.2) implies that $< L^G >$ does not act unipotently on V. Thus $\underline{U}_V(< L^G >) = 0, \underline{U}_V(L) = 0$ and $\overline{U}_V(L)$ is a non-zero finite dimensional H -submodule in V; this contradiction completes the proof of (5.1).

Lemma 5.2 *Let h be any element of G. Then there exists a subnormal subgroup H in G with $h \in H$ such that H has a finite u-composition series on V.*

Proof: Let S be any composition series for G on V and let T be the set of factors in S which are not centralized by h. Since [V,h] is finite dimensional, T is finite. Let $H_1 =< h^G >$ and inductively $H_{i+1} =< h^{H_i} >$. Pick t in T. If H_i acts irreducibly on V on t for all i, let n(t) = 0. Otherwise let n(t) be minimal such that $H_{n(t)}$ acts reducibly on t. Note that in this case by (5.1) $H_{n(t)}$ acts completely reducibly on t. Since $H_{n(t)-1}$ acts irreducibly and finitary on t it is easy to see that all the irreducible $H_n(t)$-submodules are finite dimensional.

Let $j \leq n(t)$. We claim that H_j has a finite u-composition series on t. If n(t) = 0, H_j is irreducible. So assume that $n(t) > 0$. Then H_{j-1} is reducible on t and so t is the direct sum of finite dimensional irreducible H_{j-1}-submodules. Note that h centralizes all but finitely many of these submodules and so $[t, H_j]$ is finite dimensional, proving our claim.

Put $n = max\{n(t)|t \in T\} + 1$ and $H = H_n$. Then H has a finite u-composition series on all elements of T and hence also on V.

6 Component-Type Subgroups for Irreducible Finitary Groups

Definition 6.1 1. *d(G) is the sum of the dimensions of the factors of a u-composition series of G in V.*

2. *d_G = min{ d(L)— 1 $\neq$ L asc G}.*

3. *Λ = { L asc G — $d(L) = d_G$}.*

We remark that d(G) and d_G may be infinite and that by 4.5, d(G) does not depend on the choice of the composition series.

Theorem 6.2 . *Suppose G is irreducible on V and and d_G is finite. Let $L \in \Lambda$. Then one of the following holds:*

(i) There exists $L^* \in \Lambda$ such that $L^* =< L^g | g \in G, L^g \leq L^* >$ and $[L^*, L^{*g}] = 1$ for all g in G with $L^* \neq L^{*g}$.

(ii) L is of order p for some prime p and $< L^G >$ is a solvable and locally finite p-group.

(iii) L has a normal subgroup B such that $|L/B| = 2, B \in \Lambda$, B is isomorphic to a subgroup of the multiplicative group of a finite field extension of K, L inverts B and $C_B(L) = 1$.

Proof: Let $G_i, i \in I$ be an ascending normal series from L to G. For $X, Y \leq G$ write $X \sim Y$ if $d(< X, Y >) = d(X) = d(Y)$. Note that

(1) If $X \sim Y$ and $A, B \leq < X, Y >$ with d(A) = d(B) = d(X), then $A \sim B$.

Next we prove that

(2) Let X and Y be in Λ such that X normalizes Y and Y normalizes X. Then $X \sim Y$ or $X \cap Y = 1$.

Suppose $X \cap Y \neq 1$ and let t be a composition factor for XY on V. If $[t, X \cap Y] = 0$, then $d = d(X \cap Y) = d(X) = d(Y)$ implies that $0 = [t, X \cap Y] = [t, X] = [t, Y] = [t, XY]$. If $[t, X \cap Y] \neq 0$, then since $X \cap Y$ is normal in XY, t is the direct sum of irreducible $X \cap Y$-submodules. It follows that $X \cap Y$ has no central composition factor on t. Hence $d(XY) = d(X \cap Y) = d$ and $X \sim Y$.

(3) Let $X \in \Lambda$ and $\Sigma \subseteq \Lambda$ such that for every S in Σ , S normalizes X and $S \sim X$, then $d(< X, < \Sigma >>) = d$ and $S \sim R$ for any S and R in Σ.

Put $T =< X, < \Sigma >$ and let t be a composition-factor for T on V. If X centralizes t, then since d(X) = d(XS) for all S in Σ, S acts unipotently on t. By part (b) of (3.2), T acts unipotently on t. If X does not centralize t, then since X is normal in T, (5.1) implies that X has no central composition factor on t. Hence $d = d(X) = d(T) = d(< S, T >)$ and (3) is proved.

(4) Suppose L is a group of prime order. Then (ii) holds.

Since L is an ascending, locally finite and locally solvable p-subgroup of G, an easy induction argument shows that $< L^G >$ is a locally finite and locally solvable p-subgroup of G. Proposition 1 in [4] implies that $< L^G >$ is solvable and this completes the proof of (4).

Let 0 be the minimal element in I. Put $L_0 = L$ and inductively define $L_i, (i \in I)$ by $L_i = \cup\{L_i | j < i\}$, if i is an limit ordinal and $L_{i+1} =< L_i^g | L_i \sim L_i^g, g \in G_{i+1} >$.

(5) If $k \in I$ and $s \in G_k$ with $d(L_k) = d$ and $L_k \sim L_k^s$, then $L_k = L_k^s$.

Let r be minimal in I with $s \in G_r$. Put $J = \{j \in I | j < k$ and $r \leq j + 1\}$. Let j in J and h in G_{j+1} with $L_j \sim L_j^h$. Then $L_j^h \leq L_{j+1} \leq L_k$. By assumption $d(< L_k, L_k^s >) = d$and so by (1) $L_j \sim L_j^{hs}$. Since $hs \in G_{j+1}, L_j^{hs} \leq L_{j+1} \leq L_k$. This holds for all such h and so $L_{j+1}^s \leq L_k$.

If k is not a limit ordinal, k-1 is in J and hence $L_k^s \leq L_k$. If k is a limit ordinal then $r < k$ and $L_k = \cup\{L_j | j \in J\}$ and again $L_k^s \leq L_k$. By symmetrie, $L_k^{s^{-1}} \leq L_k$ and $L_k = L_k^s$.

(6) Suppose that neither (ii) nor (iii) hold. Then each of the following holds for all i in I:
(a) For each g in $G_i, L_i = L_i^g$ or $[L_i, L_i^g] = 1$.
(b) If i is not a limit ordinal, L_{i-1} is normal in L_i.
(c) $L_i \in \Lambda$.

Assume (6) is false and let i be minimal in I such that (6) fails. Put $J = \{j \in I | j < i\}$.

Suppose that (a) and (b) hold. Then by (a), L_i is normal in $< L_i^{G_i} >$ and so L_i is an ascending subgroup of G. If i is not limit ordinal, then by (b), (2) and the definition of L_i, $d(L_i) = d$. Thus (c) hold in this case, a contradiction. Hence i is a limit ordinal. Let t be a non-central composition factor for L_i on V. Since $d = d(L_i)$ for all j in J and since L_j is normal in L_{j+1}, we conclude that $\underline{U}_t(L_j) = \underline{U}_t(L_{j+1})$. An easy induction argument shows that $\underline{U}_t(L_j) = \underline{U}_t(L_i) = 0$. Since d is finite, t is finite dimensional and so every L_j normalizes a non-central irreducible submodule in t. Let m = max { dim X | there exists j in J such that X is a L_j- submodule with no central L_j -composition factor}. Note that $0 < m \leq d$. Choose j and X as above with dim X = m. Then L_j as no central composition factor in $< X^{L_{j+1}} >$ and so L_{j+1} normalizes X and L_{j+1} has no central composition factor in X. By induction, L_i normalizes X and so X = t, and L_j has no central composition factor in t. This implies that $d(L_i) = d$ and $L_i \in \Lambda$.

Hence (a) or (b) fails. Suppose first that i is a limit ordinal. Then (a) must fail. Pick $g \in G_i$ with $L_i^g \not\leq L_i$ and $[L_i, L_i^g] \neq 1$.Then there exist j,k,l,m in J such that $L_j^g \leq L_i, [L_k, L_l^g] \neq 1$ and $g \in G_m$. Put q =max{j,k,l,m}. Then $L_q^g \not\leq L_q, [L_q, L_q^g] \neq 1$ and $g \in G_q$, a contradiction to the minimal choice of i. Hence i = k +1 for some k in I. Set $R = L_k$. We will first prove that

(*) R is normal in $< R^{G_i} >$.

Suppose not and pick h in G_i so that $S = R^h$ does not normalize R. Pick s in S with $R^s \neq R$. Since $s \in G_k, [R, R^s] = 1$. By (4) $d(RR^s) \neq d$. Hence there exists a composition factor t for $< R, S >$ in V such that R has a central and a non-central composition factor on t. Note that R is normal in $< R^{G_k} >$ and so also S is normal in $< S^{G_k} >$. In particular, $R \cap S$ is subnormal in $< R, S >$. Suppose that $R \cap S \neq 1$. Since R, S and $R \cap S$act completely reducibly on t and $d = d(R \cap S) = d(R) = d(S)$ we conclude that $[t, R] = [t, R \cap S] = [t, S] = t$ and so R has no central composition factor on t, a contradiction. We conclude that $R \cap S = 1$. Similarly, since $R \cap R^s$ is centralized by RR^s and $d(RR^s) \neq d, R \cap R^s = 1$.

Suppose that R is abelian and let C be the set of factors of some u-composition series of $< R^{<s>} >$ on V. Let $1 \neq r \in R$. Then $< r >$ is normal in R and so ascending in G. Thus $d(< r >) = d$ and it follows that $\overline{U}_V(R) \leq [V, r]$. Hence $\overline{U}_V(< R, s >)$ is finite dimensional and so C is finite. Let E be a subset of C and let R_1 and R_2 be in $R^{<s>}$ with $[e, R_i] \neq 0$ and $[f, R_i] = 0$ for all e in E, $f \in C \setminus E$ and $i \in \{1, 2\}$. Then since R_i is normal in $R^{<s>}$ we conclude that $[e, R_i] = e$ for all e in E and so $d(R_1 R_2) = d$. Thus by (5) $R_1 = R_2$. Since C has only finitely many subsets we conclude that $|R^{<s>}|$ is finite. So s^k normalizes R for some positive integer k. Since $< R^S >$ and $< S^R >$ are abelian, $[R, S] \leq Z(< R, S >)$ and so $[r^k, s] = [r, s^k] \in R$. Hence $(r^k)^s \in R \cap R^s = 1$ and R and S are of exponent k. Choosing s such that $s^p \in N_S(R)$ for some prime p we see that R and S are of exponent p. It follows that $|R/C_R(u)| = p$ for every non-central composition factor u of R on V. Then $C_R(u)$ is an ascending subgroup of G with $d(C_R(u)) < d, C_R(u) = 1$ and $|R| = p$, a contradiction to (4).

Thus $R' \neq 1$. Suppose that R normalizes S. By (2.3), $R' \leq [R, S]$ and so $R' \leq R \cap S = 1$, a contradiction. So R does not normalize S and we can apply (2.4). Let $A = N_R(S)$, then A is a abelian, R inverts A and $C_A(R) = 1$. Moreover, since d(A) = d and A asc G, A acts faithful on each of its u-composition factors on V. Thus by Schur's Lemma, A is isomorphic to a subgroup of the multiplicative group of a finite field extension of K. Suppose $L \leq A$ and pick n in I minimal with $L_n \not\leq A$. Then clearly n = j+1 for some $j \in I$. Pick $g \in G_r$ with $L_j \sim L_j^g$ and $L_j^g \not\leq A$. Since $L_j \leq A, L_j^g$ is abelian and so $A \cap L_j^g \leq C_A(R) = 1$. Thus $|L_j^g| = 2$ and since R inverts T, $L_j \leq C_T(R) = 1$, a contradiction. Thus $L \not\leq T$. Put $B = L \cap A$. Then (iii) holds.This contradiction to the assumptions establishes (*).

By (*), L_k is normal in L_i and so (a) must fail. Let $g \in G_i$. If $L_i \sim L_i^g$, then by (5) $L_i = L_i^g$. If $L_i \not\sim L_i^g$ then (2) implies $[L_i, L_i^g] = 1$. This completes the proof of (6).

By (6) either (ii) or (iii) holds or (i) holds with $L^* = L_a$, where a is in I with $G = G_a$.

Remark 6.3 *Retain the assumption of (6.2). Replacing L by B in case (iii) it follows that there always exists L in Λ which fulfills (i) or (ii).*

7 Ascending Subgroups of Finitary Groups

Throughout this section H is a ascending subgroup of G, $G_i, i \in I$ be an ascending normal series from L to G,0 is the minimal element in I and a is the maximal element. Let $\mathcal{F}$ be the class of finite dimensional KH-modules and put $m_V(H) = m_{V,\mathcal{F}}(H)$ and $M_V(H) = M_{V,\mathcal{F}}(H)$, where $m_{V,\mathcal{F}}(H)$ and $M_{V,\mathcal{F}}(H)$ are defined in (2.2) and (4.8), respectively.

Lemma 7.1 *(a) Let U be a KH-submodule in V and $M_U^V(H)$ the inverse image of $M_{V/U}(H)$ in V. Then G normalizes $m_U(H)$ and $M_U^V(H)$.*

(b) If G is irreducible and H reducible on V, then every composition series of H in V is finite dimensional. In particular, $d(< h^H >)$ is finite for all h in H.

Proof: Suppose (a) is not true and choose i mimimal in I so that G_i does not normalize both of $m_U(H)$ and $M_U^V(H)$. Then clearly i is not a limit ordinal and so G_{i-1} is normal in G_i and normalizes $m_U(H)$ and $M_U^V(H)$. Let $g \in G_i$. Then $m_U(H) \cap m_U(H)^g$ is of finite codimension in $m_U(H)$. Moreover, G_{i-1} and therefore also H normalizes $m_U(H) \cap m_U(H)^g$. It follows immediately from the definition of $m_U(H)$ that $m_U(H)$ has no proper H-submodule of finite codimension. Hence $m_U(H) \cap m_U(H)^g = m_U(H) = m_U(H)^g$. Thus G_i normalizes $m_U(H)$. Similarly, $M_U^V(H)^g + M_U^V(H)/M_U^V(H)$ is a finite dimensional KH-submodule of $V/M_U^V(H)$ and G_i normalizes $M_U^V(H)$, a contradiction which establishes (a).

(b) Let U be a proper H-submodulein V. Since G acts irreducibly on V, (a) implies that $M_U^V(H) = V$ and $m_U(H) = 0$. Thus H has finite dimensional composition series on U and on V/U and therefore also on V. By (4.5) every composition series for H on V is finite dimensional. Note that $< h^H >$ centralizes all but finitely many of those composition factors and (b) is verified.

Note that part (b) of (7.1) implies:

Corollary 7.2 *Suppose G is irreducible on V and has a reducible ascending subgroup. Then d_G is finite.*

Definition 7.3
1. *A set Π of proper K-subspaces in V such that $V = \oplus\Pi$ and $P^g \in \Pi$, for all $P \in \Pi, g \in G$, is called a system of imprimitivity for G on V. A K-subspace P of V is called an subspace of imprimitivity for G on V if P^G is a system of imprimitivity.*
2. *$H \leq GL_K(V)$ is called imprimitive, if there exists a system of imprimitivity for G on V. Otherwise G is called primitive.*

Proposition 7.4 *If G is primitive and V infinite dimensional, then every non-trivial ascendent subgroup of G acts irreducibly and primitively on V.*

Proof: Suppose that H is reducible. Using (5.1) the Wedderburn components of a non-trivial reducible normal subgroup of G form a system of imprimitivity. So G has no non-trivial reducible normal subgroup. In particular, G has no non-trivial abelian or solvable normal subgroup. By (7.2) d_G is finite and this permits us to apply (6.2). Note that case (ii) cannot occur. In case (iii) we replace L by B. Thus (i) holds for some $L \in \Lambda$. Without loss $L^* = L$. Then L is subnormal in G and so by (5.1) V is completely reducible for L. Since $d(L) = d_G$ is finite we conclude that [V,L] is finite dimensional. Therefore $< L^G >$ is reducible, a contradiction.

So H is irreducible. Suppose that Π is a system of imprimitivity for H on V. By the previous paragraph, H has no reducible normal subgroup and so H acts faithfully on Π as a transitive group of finitary transformations. Suppose that H has no maximal system of imprimitivity on Π. Then any element in H acts trivally on some system of imprimitivity, a contradiction. Thus we may assume that H acts primitively on Π. By Satz 9.4 in [9], $H = Alt(\Pi)$ or $FSym(\Pi)$. Let $P \in \Pi$. Suppose that [P, h] $\neq 0$. for some $h \in N_H(P)$. Then $|P^{C_H(h)}|$ is infinite and so [V,h] is infinite dimensional, a contradiction. Hence $[P, N_H(P)] = 0$ and V is the natural permutation module for H, but then the even permutation module is a proper H-submodule in V, a contradiction.

Proposition 7.5 *(a) If Π is a infinite system of imprimitvity for G on V, and H is an ascending subgroup of G acting transitively on Π, then $G' \leq H$.*

(b) Suppose that G is irreducible and imprimitive on V and V that is infinite dimensional; then G' is the unique mimimal irreducible ascending subgroup of G.

Proof: (a) Suppose $G' \not\leq H$ and pick j in I minimal with $G'_j \not\leq H$. Then clearly j is not a limit ordinal and so j = t+1 for some t in I. Let $L = G'_t$. Then L is a normal subgroup of G_j and $L \leq H$. Moreover, since G_k is transitive on Π, and Π is infinite, L is transitive on Π. Let $x, y \in G_j$ and $A =< x, y >$. Since [V,A] is finite dimensional, there exists a finite subset Σ of Π with [P, A] = 0 for all $P \in \Pi \setminus \Sigma$. By (2.3) in [6], there exists l in L with $\Sigma \cap \Sigma^l = \emptyset$. Hence $[A, A^l] = 1$. Thus by (2.3) $A' \leq [A, l] \leq [G_j, L] \leq L$ and so $G'_j \leq L \leq H$, a contradiction.

(b) By (a) it remains show that G' acts irreducibly on V. Otherwise the abelian group G/G' acts transitively and finitary on the infinite set of Wedderburn components of G' in V. This is impossible.

Theorem 7.6 *If G is irreducible on V, then every ascending subgroup of G acts completely reducibly on V.*

Proof: Suppose H is not completely reducible on V. Let i in I be minimal such that G_i is completely reducible. If i = k+1 for some k in I, then G_k is normal in G and by (5.1), G_k is completely reducible, a contradiction. So $G = \cup\{G_k | k < i\}$. Suppose first that V is finite dimensional. For k in I let A_k be the K-subalgebra of $End_K(V)$ generated by the elements of G_k. Under the elements less than i, choose k such that $dim_K A_k$ is maximal. Then $A_k = A_j$ for all $k \leq j < i$. Thus $A_i = \cup\{A_j | j < i\} = A_k$. It follows that G_k and G_i have the same submodules in V, a contradiction to the fact that G_i is completely reducible on V and G_k is not.

Suppose next that V is infinite dimensional. Since H is reducible, (7.4) implies that G is imprimitive. Let Π be a system of imprimitivity for G om V.

Assume first that H does not act transitively on Π. Then by (2.6), V is the direct sum of H-invariant subspaces of imprimitivity for G on V. Let U be any H-invariant subspaces of imprimitivity for G on V. Then U is finite dimensional, $N_G(U)$ acts irreducibly on U and H is an ascending subgroup of $N_G(U)$. By the finite dimensional case, H is completely reducible on U. Since this holds for all such U, H is completely reducible on V.

Assume next that H is transitive on Π. Then by 7.5, $G' \leq H$, and H acts irreducibly on V, a contradiction, which completes the proof of (7.6).

We are now able to prove an improved version of (5.2)

Lemma 7.7 *Let h be in G, then* $< h^{<h^G>>}$ *has a finite u-composition series on V.*

Proof: Retain the notation established in the proof of (5.2). By that proof it is enough to show that $n(t) \leq 1$ for all t in T. Suppose first that G acts primitively on t. Then by (7.4), n(t) = 0. Suppose next that G has a system of imprimitivity Π on t and let D be the subgroup of G which is maximal with respect to acting trivial on Π. Now assume that h is in D. Then $H_1 \leq D$ and n(t) = 1. So we may assume that h is not in D and that G acts primitively on Π. But then by Satz 9.2 im [9], G/D = Alt(Π) or FSym(Π) and so $G'D \leq H_1 D$. It follows by induction that $G'D \leq H_i D$ for all i . Thus by (7.5), $G' \leq H_i C_G(t)$ and H_i acts irreducibly on t. Thus n(t) =0, and (7.7) is proved.

8 Free Groups as Infinite Dimensional Finitary Groups

Example 8.1 *Let I be a set with at least two elements, F a free group with generators f_i, i in I, and M a free abelian group with generators v_i, i in I. For i in I let z_i be an integer with $|z_i| \leq 6$. Then the representation of F on M defined by*

$$v_i^{f_i} = v_i \text{ and } v_j^{f_i} = v_j + z_i v_i$$

for all i,j in I with $i \neq j$, is faithful. Moreover, if K is a field with $z_i \neq 0$ (in K) for all i in I, then $K \otimes M$ is an irreducible KF-module. If char K = 0, then $K \otimes M$ is a faithful F -module.

Proof: For i in I let $M(i) = \{\sum a_k z_k \in M \| a_i| > 2(\sum_{k \neq i} |a_k|)\}$. We will first prove

(*) Let i,j in I with i $\neq$ j and t a nonzero integer. Then $M(j)^{(f_i^t)} \subseteq M(i)$.

Let $m = \sum a_k v_k \in M(j)$ and $m^{(f_i^t)} = \sum b_k v_k$. Since

$$v_i^{(f_i^t)} = v_i \text{ and } v_j^{(f_i^t)} = v_j + t z_i v_i$$

we conclude that $b_k = a_k$ for k $\neq$ i and $b_i = t z_i (\sum_{k \neq i} a_k) + a_i$. Thus

$$\sum_{k \neq i} |b_k| = \sum_{k \neq i} |a_k| = \sum_{k \neq i,j} |a_k| + |a_j| < |a_j|/2 + |a_j| = 3/2(|a_j|).$$

On the other hand,

$$|b_i| = |t z_i (\sum_{ki} a_k) + a_i| \leq |t z_i||a_j| - |t z_i|(\sum_{k \neq i,j} |a_k|) - |a_k| \leq |t z_i||a_j| - |t z_i|(\sum_{k \neq i} |a_k|) =$$

$$= |t z_i|(|a_j| - \sum_{k \neq i} |a_k|) \leq 6(|a_j|/2) = 3|a_j|.$$

Hence $|b_i| > 2(\sum_{k \neq i} |b_k|), m^{(f_i^t)} \in M(i)$ and (*) is established.

Let $f = f_{i(1)}^{t(1)} \cdot \ldots \cdot f_{i(s)}^{t(s)} \in F$ with i(r) $\neq$ i(r+1) and t(r) $\neq$ 0. Let $j \in I$ with $j \neq i(1)$. We claim that $M(j)^f \subseteq M(i(s))$. Indeed, put $x = f_{i(s)}^{t(s)}$ and $h = f x^{-1}$. Then by induction on s, $M(j)^h \subseteq M(i(s-1))$ and so by (*),

$$M(j)^f = (M(j)^h)^x \subseteq M(i(s-1))^x \subseteq M(t(s))$$

.

We are now able to prove that f acts non-trivially on M. If i(1) = i(s) pick $j \in I \setminus i(1)$ and if i(1) $\neq$ i(s) let j = i(1). Put $y = f_{i(1)}^{t(1)}$. Note that $v_j \in M(j)$ and so in any case $v_j^y \in M(i(1))$. Since i(1) $\neq$ i(2) we conclude that $v_j^f \in M(t(s))$. Since j $\neq$ t(s), $v_j \notin M(t(s))$. Thus $v_j \neq v_j^f$ and F acts faithfully on M.

Let K be a field with $z_i \neq 0$ for all i in I. Let $V = K \otimes M$ and U a nonzero KF-submodule in V. Clearly $C_V(F) = 0$. Hence $[U, f_i] \neq 0$ for some i in I and so $v_i \in [U, f_i] \leq U$. It follows that for j $\neq$ i, $v_j \in [K v_i, f_j] \leq U$. Thus U = V and V is irreducible as a KF-module.

Remark 8.2 *(a) The first assertion of (8.1) is also an easy consequence of the main theorem in [3]. Indeed Humphrey proves a much stronger version of (8.1) for finite I's. (8.1), F has an irreducible, infinite dimensional, and finitary representation over K. It is easy to see that any such representation must be primitive. It follows from (7.4) that every non-trivial ascending subgroup of F acts primitively. In particular, the intersection of the normal and irreducible subgroups of F is trivial and (7.5) does not hold in general for a finitary primitive groups.*

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