The maximal 2-local subgroups of the Monster and Baby Monster, II

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Abstract

In this paper the maximal 2-local subgroups in the Monster and Baby Monster simple groups which are not of characteristic 2 are determined.

1 Introduction

Let p be a prime and G a finite group. We say G is of characteristic p if $C_G(O_p(G)) \leq O_p(G)$. A subgroup of G is called a p-local subgroup if its the normalizer of a non-trivial p-subgroup of G. In [MS] all maximal 2-local subgroups of the Monster M and the Baby Monster BM which are of characteristic 2 have been classified. As a follow up in this paper we determine whose maximal 2-local subgroups of the Monster and the Baby Monster which are not of characteristic 2.

Theorem A *The Monster group M contains exactly 2 conjugacy classes of maximal 2-local subgroups which are not of characteristic 2 with the structures as follows:*

(1) $2.BM$;

(2) $2^2.{}^2E_6(2).S_3$

Theorem B *The Baby Monster group BM contains exactly 3 conjugacy classes of maximal 2-local subgroups which are not of characteristic 2 with the structures as follows:*

(1) $2.{}^2E_6(2).S_3$;

(2) $2^2.F_4(2).2$;

(3) $S_4 \times {}^2F_4(2)$

Let G be a finite group. For $g \in G$ let $C_g = C_G(g)$. An element z in G is called singular that if there exists a normal subgroup Q_z of C_z such Q_z is an extraspecial p-group for some prime p and $C_G(Q_z) = \langle z \rangle$. So Q_z is a large extra special subgroups of G. Note that this implies that z has order p and that C_z is of characteristic p. If $x, y \in G$ and x is singular we say y is perpendicular to x provided that $y \in Q_x$. A subgroup E of G is called singular if all the elements in $E^\#$ are singular and pairwise perpendicular. If $E \leq G$, put $C_E = C_G(E)$ and $Q_E = \bigcap\{Q_z \mid z \in E, z \text{ singular }\}$. If X, Y are subgroups of G with Y singular, we say that X is perpendicular to Y if $X \leq Q_Y$.

With the Monster we mean a finite group M such that M has a singular involution z with $C_z/Q_z \cong Co_1$ and $|Q_z| = 2^{25}$ Then M has two classes of involutions, see for example [MS, 7.6]. Let t be a non-singular involution in M and put $T = \langle t \rangle$ and $B = C_t$. With the Baby Monster we mean the group $BM = \overline{B} = B/T$.

2 Subgroups of p-type

In this section G is a finite group and p a prime.

Lemma 2.1 *Let A be p-subgroup of G. Then the following two statements are equivalent.*

(a) $C_G(O_p(C_G(A))A)$ is a p-group.

(b) $N_G(A)$ is of characteristic p.

Proof: Let $Q^* = O_p(N_G(A))$ and $Q = O_p(C_G(A))$.

Suppose (a) holds, i.e. that $C_G(QA)$ is a p-group. Since AQ is a normal p-subgroup of $N_G)A)$, $AQ \leq Q^*$. Thus $C_G(Q^*) \leq C_G(AQ)$. So $C_G(Q^*)$ is a normal p-subgroup of $N_G(A)$ and $C_G(Q^*) \leq Q^*$. Thus by definition of "characteristic p", (b) holds.

Suppose (b) holds. Let $E = O^p(C_G(QA))$. Then (a) is equivalent to $E = 1$. Note that Q^* normalizes E and E normalizes Q^*. Thus $[Q^*, E] \leq Q^* \cap E \leq O_p(E)$. Since $O_p(E)$ is a normal p-subgroup of $C_G(A)$, $O_p(E) \leq Q$ and so $[O_p(E), E] = 1$ and $[Q^*, E, E] = 1$. Since $E = O^p(E)$ and Q^* is a p-group, we get $E \leq C_G(Q^*)$. Since $N_G(A)$ is of characteristic p, $C_G(Q^*)) \leq Q^*$. Hence $E \leq Q^*$ and $E = 1$. □

A p-subgroup A of G is called of *p-type in G* provided that it fulfills one of the equivalent conditions of the previous lemma.

Lemma 2.2 *Let A be p-subgroup of G and $D \leq A$. If D is of p-type then A is of p-type.*

Proof: Put $Q = O_p(C_G(D))$ and $T = O_p(C_G(A))$. By induction on $|A/D|$ we may assume that A normalizes D. Let x be a p'-element in G centralizing AT. By 2.1a it suffices to show that $x = 1$. Since $C_G(A) \leq C_G(D)$, $C_G(A)$ normalizes Q. So $C_Q(A)$ is a normal p-subgroup of $C_G(A)$ and hence $C_Q(A) \leq T$. Thus x centralizes $C_Q(A)$. Since A and x normalize Q, the $P \times Q$-lemma implies, $[Q, x] = 1$. So $x \in C_G(QD)$. By 2.1a $C_G(QD)$ is a p-group. Since x is a p' element we get $x = 1$. $\square$

Lemma 2.3 *Let L be a finite group, $X \leq L$ and suppose:*

(a) $C_L(t) \leq X$ for all involutions $t \in X$.

(b) There exists an involution $s \in X$ with $s^L \cap X = s^X$.

(c) L has at least two classes of involutions.

Then $X = L$.

Proof: Let t be an involution in X and r an involution in L not conjugate to t. Since $\langle r, t\rangle$ is a dihedral group and r and t are not conjugate, $|rt|$ is even and there exists a unique involution u in $\langle rt\rangle$. Then by (a) $u \in C_L(t) \leq X$ and $r \in C_L(u) \leq X$.

Applying this to $t = s$ we see that X contains all the involutions in L not conjugate to X. So by (c) there exists a involution $t \in X$ with $t \notin s^L$.

So by the first paragraph $s^L \subseteq X$. Thus by (b) $s^L = s^X$ and so by the Frattini argument, $L = XC_L(s) = X$. $\square$

3 Purely non-singular subgroups

Define arks as in [MS]. In this section we determine the purely non-singular elementary abelian subgroups of M and their normalizers.

Lemma 3.1 *Let z be a singular involution and t a non-singular involution. Suppose that $t \in C_z \setminus Q_z$. Then*

(a) *tz is singular.*

(b) *Let $V = (Q_z \cap Q_{tz})\langle t, z\rangle$. Then V is ark and so V is elementary abelian of order 2^{10}, $|C_M(V)/V| = 2^{16}$ and $N_M(V)/C_M(V) = \Omega_{10}^{+}(2)$.*

(c) *There exists a $N_M(V)$ invariant non-degenerate quadratic form f on V. Let s be the bilinear form associate to f and let $a, b \in V^{\#}$. Then a is singular if and only if $f(a) = 0$. If a is singular, then b is perpendicular to a if and only if $s(a, b) = 0$,*

(d) *$VQ_z = \langle t\rangle Q_z$ and tQ_z is in Class 2A of $C_z/Q_z \cong Co_1$.*

(e) *V is the unique conjugate of V in M containing $\langle t, z\rangle$.*

Proof: By [MS, 7.7](and using the notations wherein) $\bar{t}$ is of type $2a_1$, $2a_3$ or $2c$. Since t is non-singular, [MS, 7.1] shows that t is of type $2a_1$ and that tz is singular. Hence (a) holds. Moreover, [MS, 7.7(2)] gives that V is the unique ark containing t and z. So (e) holds. (b) follows from [MS, 5.1,5.8] and (c) from [MS, 5.7, 5.9]. Note that z is singular, t is non-singular and $t \notin Q_z$. From (c), $V = \langle t\rangle(V \cap Q_z)$ and so (d) follows from [MS, 5.6]. $\square$

Lemma 3.2 *(a) For each $1 \leq i \leq 3$, M has a unique orbit on pairs (E_i, D_i) such that E_i is a singular 2^i, D_i is a purely non-singular fours group and D_i is perpendicular to E_i.*

(b) *M has a unique orbits on pairs (D_4, V) such that D_4 is a purely non-singular fours group, V is an ark and $D_4 \leq V$.*

(c) *No purely non-singular fours group is perpendicular to a singular 2^4 or 2^5.*

(d) *Representatives (E_i, D_i), $1 \leq i \leq 3$ and (D_4, V) for the above orbits can be chosen such that $E_1 \leq E_2 \leq E_3 \leq V$ and $D := D_1 = D_2 = D_3 = D_4 \leq V$. The stabilizers in $\bar{L} := N_M(D)/D$ are as follows:*

  (a) *$N_{\bar{L}}(E_1) \sim 2^{1+[20]}.U_6(2)$.*
  (b) *$N_{\bar{L}}(E_2) \sim 2^{2+[27]}.(S_3 \times L_3(4))$.*
  (c) *$N_{\bar{L}}(E_3) \sim 2^{3+[28]}.(L_3(2) \times Alt(5))$.*
  (d) *$N_{\bar{L}}(V) \sim 2^{8+[16]}.\Omega_8^{-}(2)$.*

Proof: Let E be a singular 2^i in M and put $R = N_M(E)$. We first determine the orbits of R on the purely non-singular fours groups F in Q_E and also $C_R(F)$.

By [MS, 4.10], a singular 2^4 is not perpendicular to a purely non-singular fours group. So (c) holds.

So suppose $i \leq 3$. Let $V = Q_E/E$, $\tilde{F} = FE/E$, $T = C_R(Q_E/E)$, $S = C_T(E)$ and $K = R/T$. By [MS, 4.2(3)], S acts transitively on the 2^{2i} complements to E in EF, and $T/S \cong SL_i(2)$. Note that $[N_T(F), F] \leq E \cap F = 1$. Thus $N_T(F) = C_T(F)$ and so $T/C_T(F)| = |S/C_S(F)| = 2^{2i} = |E|^2$. In particular, $T = C_T(F)S$. In particular $C_T(F)/C_S(F) \cong C_T(F)S/S = T/S \cong SL_i(2)$. We proved:

$$C_T(F)/C_S(F) \cong SL_i(2) \tag{1}$$

Since $|S/C_S(F)| = 2^{2i}$ and $|EF| = 2^{i+2}$

$$|C_S(F)| = \frac{S}{2^{3i+2}}. \tag{2}$$

From [MS] the order of $|S|$ is as follows:

i	1	2	3
$\lvert S\rvert$	2^{25}	2^{35}	2^{39}

(3)

From (2) and (3) we obtain the order of $C_S(F)/FE$:

i	1	2	3
$\lvert C_S(F)/FE\rvert$	2^{20}	2^{25}	2^{28}

(4)

Since $C_T(F)$ induces $\mathrm{Aut}(E)$ on E, $C_R(F) = C_T(F)C_R(EF)$. Since $C_T(F) \cap C_R(EF) = C_S(F)$ we conclude

$$C_R(F)/C_S(F) \cong C_T(F)/C_S(F) \times C_R(F)/C_T(F) \tag{5}$$

Also $C_R(F) \leq C_R(\tilde{F})$, where $\tilde{F} = FE/E$, and by the Frattiargument, $C_R(\tilde{F}) = TC_R(F)$ so

$$C_R(F)/C_T(F) \cong C_R(\tilde{F})/T = C_K(\tilde{F}). \tag{6}$$

In view of (1),(4),(5) and (6), the structure of $C_R(F)/F$ will be determined once we know $C_K(\tilde{F})$. Also the orbits of R on the possible F are in one to one correspondence with the orbits of K on the possible $\tilde{F}$.

Suppose that $i = 1$. Let Λ be the Leech lattice and $\overline{\Lambda} = \Lambda/2\Lambda$. Let $(x, y) = \frac{1}{8}\sum_{i=1}^{24} x_i y_i$ be the unimodular inner product on Λ. $\frac{1}{2}(x, x)$ is called the type of x. The type of $\overline{x} = x + 2\Lambda$ is the minimum type of a vector in $x + 2\Lambda$. By our definition of the Monster, V is as a $C_z/Q_z \cong Co_1$-module isomorphic to $\overline{\Lambda}$. By [MS, 4.4] the non-singular elements in Q_z corresponds to the vectors of type 2 in $\overline{\Lambda}$. By [ATLAS], Co_1 acts transitively on the fours groups in $\overline{\Lambda}$ all of whose non-trivial elements have type 2. Moreover, the centralizer of such a fours group is $U_6(2)$. So (a) and (d:a) hold for $i = 1$. holds in this case.

Suppose next that $i = 2$. By [MS, 4.5], $V = Q_E/E$ is the Todd-module for $K \cong M_{24}$ and if t is a non-singular involutions in Q_E, tE corresponds to a pair in $\Omega := \{1, 2, \dots, 24\}$. Hence $\tilde{F}^{\#}$ corresponds to three pairs in a subset of size three of Ω. Since M_{24} is five transitive on Ω we conclude that $C_K(\tilde{F}) \cong M_{21} \cong L_3(4)$ and so (a) and (d:b) holds for $i = 2$.

Suppose now that $i = 3$. Then by [MS, 4.8], $K \cong 3.Sym(6)$, V is irreducible of order 2^6 and the non-singular involutions in Q_E lie in the orbit of length for 18 for K on $V^{\#}$. Note that $Z(K')$ has order three and acts fixed-point freely on V. For $v \in V$ let $v^* = Z(K')v \cup \{1\}$. It follows that v^* is a fours group in V. Let $a, b \in \tilde{F}^{\#}$. Suppose that $a^* \neq b^*$ and observe that $(ab)^* \leq a^*b^*$ and $a^* \neq (ab)^* \neq b^*$. Let $I = \{(a^*)^k \mid k \in K\}$. Since $|a^K| = 18$, $|I| = 6$. Hence K induces $Sym(I)$ on I and $N_K(a^*) \cap N_K(b^*)$ acts transitively on $I \setminus \{a^*, b^*\}$ and so $d^* \leq a^*b^*$ for all $d^* \in I$. But then $\langle I \rangle \leq a^*b^*$, contradicting the irreducible action of K on V (and $|a^*b^*| = 2^4 < |V|$). Hence $a^* = b^*$ and $\tilde{F} = a^*$. Thus $N_K(\tilde{F}) = 3.Sym(5)$. Since $[Z(K'), K] \neq 1$, $N_K(\tilde{F})$ induced $Sym(3)$ on $\tilde{F} = a^*$ and so $C_K(\tilde{F}) = Alt(5)$. Thus (a) and (d:c) holds for $i = 3$.

Now let V be an ark with $D \leq V$. Then by 3.1, D is a non-degenerate 2-space in V. Thus by 3.1, D is unique up to conjugacy in $N_M(V)$, $N_M(V) \cap N_M(D)$ induces $Sym(3)$ on D and $N_M(V) \cap C_M(D) \sim 2^{10+16}\Omega_8^+(2)$. So (b) and (dd) hold. Finally that exists a chain $E_1 < E_2 < E_3$ of subgroups of order $2, 4$ and 8 in V which are (with respect to the quadratic form on V) singular and perpendicular to D. Thus by 3.1 E_i is (in M) singular and perpendicular to D. $\square$

Lemma 3.3 *Let z be a singular involution in M. Then C_z is transitive on the purely non-singular fours group in Q_z. Moreover, there does not exist any purely non-singular subgroups of order larger than four in Q_z.*

Proof: The first statement follows from 3.2(a). For the second we use the use the same notations for $Q_z/ < z > \cong \overline{\Lambda}$ as in the the previous lemma.

Suppose that exist vectors a, b, c of type 2 in Λ such that $\langle \overline{a}, \overline{b}, \overline{c} \rangle$ has order eight and only contains vectors of type 2. Since $\overline{a+b}$ has type two it is easy to see that $(a, b) = \pm 2$. Moreover, if $(a, b) = 2$, then $a - b$ has type 2, and if $(a, b) = -2$ then $a + b$ has type 2. Replacing b and c by their negatives, if necessary, we may assume $(a, b) = (a, c) = 2$. Thus $(a, b + c) = 4$ and $(a, b - c) = 0$. Since either $b + c$ or $b - c$ has type 2, we get that $\overline{a + b + c} = \overline{a + b - c}$ is not of type 2, a contradiction. $\square$

Lemma 3.4 *There exists a unique class of purely non-singular fours groups in M.*

Proof: Let $F = \{1, a, b, c\}$ be a purely non-singular fours group in M. By 3.3 it suffices to show that $F \leq Q_z$ for some singular involution. For this let z be a singular involution with $a \in Q_z$. Then $C_a \cap C_z$ contains a Sylow 2-subgroup of C_a and so we may choose z such that $F \leq C_a \cap C_z$. If $F \leq Q_z$ we are done. So we may assume that $F \not\leq Q_z$. Then $b, z \in C_z \setminus Q_z$ and by 3.1 bz and cz are singular and there exists a unique ark V containing b and z. Since $a \in Q_z$, az is conjugate to a and so az is non-singular. Also $(az)(bz) = c$ is non-singular and thus 3.1a applied with to bz in place of z shows that $az \in Q_{bz}$. Thus $az \in Q_z \cap Q_{bz} \leq V$ and so $F \leq V$. By 3.1c there exists a singular involution in V perpendicular to F. $\square$

Lemma 3.5 *Let D be purely non-singular fours group. Then there exist eights groups D_1, D_2 and D_3 containing F such that*

(a) D_1 contains a unique singular involution z and $D_1 \leq Q_z$.

(b) $D_2 \setminus D$ contains a unique non-singular involution and D_2 lies in an unique ark.

(c) All elements of $D_3 \setminus D$ are singular, D_3 is not contained in a ark and the singular involutions in M perpendicular to D_3 generate a non-trivial singular subgroup of M.

Proof: By 3.1c groups D_1 and D_2 as in (a) and (b) can be found in any ark. It remains find a group D_3 as in (c). Let E be a singular fours group with $D \leq Q_E$. By [MS, 4.5] Q_E/E is the Todd-module for $C_E/O_2(C_E) \cong M_{24}$. Moreover, the pairs represent the non-singular involutions, while the sextetts represent singular involutions. Let $\mathcal{S} = (T_i \mid 1 \leq i \leq 6)$ be a sextett and $k_i \in T_i$. Let a and b be non-singular involutions in Q_E such that Ea and

Eb correspond to $\{k_1, k_2\}$ and $\{k_1, k_3\}$ respectively. Then Eab correspond to $\{k_2, k_3\}$ and so $D := \langle a, b\rangle$ is a purely non-singular fours group. Let Ez correspond to $\mathcal{S}$ and put $D_3 = \langle a, b, z\rangle$. Then az corresponds to the sextett determined by the $T_1 \cup \{k_2\} \setminus \{k_1\}$ and so az is singular. Similarly bz and abz are singular. In particular, no element in $D^{\#}$ is perpendicular to z and so D_3 cannot be contained contained in an arc. Also E is perpendicular to D_3. So it remains to show that if d and e are singular involutions perpendicular to D_3, then d is perpendicular to e. Note that d and e commute. [MS, 7.7] lists the orbits of M on pairs of commuting singular involutions and their common perp. We apply this to (c, d). Since $D_3 \leq Q_c \cap Q_d$, $Q_c \cap Q_d$ contains a purely non-singular fours group (namely D). This rules out Cases (3) and (4) of the list in [MS, 7.7]. In Case (1) we would conclude that D_3 lies in the ark $(Q_c \cap Q_d)\langle c, d\rangle$. So Case (2) holds, d and e are perpendicular and (c) is established. □

Lemma 3.6 *Let D be a purely non-singular fours group and put $L = C_M(D)$ and $\bar{L} = L/D$.*

- *(a) $\bar{L} \cong {}^2E_6(2)$ and $N_M(A)/A \cong Aut({}^2E_6(2)) \sim {}^2E_6(2).Sym(3)$*
- *(b) M has exactly three classes of eights groups containing a pure non-singular fours group. Representatives are as given in 3.5. In particular, any purely non-singular subgroup has order at most four.*
- *(c) If $x \in L$ with $x^2 \in D$ then $x^2 = 1$.*

Proof: (a) Let E_1, E_2, E_3, D and V be as in 3.2(d). Define $X = \langle N_L(E_i), N_L(V) \mid 1 \leq i \leq 3\rangle$. It is straightforward from 3.2(d) to show that $\overline{X}$ acts faithfully and flag transitively on a geometry $\mathcal{B}$ with F_4-diagram. By [Ti2, Theorem 1] $\mathcal{B}$ is covered by a building. By the classifications of spherical buildings [Ti1] $\mathcal{B}$ is the building associate to ${}^2E_6(2)$. Thus $X \leq \mathrm{Aut}(\mathcal{B} \cong \mathrm{Aut}({}^2E_6(2)) \sim {}^2E_6(2).Sym(3)$. It is now easy to see that $N_{\bar{L}}(E_1) \leq {}^2E_6(2)$ and is a maximal parabolic subgroup of ${}^2E_6(2)$. Hence $\bar{X} \cong {}^2E_6(2)$.

We may choose the D_i such that E_1 contains the unique singular involution in D_1, V is the unique ark containing D_2 and for some i, E_i is the subgroup generated by the singular involutions perpendicular to D_3. It follows that $C_{\bar{L}}(D_1/D) \leq N_{\bar{L}}(E_1) \leq \bar{X}$, $C_{\bar{L}}(D_2/D) \leq N_{\bar{L}}(V) \leq \bar{X}$ and $C_{\bar{L}}(D_3/D) \leq N_{\bar{L}}(E_i) \leq \bar{X}$..

By [AS] $\bar{X}$ has three classes of involutions. Thus $D_1/D, D_2/D$ and D_3/D are representatives for the classes of subgroups of order two in $\bar{X}$. Since for

$i \neq j$, D_i is not conjugate to D_j in M, two involutions in $\bar{X}$ are conjugated in $\bar{X}$ if and only if they are conjugate in $\overline{L}$. Hence all assumptions of 2.3 are fulfilled and so $\overline{X} = \overline{L}$.

In $N_M(V)$ we see that $N_M(A)/C_M(A) \cong Sym(3)$ and $C_M(\overline{L}) = 1$. Thus $N_M(A)/A \cong \mathrm{Aut}({}^2E_6(2))$. So (a) holds.

(b) As $L = X$ this was proved in (a).

(c) By the prove of (a), $D\langle x\rangle/D$ is conjugate in $\bar{L}$ to some D_i/D. Since D_i is elementary abelian, (c) holds. □

4 Proof of Theorem A

In this section we prove Theorem A. So let P be a maximal p-local subgroup of M which is not of characteristic p. Let $A = \Omega_1 Z(O_p(P))$. Then $N_M(A) = P$ and as P is not of characteristic p, A is not of p-type. Thus by 2.2 none of the involutions are of 2-type and so all the involutions in A are non-singular. By 3.6 $|A| \leq 4$. If $|A| = 2$ then by [MS, 7.6], A is conjugate to T and so Case (1) of Theorem A holds. If $|A| = 4$ then by 3.4 A is unique up to conjugacy and by 3.6(a), Case (2) of Theorem A holds.

5 Involutions in the Baby Monster

In this section we determine the conjugacy classes of involution in the Baby Monster. The conjugacy classes of involution in the Baby Monster which lift to involution in the monster already have been determined. So we need to investigate the set $\mathcal{F}$ of elements of order four in M which square to a non-singular involution. We start with a technical lemma used later to show that elements in $\mathcal{F}$ normalize a purely non-singular fours group.

Lemma 5.1 *Let H be a finite group, Q a normal subgroup of H and $f \in H$. Suppose that Q is an extra-special 2-group, $f^2 \in Q \setminus Z(Q)$ and that there exists an involution in Qf. Then one of the following holds:*

- *(a) There exists an involution $q \in Q$ with $[q, f] = f^2$.*
- *(b) $N_H(Qf)$ normalizes $f^2 Z(Q)$.*

Proof: Recall the commutator formulas:

$$[a, bc] = [a, c][a, b]^c \tag{1}$$

$$[ab, c] = [a, c]^b[b, c] \tag{2}$$

Let i be an involution in Qf. Set $s = fi$ and $z = s^2$. Then $s \in Q$, $f = si$ and since Q is extra special, $z \in Z(Q)$. By (1), $[s, f] = [s, si] = [s, i]$ and so $f^2 = sisi = s^2[s, i] = z[s, f]$. We record

$$f^2 = z[s, f] \tag{3}$$

If $z = 1$, (a) holds with $q = s$. So suppose that $z \neq 1$. Then $Z(Q) = \{1, z\}$.

Assume first that there exist an involutions $r \in [Q, i]$ with $[r, s] \neq 1$. Since $[Q, i] = \{[u, i], z[u, i] \mid u \in Q\}$ we may assume that $r = [u, i] = i^u i$ for some $u \in Q$. Then$\langle i^u, i\rangle$ is a Dihedral group of order four and so

$$[r, i] = 1 \tag{4}$$

Since Q is extraspecial and $r, s \in Q$, $[r, s] = z$. Hence r inverts s and $q := rs$ is an involution. Moreover, by (2)

$$[q, s] = [rs, s] = [r, s]^s[s, s] = z^s = z \tag{5}$$

by (2) and (4)

$$[q, i] = [rs, i] = [r, i]^s[s, i] = [s, i] \tag{6}$$

and so by (1),(5),(6) and (3)

$$[q, f] = [q, si] = [q, i][q, s]^i = [s, i]z = f^2. \tag{7}$$

Thus (a) holds in this case.

Assume finally that s centralizes $E := \Omega_1([Q, i])$. Note that $[Q, i] = [Q, Qf]$ and so $N_H(Qf)$ normalizes E. Moreover, $[Q, i]$ is abelian and so E has index at most two in $[Q, i]$. Let $D := C_Q([Q, i])$ and $R := C_Q(E)$. Then $s \in R$, $|R/D| = |[Q, i]/E| \leq 2$ and $D/Z(Q) = C_{Q/Z(Q)}(i)$. If $s \in D$ we get $[s, i] \in Z(Q)$ and $f^2 = z[s, i] \in Z(Q)$, a contradiction to the assumptions of the lemma. Thus $s \notin D$, E has index two in $[Q, i]$ and $\langle s\rangle D = R$. Thus

$$\langle f^2\rangle Z(Q) = \langle [s, i]\rangle Z(Q) = [R, i]Z(Q) = [R, Qi]Z(Q) = [R, Qf]Z(Q).$$

As $N_H(Qf)$ normalizes Q, E, R and $Z(Q)$ we conclude that (b) holds. □

Lemma 5.2 *Let $f \in \mathcal{F}$. Then there exists a dihedral group H of order eight in M with $f \in H$ and such that H contains exactly two singular involutions.*

Proof: Let z be a singular involution with $f^2 \in Q_z$. Then z is 2-central in C_{f^2} and so we may assume $f \in C_z$. Since f^2 is non-singular, $\widetilde{f^2}$ is of 2-type in $\widetilde{Q_z} = Q_z/\langle Z \rangle \cong \Lambda/2\Lambda$. By [MS, 2.5] involutions of type $2B$ in Co_1 do not centralizes elements of type 2 in $\Lambda/2\Lambda$. Hence $Q_z f$ is of $2A$ or $2C$. By [MS, 7.1] in both case there exists an involution $i \in Q_z f$. Moreover by [MS, 2.4,2.5] $N_{C_z}(Q_z f)$ normalizes no elements type 2 in the Leech lattice $\widetilde{Q_z}$. Thus by 5.1 there exists an involution $q \in Q_z$ with $[q, f] = f^2$.

Suppose that all such q's are non-singular. Let $X = Q_z \cap C_f \cap C_q$. $\overline{X}$ has index at most 4 in $C_{\overline{Q}_x}(f)$ and $C_{\overline{Q}_x}(f)$ has order at least 2^{12}. Thus $|\overline{X}| \geq 2^{10}$. Hence X contains elementary abelian subgroup of order 2^5 and so also 2^4 E with $z \notin E$. Since M contains no pure non-singular 2^4, there exists a singular involution s in E. Then $s \neq z$, $s \in Q_z$ and s centralizes q and f. In particular, sq is an involution in Q_z with $[sq, f] = f^2$ and so by assumption sq is non-singular. Then as q and sq are non-singular, $q \in Q_s$. If sqf is non-singular, then $qf \in Q_s$ and so also $f = q^{-1}qf \in Q_s$ and $f^2 \in \langle s \rangle$, a contradiction since s is singular and f^2 is not. Thus sqf is singular and we can choose $H = \langle sq, sqf \rangle$.

So we may assume now that q is singular. If qf is non-singular we can choose $H = \langle q, qf \rangle$. So we may assume that qf is singular.

Put $X = \{1, f^2, q, q^f\}$. Then X is a fours group normalized by f and $[X, f] = \langle f^2 \rangle$. Put $Y = Q_X$. Since q and q^f are singular and f^2 is not, 3.1 implies that X lies in a unique ark V, and that Y is a non-degenerate subspace of order 2^8 in V. Namely Y is the orthogonal complement to X in V. Suppose that $C_Y(f)$ is singular. Since $C_Y(f)$ has order at least 2^4 we get $|C_Y(f)| = 2^4$ and $|C_V(f)| = 2^5$. But this contradicts the fact that any involution in $\Omega(V)$ centralizes an even dimensional subspace of V.

Hence $C_Y(f)$ is not singular. Let b be any non-singular involution in $C_Y(f)$. Since $b \in Y \leq Q_q$ also bq is non-singular. If bqf is singular, we can choose $H = \langle bq, bqf \rangle$. So we may assume that bqf is non-singular for all such b. Since b and bqf are non-singular we conclude that $b \in Q_{qf}$. Let E be the group of generated by the non-singular involution in $C_Y(f)$. Then $E \leq Q_{qf}$. If E contains a singular involution s we get $\langle q, qf \rangle \in Q_s$. But then $f^2 = [q, f] \in \langle s \rangle$, a contradiction, as f^2 is non-singular. We conclude that E is purely non-singular and so $|E| \leq 4$. But then also $|C_Y(f)| \leq 4$, a contradiction. $\square$

Lemma 5.3 *Let V be an ark and $t, z \in V^\#$ with z singular, t non-singular and $t \in Q_z$. Then $\tilde{V}$ is in class $2A$ of $\widetilde{C_z \cap C_t} \cong Co_2$.*

Proof: V is the natural module for $N_M(V)/C_M(V) \cong \Omega_{10}(2)$ we get $C_z \cap C_t \cap N_M(V)$ has factor group isomorphic to $Sp_6(2)$. Thus the lemma follows from Lagrange's Theorem and [ATLAS]. □

Lemma 5.4 *There exist subgroups D and D° of M such that with $X = N_M(D)$:*

(a) $D \cong D_8$ and all involutions in D are non-singular.

(b) X interchanges the two fours groups in D.

(c) Let f be the element of order four in D and J a Sylow 2-subgroup of $N_G(\langle f\rangle)$ containing D. Then D is the unique conjugate of D under $N_G(\langle f\rangle)$ contained in J.

(d) Let e be an involution in $C_M(D) \setminus D$. Then e or ef^2 is singular.

(e) $X/D \cong Aut(F_4(2))$.

(f) $f \in D^\circ$, $D^\circ \cong D_8$ and D° contains exactly two singular involutions.

(g) D and D° is a complete set of representatives for the conjugacy classes of subgroups isomorphic to D_8 in M which contain a purely non-singular fours group. In particular, any two D_8's in M containing only non-singular involutions are conjugate.

(h) X/D has a unique class of subgroup of order two not contained in $(X/D)'$. Moreover if S is the inverse image in X of such a group, then $X \cong SD_{16}$ and $N_X(S)/S \cong {}^2F_4(2)$.

Proof: Let $t = f^2$.

(a) Let A be a purely non-singular fours group in G. Then by 3.6 $N_M(A)/A \cong \mathrm{Aut}({}^2E_6(2))$ and so by [AS] there exists a unique class of subgroup $D \leq N_M(A)$ with $A \leq D$, $|D/A| = 2$, $(N_M(A) \cap N_M(D))/D \cong F_4(2)$ and $D \not\leq C_G(A)$. Then $D \cong D_8$. Let t be an involution in D. As 17 divides the order of $F_4(2)$ and so also of $C_M(t)$, t is non-singular. Thus (a) holds.

(b) By (a) the fours group B in D distinct from A is also pure non-singular. Hence $B = A^s$ for some $s \in M$. But then D and D^s are conjugate in $N_M(B)$ and we may and do choose s such that $D^s = D$. Let $Y = N_M(A) \cap N_M(D)$. Then $X = Y\langle s\rangle$.

(c) Suppose that is D is the unique conjugate of D under $N_G(\langle f\rangle)$ normalizing D. Then $N_{N_J(D)}(N_J(D)) \leq N_J(D)$, D is normal in J and (c) holds in this case.

So to prove (c) we may assume (for a contradiction) that there exists a conjugate D^* of D under $N_G(\langle f\rangle)$ normalizing D. Put $U = \langle D, D^*\rangle$. As $f \in D \cap D^*$, $|U| = 16$.

Assume that D^* induces only inner automorphism on D. Then $U = DC_U(D)$, $|C_U(D)| = 4$, $Z(U) = C_D(U)$ and either $Z(U)$ is cyclic or $Z(U)$ is a fours group. Note that $D \cup D^*$ contains eight involutions from $U \setminus Z(U)$. If $Z(U) \cong C_4$, every non -trivial coset of $Z(U)$ contains exactly two involutions and $U \setminus Z(U)$ contains six involutions, a contradiction. Thus $Z(U)$ is a fours group and there are exactly eight involution in $U \setminus Z(U)$. We conclude that all the involutions of $U \setminus Z(U)$ are in $D \cup D^*$ and so are non-singular. Thus U contains at most two singular involutions. Let E be an eights group in U. Then $D \cap E$ is a purely non-singular fours group in E and E contains at most two singular involutions. By 3.6b and 3.5, E contains a unique singular involution z. But then z is the unique singular involution in U, all involutions in U are in Q_z, $U \leq Q_z$ and $f^2 = z$ a contradiction, a contradiction as $f^2 = t$ is non-singular.

Hence D^* induces an outer automorphism on D and so interchanges the two fours groups in D. Let a and a^* be non central involutions in D and D^* respectively. Put $b = a^{a^*}$. Then $D = \langle a, b\rangle$ and $U = \langle a, a^*\rangle \cong D_{16}$. Put $t = f^2$ and let z be a singular involution perpendicular with $t \in Q_z$. As z is 2-central in C_t we may assume that $U \leq C_z$. Since $f^2 \notin \langle z\rangle$, $f \notin Q_z$. Let N be a normal subgroup of U with $N \not\leq Z(U) = \langle t\rangle$. Then $|N| \geq 4$, $|U/N| \leq 4$ and so $\langle f\rangle = U' \leq N$. Thus $U \cap Q_z = \langle t\rangle$. Let $\widetilde{C_z} = C_z/Q_z \cong Co_1$. As U centralizes t, $\widetilde{U}$ is contained in $\widetilde{C_z \cap C_t} \cong Co_2$. In particular, $\widetilde{f}$ is not of $G_2(4)$-type.

Since $U \cap Q_z = \langle t\rangle$ we have $a \notin Q_z$ and so az is singular. By 3.1 there exists a unique ark V_a containing z and a. Consider the eights group $E = \langle t, a, z\rangle$ and the purely non-singular subgroup $B = \langle a, t\rangle$. Also tz is non-singular and so Case (b) of 3.5 must hold for E. So E lies in an ark. Since V_a is the unique arl containing z and a, we get $t \in E \leq V_a$. Thus 5.3 implies that $\widetilde{a} \in \widetilde{V_a}$ is a $2A$ involution in Co_2. By symmetry all involution in $\widetilde{U}$ except maybe $\widetilde{f}$ are in $2A$.

Suppose first that $\widetilde{f}$ is in one of the classes $2A$ or $2B$ of Co_2. Then $\widetilde{f}$ is 2-central in Co_1 and there exists an ark V_f containing z and such that $\widetilde{f} \in \widetilde{V_f}$. Since Q_zV_f/V_f is elementary abelian, $t = f^2 \in V_f$. So 5.3 implies that $\widetilde{f} \in \widetilde{V_f}$ is $2A$ in Co_2. Hence $\{1, \widetilde{f}, \widetilde{a}, \widetilde{b}\}$ is a pure $2A$-subgroup of Co_2. But no such fours groups exists. (For example by [ATLAS] the $2A$ involutions have trace -9 on the complex 23-space and $23 - 9 - 9 - 9$ is negative, contradicting the orthogonally relations)

Thus $\widetilde{f}$ is in the class $2C$. Let x be an element of order four in $\widetilde{U}$. Let m be the trace of x on the complex 23-space. By [ATLAS] the $2A$ elements have trace -9 and the $2C$-elements have trace -1. Thus the sum of the traces of the elements in $\widetilde{U}$ is $23 + 4 \cdot (-9) + (-1) + 2m = 2m - 14$. Hence $m \geq 7$. From [ATLAS] we conclude $m = 7$, x is in the class $4A$ and x^2 is in the class $2A$. A contradiction, since $x^2 = \widetilde{f}$.

(d) Suppose that both e and et are non-singular. Then by 3.6b and 3.5 applied to $A\langle e\rangle$, there exist a unique singular involution z in $A\langle e\rangle$. As D normalizes $A\langle e\rangle$ we conclude that D centralizes z. But then D centralizes $\langle t, e, z\rangle = A\langle e\rangle$, a contradiction.

(e) Let $L = \langle t\rangle X^\infty$. Then $L \sim 2.F_4(2)$. So by [AS] there exists $e \in L$ with $e^2 \in \langle t\rangle$ such $C_{L/\langle t\rangle}(e) \sim [2^{15}]Sp_6(2)$. By [ATLAS] e has order two and e is not conjugate to et in L. Thus $C_L(e) \sim [2^{16}].Sp_6(2)$. By (d) (and replacing e be et if necessary) we may assume that e is singular. Note that $f \notin Q_e$. Let $s = f$ if $f^2 \in Q_e$ and $s = f^2$ if $f^2 \notin Q_e$. Then $\tilde{s}$ is an involution in $\widetilde{C}_e = C_e/Q_e$. Also $[DC_L(e), s] \leq \langle s^2\rangle \leq Q_e$. Since $\widetilde{C_L(e)}$ has factor group $Sp_6(2)$ we conclude form [ATLAS] that $C_{\widetilde{C_e}}(\tilde{s}) \sim 2^{1+8}\Omega_8^+(2)$. Since no fours group in $2^{1+8}\Omega_8^+(2)$ has a centralizer involving $Sp_6(2)$ we conclude that $\tilde{D}$ is neither isomorphic to D_8 nor a fours group. Since $\widetilde{f} \neq 1$ we conclude that $D \cap Q_e \in \{A, B\}$. If X induces only inner automorphisms on $L/\langle t\rangle$. Since $L/\langle t\rangle$ is perfect, we conclude that $X = C_X(L)L$ and so there exists $x \in C_X(L) \setminus DL$. Then by the proven part of (b), $A^x = B$, a contradiction as x normalizes $D \cap Q_z$. Thus (e) holds.

(f) Moreover, $a \in A \setminus Z(D)$ and $b \in B$ with $ab = f$. As $a \in Q_e$, ae is non-singular and as $b \notin Q_e$, be is singular. Put $D^\circ = \langle ae, be\rangle$. Then (f) holds.

(g) By [AS] ${}^2E_6(2).2$ has exactly two classes of involutions outside ${}^2E_6(2)$. Thus (g) follows from (a) and (f).

(h) The uniqueness part of (h) and the structure of $N_X(S)/S$ follows from [AS]. Suppose that there exists an involution d in $S \setminus D$ and let a be an involution D with $a \neq t$. Then $D = \langle a, a^d\rangle$ and so $S = \langle a, d\rangle \cong D_{16}$ and $D^* := \langle d, d^a\rangle \cong D_8$. Moreover, $C_X(d)$ involves ${}^2F_4(2)'$ and so $C_M(d)$ is divisible by 13. Thus d is non-singular. By (e) D and D^* are conjugate in M and as $\langle f\rangle$ is the unique cyclic subgroup of order four in D (in D^*), D and D^* are conjugate in $N_G(\langle f\rangle)$. But this contradicts (c). Hence $S \setminus D$ contains no involutions, $S \cong SD_{16}$ and all part of 5.4 are proved. □

Lemma 5.5 *Let $\mathcal{F}$ be the set of elements of order four squaring to a non-singular involution, $f \in \mathcal{F}$ and $F = N_M(\langle f\rangle)$. Then* *(a) M acts transitively on $\mathcal{F}$.*

(b) $F \sim D_8.Aut(F_4(2))$.

(c) $F/O_2(F)$ has a unique class of subgroup of order two not contained in $(F/O_2(F))'$. Moreover if S is the inverse image in F of such a group, then $S \cong SD_{16}$ and $N_F(S)/S \cong {}^2F_4(2)$.

Proof: (a) Let $f \in \mathcal{F}$ and choose H as in 5.2. By 5.4 H is conjugate to D°. Since the two elements of order four in D° are conjugate in D°, (a) is proved.

To prove (b) and (c) let D, X and f be as in 5.4. Put $L = N_M(\langle f \rangle)$. Then (b) and (c) follow from 5.4a,g once we establish that $L = X$. By 5.4c and the Z^* theorem ([Gl]) applied to $L/\langle f \rangle$ we have $L = O(L)X$, where $O(L)$ is the largest normal subgroup of order odd order in L. Let z be an involution in $C_M(D)$ with $z \neq f^2$. By 5.4d z or zf^2 is singular and so by 2.2 $\langle f, z \rangle$ is of 2-type. Thus $O(C_M(\langle f, z \rangle)) = 1$ and z inverts $O(L)$. Since this is true for any involution in $C_M(D)$ distinct from f^2 and as $C_M(D)$ contains elementary abelian groups of order eight we conclude that $O(L) = 1$ and $L = X$. □

6 Proof of Theorem B

In this section we prove Theorem B. So let $\overline{P}$ be a maximal 2-local of $BM = \overline{B}$ such that $\overline{P}$ is not of characteristic 2. Let P its preimage in B, $R = O_2(P)$ and $T = \langle t \rangle$. Then $P = N_B(R)$. We claim that R is not of 2-type in M.

Let U be the preimage of $C_{\overline{B}}(\overline{Q})$ in B. Since $\overline{P}$ is not of characteristic p, $U \not\leq R$ and U is not a 2-group. Since $[R, U] \leq T$ and $[T, U] = 1$ we get $[R, O^2(U)] = 1$. Hence $C_M(R)$ is not a 2-group. Since $t \in R$,$C_M(R) = C_B(R) = C_P(R)$ and so $O_2(C_M(R)) \leq R$. Thus by 2.1(b) R is not a of 2-type, proving the claim.

So 2.2 none of the involutions in R are of 2-type. That is all the involutions in R are non-singular.

Let A be a normal subgroup of P minimal with respect to $A \not\leq T$. Then $P = N_B(A)$. Note that AT/T

is elementary abelian and $AT \leq R$. So if $x \in TA^\#$ then x is a non-singular involution or x has order 4.

Suppose that A contains an involution a with $a \neq t$. Let $D = \langle a, t \rangle$. Then D is a purely non-singular fours group. Note that $b^2 \in T \leq A$ for all $b \in C_{AT}(D)$. Thus by 3.6c, $C_{AT}(D)$ is elementary abelian and by 3.6b, $|C_{AT}(D))| \leq 4$. Thus $C_{AT}(D) = D$. Since D is normal in AT and $|D| = 4$ we get $|AT/D| \leq 2$. Hence AT is a fours group or $AT \cong D_8$. If AT is a fours group, then by 3.4 AT is unique up to conjugacy in M. By 3.5 we conclude that $N_M(AT)$ induces $Sym(3)$ on AT. Thus $N_B(AT)$ induces C_2 on AT, $A = AT$ and A is unique up to conjugacy in B. So by 3.5, Case (1) of Theorem B holds. If $AT \cong D_8$, then $A = AT$ and P normalizes the unique cyclic group of order four in A, a contradiction to the minimal choice of A.

Suppose next that t is the only involution in A. Then $A \cong C_4$ or Q_8. If $A \cong C_4$, then by 5.5, Case (2) of Theorem B holds.

So suppose $A \cong Q_8$. Let R be one of the three cyclic subgroups of order four in A. Then $T \leq R$. Put $D = O_2(N_M(R))$. Then by 5.5, $D \cong D_8$. Note that $A \leq N_M(R)$ and so A normalizes D. Let F be a fours group in D and suppose that A normalizes F. Then $|A/C_A(F)| \leq 2$. So there exists $H \leq C_A(F)$ with $H \cong C_4$. Then $\Phi(H) = T \leq A$ and $|HA| = 8$, a contradiction two 3.6c.

Thus A does not normalize F and so $A \not\leq DC_M(D)$. Put $S = DA$. Then S/D has order two and by 5.5 S is unique up to conjugation in $N_M(D)$ and $N_M(S)/S \cong {}^2F_4(2)$. Note that A (D) is the unique subgroup of S isomorphic to Q_8 (D_8). . So A,D and $R = A \cap D$ are all normal subgroups of $N_M(S)$. Since $S = DA$ and $N_M(D) = N_M(R)$ we get $N_M(S) = N_M(D) \cap N_M(A) = N_M(R) \cap N_M(A)$. Moreover, $N_M(S) = D(N_M(S) \cap C_M(A)) = DC_M(A)$. Since $D \cap C_M(A) = T$ we conclude $C_M(A)/T \cong {}^2F_4(2)$ and $C_M(A) \leq N_G(S)$.

Let L be the subgroup of M generated by the various S as R runs through the three subgroups of order four in A. Since $[S, C_M(A)] \leq C_S(A) = T$ we get $[L, C_M(A)] \leq T$ and so $C_L(A)/T \leq Z(C_M(A)/A) = 1$. Thus $C_L(A) = T$. Since $[S, A] = R$, L induces $Sym(3)$ on A/T and so $L/T = L/C_L(A) \cong Aut(A) \cong Sym(4)$. Thus $N_M(A) = LC_M(A)$ and $N_M(A)/T \cong Sym(4) \times {}^2F_4(2)$. Thus Case (3) of Theorem B holds.

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