Lie Groups and Algebras I,II
Lecture Notes for MTH 915
03/04

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Contents

Chapter 1

Basic Properties of Lie Algebras

1.1 Definition

Let $\mathbb{K}$ be a field. With a $\mathbb{K}$-space we mean a vector space over $\mathbb{K}$. For $\mathbb{K}$-space V, $\mathrm{End}(V)$ denotes the ring of $\mathbb{K}$-linear maps from V to V. For $a, b \in \mathrm{End}(V)$ define $[a, b] := ab - ba$. $[a, b]$ is called the commutator or bracket of a and b. The bracket operation has an amazing property

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all $a, b, c \in \mathrm{End}(V)$. Indeed,

$$\begin{aligned}
&[a, [b, c]] + [b, [c, a]] + [c, [a, b]] \\
&= a(bc - cb) - (bc - cb)a + b(ca - ac) - (ca - ac)b + c(ab - ba) - (ab - ba)c \\
&= abc - acb - bca + cba + bca - bac - cab + acb + cab - cba - abc + bac \\
&= 0
\end{aligned}$$

Also note that $[,]$ is $\mathbb{K}$-bilinear and that $[a, a] = 0$. These observations motivate the following definitions:

Definition 1.1.1 [**def:algebra**] *Let $\mathbb{K}$ be a field, A a (left) vector space over $\mathbb{K}$ and $\cdot : A \times A \to A$ a K-bilinear map. Then $(A, \cdot)$ is called a* $\mathbb{K}$-algebra. *If $\cdot$ is associative, then A is called an* associative *algebra.*

Definition 1.1.2 [**def: lie algebra**] *A $\mathbb{K}$-algebra $(A, [,])$ is called a* Lie algebra *over $\mathbb{K}$ provided that*

(i) [**a**] *$[,]$ is symplectic, that is $[a, a] = 0$ for all $a \in A$.*

(ii) [**b**] *$[,]$ fullfills the Jacobi identity*

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for all $a, b, c \in A$.

From now on $\mathbb{K}$ is always a field and L a Lie algebra over $\mathbb{K}$.

The prime example for a Lie algebra is $(\mathrm{End}(V), [,])$. We denote this Lie algebra by $\mathfrak{gl}(V)$. L is called abelian if $[a,b] = 0$ for all $a, b \in L$. Any $\mathbb{K}$-space V becomes an abelian Lie algebra if one defines $[a,b] = 0$ for all $a, b \in V$.

Let $(A, \cdot)$ be any associative $\mathbb{K}$-algebra and define $[a,b] := ab - ba$ for all $a, b \in A$. Just as for $\mathrm{End}(V)$ none shows that $(A, [,])$ is a Lie algebra over $\mathbb{K}$. We denote this Lie algebra by $\mathfrak{l}(A)$.

Similar as for groups, rings and modules one defines homomorphisms, subalgebras, generations, ideals, ... For example a subalgebra of an algebra A is a $\mathbb{K}$-subspace I of A such that $i \cdot j \in I$ for all $i, j \in I$. Note that this equivalent to requiring that $(I, \cdot)$ is $\mathbb{K}$-algebra. If I is a $\mathbb{K}$-subspace of A with $i \cdot a \in I$ and $a \cdot i \in I$ for all $a \in A, i \in I$ then I is called an ideal. In this case the quotient A/I is a $\mathbb{K}$-algebra. The kernel $\ker \phi$ of an homomorphism $\phi : A \to B$ of $\mathbb{K}$-algebras is an ideal in A and the First Isomorphism Theorem holds: $A/\ker \phi \cong \phi(A)$ as $\mathbb{K}$-algebras.

In general one needs to distinguish between left and right ideals. This is not necessary for Lie algebras:

Lemma 1.1.3 [**alternating**]

(a) [**a**] $[,]$ *is alternating, that is* $[x,y] = -[y,x]$ *for all* $x, y \in L$.

(b) [**b**] *Let* I *be a* $\mathbb{K}$*-subspace of* L*. Then* I *is an ideal (in* L*) iff* I *is a right ideal and iff* I *is a left ideal.*

Proof: (a) $0 = [x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] = [x,y] + [y,x]$.

(b) follows immediately from (a). □

We remark that if $\mathrm{char}\, \mathbb{K} \neq 2$, then $xy = -yx$ for all x, y in an algebra A implies $xx = 0$. Indeed $xx = -xx$ and so $2xx = 0$. As 2 is invertible we get $xx = 0$.

Let V be a $\mathbb{K}$-space and $\mathcal{W}$ a set of subspaces of V with $0 \in \mathcal{W}$ and $V \in \mathcal{W}$. Put

$$\mathrm{End}(\mathcal{W}) = \{\phi \in \mathrm{End}(V) \mid \phi(W) \leq W \ \forall W \in \mathcal{W}\}.$$

Note that $\mathrm{End}(\mathcal{W})$ is a subalgebra of $\mathrm{End}(V)$. We denote the corresponding Lie algebra by $\mathfrak{gl}(\mathcal{W})$. Suppose that V has a finite basis $(v_1, v_2, \ldots, v_n)$ and $\mathcal{W}$ consist of the $n+1$ subspace $\mathbb{K}v_1 + \mathbb{K}v_2 + ... + \mathbb{K}v_i$, $0 \leq i \leq n$. The reader should verify that $\mathfrak{gl}(\mathcal{W})$ now consist of all the upper triangular matrices (with respect to the given basis).

Let f be a bilinear form on V, that is a $\mathbb{K}$-bilinear function $f : V \times V \to K$. Define

$$\mathfrak{cl}(f) = \{\alpha \in \mathfrak{gl}(V) \mid f(\alpha v, w) + f(v, \alpha w) = 0 \ \forall v, w \in V\}$$

We claim that $\mathfrak{cl}(f)$ is a Lie subalgebra of $\mathfrak{gl}_(V)$. Clearly it's a $\mathbb{K}$-subspace. Let $\alpha, \beta \in \mathfrak{cl}(f)$ and $v, w \in V$. Then

$$\begin{aligned} f([\alpha,\beta]v,w) &= \quad f(\alpha\beta v,w) - f(\beta\alpha,w) \\ &= -f(\beta v,\alpha w) + f(\alpha v,\beta w) \\ &= \quad f(v,\beta\alpha w) - f(v,\alpha\beta w) \\ &= -f(v,[\alpha,\beta]w) \end{aligned}$$

So $[\alpha,\beta] \in \mathfrak{cl}(f)$ and $\mathfrak{cl}(f)$ is a Lie subalgebra of $\mathfrak{gl}(V)$.

1.2 Structure constants

Let L be a Lie algebra over $\mathbb{K}$ and $\mathcal{B}$ a basis for L. So every $l \in L$ can be uniquely written as $l = \sum_{b\in\mathcal{B}} k_b b$, where $k_b \in \mathbb{K}$ and all but finitely many of the $k_b's$ are zero. Hence we can define $a_{ij}^k \in \mathbb{K}$, $i,j,k \in \mathcal{B}$, by

$$[i,j] = \sum_{k\in B} a_{ij}^k k.$$

The a_{ij}^k's are called the structure constants of L with respect to $\mathcal{B}$. Since $[,]$ is bilinear the structure constants uniquely determine $[,]$. Since $[,]$ is symplectic, alternating and fulfils the Jacobi identity we have for all $i,j,k,l \in \mathcal{B}$.

$$a_{ii}^k = 0$$

$$a_{ij}^k + a_{ji}^k = 0$$

$$\sum_m a_{ij}^m a_{km}^l + a_{jk}^m a_{im}^l + a_{ki}^m a_{jm}^l = 0.$$

Conversely, given a set $\mathcal{B}$ and $a_{ij}^k \in \mathbb{K}$, $i,j,k \in \mathcal{B}$ which fulfill the above three identities one easily obtains a Lie algebra with basis $\mathcal{B}$ and the a_{ij}^k as structure constants.

As an example consider the case of a 2-dimensional Lie-algebra L with basis x,y. Put $a := [x,y]$. Then $[L,L] = \mathbb{K}a$. If $a = 0$ then L is abelian.

Suppose that L is not abelian and choose $b \in L \setminus \mathbb{K}a$. Then also (a,b) is a basis for L and $[a,b] = ka$ for some $0 \neq k \in \mathbb{K}$. Replacing b by $k^{-1}b$ we may assume $[a,b] = a$. So up to isomorphism there exists at most one 2-dimensional non abelian Lie Algebra. For later use we record:

Lemma 1.2.1 [**2 dim**] *If L is 2-dimensional and non-abelian, then L has a basis (a,b) with $[a,b] = a$.* □

To show existence of such a Lie-algebra we could compute the structure constant and verify the above identies. But its easier to exhibit such Lie-algebra as a subalgebra of $\mathfrak{gl}(\mathbb{K}^2)$. Namely choose

$$a := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad b := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

1.3 Derivations

Definition 1.3.1 [**def:derivation**] *Let A be a $\mathbb{K}$-algebra. Then a* derivation *of A is a map $\delta \in \mathfrak{gl}(A)$ such that*

$$\delta(ab) = \delta(a)b + a\delta(b)$$

for all $a, b \in A$. $\mathfrak{der}(A)$ denotes the set of all derivations of A.

Lemma 1.3.2 [**derivations are lie**] *Let A be a $\mathbb{K}$-algebra. Then $\mathfrak{der}(A)$ is a subalgebra of $\mathfrak{gl}(A)$ as a Lie algebra.*

Obviously $\mathfrak{der}(A)$ is a $\mathbb{K}$-subspace of $\mathfrak{gl}(A)$. Now let $\gamma, \delta \in \mathfrak{der}(A)$ and $a, b \in A$. Then

$$\begin{aligned}
[\gamma, \delta](ab) &= \gamma\delta(ab) - \delta\gamma(ab) \\
&= \gamma(a\delta(b)) + \gamma(\delta(a)b) - \delta(a\gamma(b)) - \delta(\gamma(a)b) \\
&= \gamma(a)\delta(b) + a\gamma(\delta(b)) + \gamma(\delta(a))b + \delta(a)\gamma(b) \\
&\qquad -\delta(a)\gamma(b) - a\delta(\gamma(b)) - \delta(\gamma(a))b - \gamma(a)\delta(b) \\
&= a(\gamma(\delta(b)) - d(\gamma(b))) + (\gamma(\delta(a)) - \delta(\gamma(\alpha)))b \\
&= [\gamma, \delta](a)b + a[\gamma, \delta](b)
\end{aligned}$$

Lemma 1.3.3 [**left multiplication**] *Let A be an assocative $\mathbb{K}$-algebra and for $a \in A$ define $\mathrm{l}(a)a : A \to A, b \to ab$, $\mathrm{r}(a) : A \to A, b \to ba$ and $\mathrm{ad}\,(a) = \mathrm{l}(a) - \mathrm{r}(a)$, i.e. $\mathrm{ad}\,(a)(b) = [a, b]$. Let $a, b, c \in A$*

(a) [**a**] $\mathrm{l}(a), \mathrm{r}(a)$ *and* $\mathrm{ad}\,(a)$ *all are $\mathbb{K}$-linear.*

(b) [**b**] $[a, bc] = b[a, c] + [a, b]c$. *That is* $\mathrm{ad}\,(a)$ *is a derivation of A*

(c) [**c**] $\mathrm{l} : A \to \mathrm{End}(A), a \to \mathrm{l}(a)$ *is a homomorphism.*

(d) [**d**] $\mathrm{r} : A \to \mathrm{End}(A), a \to \mathrm{r}(a)$ *is an anti-homomorphism.*

Proof: (a) Obvious.

(b) We compute

$$b[a, c] + [a, b]c = bac - bca + abc - bac = a(bc) - (bc)a = [a, bc]$$

Also $\mathrm{ad}\,(a)(b) = ab - ba = [a,b]$ and the preceeding equation says that $\mathrm{ad}\,(a)$ is a derivation of A.

(c) and (d) are readily verifed as A is associative. □

Lemma 1.3.4 [**inner derivations**] *Let $a \in L$. Define $(\,ad)(a) : L \to L, l \to [a,l]$*

(a) [**a**] $\mathrm{ad}\, a$ *is a derivation of L.*

(b) [**b**] *Let $\delta \in \mathfrak{der}(L)$. Then $[\delta, \mathrm{ad}\, a] = \mathrm{ad}\,(\delta(a))$.*

(c) [**c**] $\mathrm{ad} : L \to \mathfrak{gl}(L)$ *is a homomorphism.*

(d) [**d**] $\mathrm{ad}\,(L)$ *is an ideal in $\mathfrak{der}(L)$.*

Proof: Let $a, b, c \in A$. Then

$$\mathrm{ad}\,(a)([b,c]) = [a,[b,c]] = -[b,[c,a]] - [c,[a,b]] = [b, \mathrm{ad}\, a(c)] + [\mathrm{ad}\,(a)(b), c]$$

and so $\mathrm{ad}\,(a)$ is a derivation.

Let δ be a derivation of A. Then

$$\begin{aligned} [\delta, \mathrm{ad}\,(a)](b) &= \delta(\mathrm{ad}\,(a)(b)) - \mathrm{ad}\,(a)(\delta(b)) \\ &= \delta([a,b]) - [a, \delta(b)] \\ &= [\delta(a), b] + [a, \delta(b)] - [a, \delta(b)] \\ &= \mathrm{ad}\,(\delta(a))(b) \end{aligned}$$

Thus (b) holds.

From (b) applied to the derivation $\mathrm{ad}\, b$ in place of δ we have $[\mathrm{ad}\, b, \mathrm{ad}\, a] = \mathrm{ad}\,([b,a])$ so (c) holds.

Finally (d) follows from (b). □

A derivation of the form $\mathrm{ad}\,(a)$ is called an *inner derivation.* All other derivations of a Lie Algebra are called *outer derivations*.

1.4 Modules

In this section A is an associative or Lie algebra over the field $\mathbb{K}$.

Definition 1.4.1 [**def:rep for associative**] *Let A be an associative $\mathbb{K}$-algebra and V a $\mathbb{K}$-space.*

(a) [**a**] *A representation for A over V is a homomorphism $\Phi : A \to \mathrm{End}(V)$.*

(b) [**b**] *An action for A on V is a bilinear map $A \times V \to V, (a, v) \to av$ such that*

$$(ab)v = a(bv)$$

for all $a, b \in A$, $v \in V$.

Definition 1.4.2 [**def:rep for lie**] *Let V be a $\mathbb{K}$-space.*

(a) [**a**] *A* representation *for L is a homomorphism $\Phi : L \to \mathfrak{gl}(V)$.*

(b) [**b**] *An* action *for L on V is a bilinear map $L \times V \to V, (a, v) \to av$ such that*

$$[a, b]v = a(bv) - b(av)$$

for all $a, b \in L$, $v \in V$.

If A is a associative algebra, then by 1.3.3 the left multiplication l is a representaion for A on A. And if L is a Lie algebra then by 1.3.4 ad is a representation of L on L.

Lemma 1.4.3 [**rep=action**] *Let A be an associative or a Lie algebra and V a $\mathbb{K}$-space.*

(a) [**a**] *Let Φ be a representation for A over V. Then $A \times V \to V, (a, v) \to \Phi(a)(v)$ is an action for A on V.*

(b) [**b**] *Suppose $A \times V \to V, (a, v) \to av$ is an action for A on V. Define $\Phi : A \to \mathrm{End}(V)$ by $\Phi(a)(v) = av$ for all $a \in A$, $v \in V$. Then Φ is a representation for A over V.*

Proof: Straightforward. □

If A acts on V we say that V is a *module* for A.

Lemma 1.4.4 [**associative to lie**] *Let V be a module for the associative algebra A. Then with the same action V is also a module for $\mathfrak{l}(A)$. In particular, left multiplication is an action of the Lie-Algebra $\mathfrak{l}(A)$ on A.*

Proof: Let $a, b \in A$ and v in v. Then

$$[a, b]v = (ab - ba)v = a(bv) - b(av).$$

□

Definition 1.4.5 [**def:centralizer**] *Let V be a module for the associative or Lie algebra A.*

(a) [**a**] $C_V(A) = \{v \in V \mid av = 0\, \forall a \in A\}$.

(b) [**b**] $C_A(V) = \{a \in A \mid av = 0\, \forall v \in V\}$

(c) [**c**] *If $C_A(V)=0$ we say that V is a* faithful *A-module.*

If Φ is the representation corresponding to the module V, then $C_A(V) = \ker\Phi$. In particular, $C_A(V)$ is an ideal in A. Note that V is also a module for $A/C_A(V)$ via $(a + C_A(V))v = av$. Even more, V is faithful for $A/C_A(V)$.

Put $Z(A) := \{a \in A \mid ab = 0 \forall b \in A\} = C_A(A)$. Then $Z(A)$ is an ideal in A called to center of A. Note that L is abelian iff $L = Z(L)$.

If X is a subsets of A and Y a subset of the A-module V, then we denote by XY the $\mathbb{K}$-subspace of V generated by $\{xy \mid x \in X, y \in Y\}$. We say that Y is X-invariant if $xy \in Y$ for all $x \in X$, $y \in Y$. $\langle Y\rangle$ denotes the additive subgroup of V generate by Y, while $\mathbb{K}Y$ denotes the $\mathbb{K}$-subspace of V generated by Y.

Lemma 1.4.6 [**product of subspaces**] *Let A be a Lie or an associative algebra, V an A-module, X a subset of A and Y an X-invariant subset of V. Then $\mathbb{K}Y$ is X-invariant.*

Proof: Let $x \in X$ and put $Z = \{z \in V \mid xz \in \mathbb{K}Y\}$. Then Z is an $\mathbb{K}$-subspace of V and since $Y \subseteq Z$, $\mathbb{K}Y \subseteq Z$. Thus $x\mathbb{K}Y \subseteq \mathbb{K}Y$ and $\mathbb{K}Y$ is X-invariant. □

Lemma 1.4.7 [**submodules and ideals**] *Let V an L-module and $I \subseteq L$.*

(a) [**a**] *I is an ideal in L if and only if I is L-submodule of L.*

(b) [**b**] *If I is an ideal in L then IV and $C_V(I)$ are L-submodule of V.*

Proof: Clearly I is a submodule iff its a left ideal. As left ideals are the same as ideals, (a) holds.

For (b) let $v \in V$, $i \in I$ and $l \in L$. Then $l(iv) = ([l,i])v + i(lv) \in IV$. In particular, IV is a L-submodule. Moreover, if $v \in C_V(I)$ we get $i(lv) = 0$ and so $lv \in C_V(I)$ and $C_V(I)$ is an L-submodule. □

1.5 The universal enveloping algebra

We assume the reader to be familiar with the definitions of tensor products and symmetric powers, see for example [La].

Definition 1.5.1 [**universal**]

(a) [**a**] *Let V be a $\mathbb{K}$-space. Then a tensor algebra for V is an associative algebra T with 1 together with an $\mathbb{K}$-linear map $\Phi : V \to T$ such that whenever T' is an associative $\mathbb{K}$-algebra with one and $\Phi' : V \to T'$ is $\mathbb{K}$-linear, then there exists an unique $\mathbb{K}$-algebra homomorphis $\Psi : T \to T'$ with $\Phi' = \Psi \circ \Phi$.*

(b) [**b**] *Let V be a $\mathbb{K}$-space. Then a symmetric algebra for V is a commutative and associative algebra T with 1 together with an $\mathbb{K}$-linear map $\Phi : V \to T$ such that whenever T' is a commuative and associative $\mathbb{K}$-algebra with one and $\Phi' : V \to T'$ is $\mathbb{K}$-linear, then there exists an unique $\mathbb{K}$-linear map $\Psi : T \to T'$ with $\Phi' = \Psi \circ \Phi$.*

(c) [**c**] *Let L be a Lie algebra over $\mathbb{K}$. Then an* universal enveloping algebra *for L is an associative $\mathbb{K}$-algebra U with one together with a homomorphism $\Phi : L \to \mathfrak{l}(U)$ such that whenever U' is an associative $\mathbb{K}$-algebra with one and $\Phi' : L \to \mathfrak{l}(U')$ is a homomorphism, then there exists a unique homomorphism of $\mathbb{K}$-algebra $\Psi : U \to U'$ with $\Phi' = \Psi \circ \Phi$.*

Lemma 1.5.2 [**existence of universal**]

(a) [**a**] *Let V be a $\mathbb{K}$-space. Then V has a tensor algebra $\mathfrak{T}(V)$ and $\mathfrak{T}(V)$ is unique up to isomorphism.*

(b) [**b**] *Let V be a $\mathbb{K}$-space. Then V has a symmetric algebra $\mathfrak{S}(V)$ and $\mathfrak{S}(V)$ is unique up to isomorphism.*

(c) [**c**] *Let L be a Lie algebra. Then L has a universal enveloping algebra $\mathfrak{U}(L)$ and $\mathfrak{U}(L)$ is unique up to isomorphism.*

Proof: The uniqueness statements follows easily from the definitions.

(a) Define $\mathfrak{T} = \bigoplus_{i=0}^{\infty} \bigotimes^i V$ and define a multiplication on $\mathfrak{T}$ by

$$(v_1 \otimes \ldots \otimes v_m)(w_1 \otimes \ldots w_n) = v_1 \otimes \ldots \otimes v_m \otimes w_1 \otimes \ldots w_n$$

The its is staighforward to check that $\mathfrak{T}$ is an associative algebra with 1. If T' is an associative algebra with 1, and $\Phi' : V \to T'$ is linear. Define $\Psi : \mathfrak{T} \to T'$ by $\Psi(v_1 \otimes \ldots \otimes v_m) = \Phi'(v_1)\Phi'(v_2)\ldots\Phi'(v_m)$.

(b) Let $\mathfrak{S}^n V$ be the n-th symmetric power of V and define $\mathfrak{S} := \bigoplus_{i=0}^{\infty} \mathfrak{S}^i V$. Proceed as in (a).

(c) Let I be the ideal in $\mathfrak{T}(L)$ generated by all the $a \otimes b - b \otimes a - [a, b]$, $a, b \in L$. Then T/I is a universal enveloping algebra. □

Lemma 1.5.3 [**basis for tensor**]

(a) [**a**] *Let I be set and for $i \in I$ let V_i be a $\mathbb{K}$-space with basis $\mathcal{B}_i$. Put $\mathcal{B} = \otimes_{i \in I} \mathcal{B}_i = (\otimes_{i \in I} b_i \mid b_i \in \mathcal{B}_i \forall i \in I)$. Then $\mathcal{B}$ is a basis for $\bigotimes_{i \in I} V_i$.*

(b) [**b**] *Let V be a $\mathbb{K}$-space with ordered basis $\mathcal{B}$. Let $n \in \mathbb{N}$. Then $(b_1 b_2 \ldots b_n \mid b_1 \leq b_2 \leq \ldots \leq b_n, b_i \in \mathcal{B})$ is a basis for $\mathfrak{S}^n V$.*

Proof: Wellknown. See for example [La]. □

Let A be any associative $\mathbb{K}$-algebra. Note that the definition of an universal enveloping algebra implies that the map

$$\operatorname{Hom}(\mathfrak{U}(L), A) \to \operatorname{Hom}(L, \mathfrak{l}(A)), \quad \alpha \to \alpha \circ \Phi$$

is a bijection. For the case that $A = \operatorname{End}(V)$ for a $\mathbb{K}$-space V we conclude:

Lemma 1.5.4 [**modules for universal**] *Let $\phi : L \to \mathfrak{U}$ be an universal enveloping alebra. Let V an L-module. Then there exists a unique action of $\mathfrak{U}$ on V with $\phi(l)v = lv$ for all $l \in L$. The resulting map between the set of L-modules and the set of $\mathfrak{U}$-modules is a bijection.*

□

Lemma 1.5.5 [**d spans u**] *Let $\phi : L \to \mathfrak{U}$ be an universal enveloping algebra for L. Also let $\mathcal{B}$ be an ordered basis for L. Put $\mathfrak{U}_m = \sum_{i=0}^{m} \phi(L)^i$. Then*

$$\mathfrak{U}_m = \mathbb{K}\langle \phi(b_1)\ldots\phi(b_i) \mid 0 \le i \le m, b_j \in \mathcal{B}, b_1 \le b_2 \le \ldots \le b_i\rangle.$$

Proof: By induction on m. Since we interpret the empty product as 1, the statement is true for $m = 0$. Suppose its is true for $m-1$. Let $b_1, b_2, \ldots, b_m \in \mathcal{B}$. For simplicity, we write the product $\phi(b_1)\phi(b_2)\ldots\phi(b_k)\forall k \in \mathbb{N}/\{0\}$ Also let $0 \le i < m$ and put $a = b_1 b_2 \ldots b_{i-1}$ and $c = b_{i+2}\ldots b_m$. Then

$$b_1 b_2 \ldots b_m = a b_i b_{i+1} c = a b_{i+1} b_i c + a[b_i, b_{i+1}]c$$

Thus

$$b_1 b_2 \ldots b_m + \mathfrak{U}_{m-1} = b_1 \ldots b_{i-1} b_{i+1} b_i b_{i+2} \ldots b_n + \mathfrak{U}_{m-1}$$

and so for all $\pi \in \operatorname{Sym}(m)$,

$$b_1 b_2 \ldots b_m + \mathfrak{U}_{m-1} = b_{\pi(1)} \ldots b_{\pi(m)} + \mathfrak{U}_{m-1}$$

Choosing π such that $b_{\pi(1)} \le b_{\pi(2)} \le \ldots \le b_{\pi(m)}$ and we see that the lemma also holds for m. □

Lemma 1.5.6 [**action of l on s**] *Let $\mathcal{B}$ be an ordered basis for the Lie algebra L. Identify L with its image in $S := \mathfrak{S}(L)$. Let $b \in \mathcal{B}$ and $s \in \mathcal{B}^n$. Define $b \le s$ if either $n = 0$ or $s = \prod_{i=1}^{n} b_i, b_i \in \mathcal{B}$ with $b \le b_i$ for all $1 \le i \le n$. Then there exists a unique action $\cdot$ of L on S such that $b \cdot s = bs$ for all $b \in \mathcal{B}$, $n \in \mathbb{N}$ and $s \in \mathcal{B}^n$ with $b < s$.*

Proof: Put $S_m = \sum_{i=0}^{m} L^m \le S$. To show the uniqueness of $\cdot$ we show by induction on m that the restriction of $\cdot$ to $L \times S_m$ is unique and that

1° [**1**] *$w(b,s) := b \cdot s - bs \in S_m$ for all $b \in L$ and $s \in S_m$.*

Note that (1°) implies that $b \cdot s = bs + w(b,s) \in S_{m+1}$

If $m = 0$, the $S_0 = \mathbb{K}$ and $b \cdot s = bs = sb$ for all $s \in S_0$. Suppose now that $m \geq 1$. Let $s = dt \in \mathcal{B}^m$ with $d \in \mathcal{B}$, $t \in \mathcal{B}^{m-1}$ and $d \leq t$. We need to compute $b \cdot s$ uniquely and show that $b \cdot s - bs \in S_m$. Note that $dt = d \cdot t$.

If $b \leq d$, then $b \leq s$. So

2° [**2**] *$b \cdot s = bs$, whenever $b \leq s$*

Also $b \cdot s - bs = 0 \in S_m$.

If $b > d$, then since $\cdot$ is an action

3° [**3**]

$$b \cdot s = b \cdot (d \cdot t) = d \cdot (b \cdot t) + [b,d] \cdot t$$

By induction on m, $b \cdot t$ and $[b,d] \cdot t$ are uniquely determined. Moreover, $b \cdot t = bt + w(b,t)$ with $w(b,t) \in S_{m-1}$ and $[b,d] \cdot t \in S_m$. Since $d \leq bt$ we have $d \cdot (bt) = dbt = bs$. Also by induction $d \cdot w$ is uniquely determined and contained in S_m. Thus the formula

4° [**4**] *$b \cdot s = dbt + d \cdot w(b,t) + [b,d] \cdot t$, whenever $b \not\leq s$*

uniquely determines $b \cdot s$. Moreover $w(b,s) = d \cdot w(b,t) + [b,d] \cdot t \in S_m$.

Thus $\cdot$ is unique and (1°) holds.

To prove existence we define $b \cdot s$ for $b \in \mathcal{B}$ and $s \in \mathcal{B}^m$ by induction on m via (2°) and (4°). Once $b \cdots s$ is defined for all $s \in \mathcal{B}^m$, define $l \cdot s$ for all $l \in L$ and $s \in S_m$ by linear extension. Note also that (1°) will hold inductively. So all terms are on the right side of (4°) are defined at the time its used to define the left side.

We need to verify that $\cdot$ is an action.

Let $a, b \in L$ and $s \in S$. We say that $\{a,b\}$ acts on v if $a \cdot (b \cdot s) - b \cdot (a \cdot s) = [a,b] \cdot s$. Note that set of $s \in S$ on which $\{a,b\}$ acts is a $\mathbb{K}$-subspace of V.

Suppose inductively that we have shown

5° [**5**] *For all $a, b \in \mathcal{L}$ and all $s \in S_{m-1}$, $\{a,b\}$ acts on s.*

Let $a, b \in \mathcal{B}$ and $s \in \mathrm{B}^m$. We need to show that $\{a,b\}$ acts on s. This is obviously the case then $a = b$. So suppose $a \neq b$

Suppose that $a \leq s$ or $b \leq s$. Without loss $a > b$. Then $b \leq s$. Using the definition of $a \cdot u$ for $u = b \cdot s$ (compare (3°)) we get

6° [**6**] *If $a \leq s$ or $b \leq s$ then $\{a,b\}$ acts on all $s \in \mathcal{B}^m$.*

Suppose next that $a > s$ and $b > s$. Let $s = dt = d \cdot t$ be as above. Then

Since $d \leq bt$ (6°) gives that $\{a, d\}$ acts on bt. By induction $\{a, d\}$ also acts on $w(b, t) \in S_{m-1}$ and so $\{a, d\}$ acts on $b \cdot t = bt + w(b, t)$. This allows us to compute (using our inductive assumption (5°) various times):

$$\begin{aligned} a \cdot (b \cdot (d \cdot t)) &= a \cdot (d \cdot (b \cdot t) + [b, d] \cdot t) \\ &= d \cdot (a \cdot (b \cdot t)) + [a, d] \cdot (b \cdot t) + [b, d] \cdot (a \cdot t) + [a, [b, d]] \cdot t \end{aligned}$$

Since the situation is symmetric in a and b the above equation also holds with the roles of a and b interchanged. Subtracting these two equations we obtain:

$$\begin{aligned} a \cdot (b \cdot dt) - b \cdot (a \cdot dt) &= d \cdot (a \cdot (b \cdot t) - b \cdot (a \cdot t)) + [a, [b, d]] \cdot t - [b, [a, d]] \cdot t \\ &= d \cdot ([a, b] \cdot t) + [a, [b, d]] \cdot t + [b, [d, a]] \cdot t \\ &= [a, b] \cdot (d \cdot t) + ([d, [a, b]] + [a, [b, d]] + [b, [d, a]]) \cdot t \\ &= [a, b] \cdot dt \end{aligned}$$

Thus $\{a, b\}$ acts on $s = dt$ and so by induction L acts on S. □

Theorem 1.5.7 (Poincare-Birkhof-Witt) [**pbw**] *Let $\phi : L \to \mathfrak{U}$ be an universal enveloping algebra of L. Let $\mathcal{B}$ be the ordered basis of L and view $\mathfrak{S}(L)$ as an L- (and so as an $\mathfrak{U}(L)$-) module via 1.5.6.*

(a) [**a**] *The map $\Psi : \mathfrak{U}(L) \to \mathfrak{S}(L), u \to u \cdot 1$ is a isomorphism of $\mathbb{K}$-spaces.*

(b) [**b**]
$$\mathcal{D} := (\phi(b_1)\phi(b_2)\ldots\phi(b_n) \mid n \in \mathbb{N}, b_1 \leq b_2 \leq \ldots \leq b_n \in \mathcal{B})$$
is a basis for $\mathfrak{U}$.

(c) [**c**] *ϕ is one to one.*

Proof: Let $b_1, b_2, \ldots b_n$ be a nondecreasing sequence in $\mathcal{B}$. The definition of the action of L on $\mathfrak{S}(L)$ implies that $\phi(b_1)\phi(b_2)\ldots\phi(b_n) \cdot 1 = b_1 b_2 \ldots b_n$. Hence $\Psi(\mathcal{D})$ is a basis for $\mathfrak{S}(L)$. Thus Ψ is onto and $\mathcal{D}$ is linearly independent in $\mathfrak{U}$. By 1.5.5 $\mathbb{K}\mathcal{D} = \mathfrak{U}$ and so $\mathcal{D}$ is a basis for $\mathfrak{U}$. Hence Ψ sends a basis of $\mathfrak{U}$ to a basis of $\mathfrak{S}(L)$ and so is an isomorphism. Also $\phi(\mathcal{B})$ is linearly independent and so ϕ is one to one. □

From now on $\mathfrak{U}$ denotes a universal envelpong algebra for L. In view of the Poincare-Witt-Birkhoff Theorem we may and do identify L with its image in $\mathfrak{U}$. In particular for $n \in \mathbb{N}$ we obtain the $\mathbb{K}$-subspace L^n of $\mathfrak{U}$ Also according to 1.5.4 we view every L-module V as an $\mathfrak{U}$-module. Indeed if $a_1, a_2, \ldots, a_n \in L$ and $v \in V$, then $a_1 a_2 \ldots a_n \in U$ just acts

$$(a_1 a_2 \dots a_n)v = a_1(a_2(\dots(a_n v)\dots)).$$

In particular the adjoint action of L on L extends to an action of $\mathfrak{U}$ on L. We denote this action by $U \times L \to L, u \to u * l$. For example $a, b, l \in L$ we have $a * l = [a, l]$ and $(ab) * l = [a, [b, l]]$. With this notations we have

$$L^n * L = \underbrace{[L, [L, \dots [L}_{n\text{-times}}, L]] \dots]].$$

Lemma 1.5.8 [**[l,l,n]**] $L^n * L \leq L^{n+1}$.

Proof: The proof is by induction on n.. The statement is clearly true for $n = 0$. Suppose now that $L^{n-1} * L \leq L^n$. Let $l \in L$ and $a \in L^{n-1} * L$. Then $a \in L^n$ and so $l * a = la - al \in L^{n+1}$. Thus $L^n * L = L * (L^{n-1} * L) \leq L^{n+1}$ and the lemma is proved. □

1.6 Nilpotent Action

Let R be a ring and $X \subseteq R$. We say that X is nilpotent if $X^n = 0$ for some $n \in \mathbb{N}$. Note that for $R = \mathrm{End}(V)$ we have $X^n = 0$ iff $X^n V = 0$.

Now let A be an associative or Lie algebra and V a module for A. Then we say that $X \subseteq A$ acts nilpotenly on V if the image of X in $\mathrm{End}(V)$ is nilpotent. Note that that X acts *nilpotently* on V if and only if $X^n V = 0$ for some $n \in \mathbb{N}$.

We say that L is nilpotent if L acts nilpotently on L, that is if $L^n * L = 0$ for some n. Note that for associative algebra A a subalgebra B is nilpotent if an only if the action of B on A by left multiplication is nilpotent. Indeed if $B^n = 0$, then $B^n A = 0$ and if $B^n A = 0$ then $B^{n+1} = 0$. The analog of this statement is not true for Lie algebras. For example consider that Lie algebra L with basis x, y such that $[x, y] = x$. Then $\mathbb{K}y$ is an abelian and so a nilpotent subalgebra of L, but y does not act nilpotently on L. On the other hand if I is an ideal in L, then I is nilpotent if and only if I acts nilpotently on L.

We remark that if X acts nilpotently on V then all elements in X act nilpotently on V. The main goal of this section is to show that for finite dimensional Lie-algebras, the converse holds. That is if all elements of the finite dimensional Lie-algebra L act nilpotently on V, then also L acts nilpotenly on V.

We say that L acts trivially on V if $LV = 0$.

Lemma 1.6.1 [**nilpotent and chains**] *Let A be an associative or Lie algebra. Let V be an A-module. Then the following are equivalent:*

(a) [**a**] *A acts nilpotently on V.*

(b) [**b**] *There exists a finite chain of A submodules $0 = V_n \leq V_{n-1} \leq \dots V_0 = V$ such that A acts trivially on each V_i/V_{i+1}.*

(c) [**c**] *There exists a finite chain of A submodules $0 = V_n \leq V_{n-1} \leq \dots V_0 = V$ such that A acts nilpotently on each V_i/V_{i+1}.*

Proof: (a)$\Longrightarrow$ (b): Just put $V_i = A^i V$.

(b)$\Longrightarrow$ (c): This holds since trivial action is nilpotent.

(c)$\Longrightarrow$ (a): For $0 \leq i < n$ choose m_i with $A^{m_i}(V_i/V_{i+1}) = 0$. Then $A^{m_i} V_i \leq V_{i+1}$. Put $m = \sum_{i=0}^{n-1} m_i$. Then $A^m V = 0$. □

Lemma 1.6.2 [**nilpotent implies nilpotent**] *Suppose L acts nilpotenly on the L-module V. Then $L/C_L(V)$ is nilpotent.*

Proof: Let $L^n V = 0$ for some $n \geq 1$. Then by 1.5.8 $(L^{n-1} * L)V = 0$. Thus $L^{n-1} * L \leq C_L(V)$ and $L/C_L(V)$ is nilpotent. □

Lemma 1.6.3 [**nilpotent + nilpotent**] *Let A be an associative or Lie algebra. Let V be an A-module, D, E subalgebras of A with $[E, D] \leq D$. If E and D acts nilpotently on V, then $E + D$ acts nilpotently on V.*

Proof: In the case that A is associative, we replace A by $\mathfrak{l}(A)$. So A is now a Lie algebra. Since $[E, D] \leq D$, D is an ideal in $E + D$. By 1.4.7(b) DV is an $E + D$-submodule. By induction, $D^n V$ is a $E + D$-submodule. D acts trivially and so $E + D$ acts nilpotenly on $D^n V/D^{n+1}V$ for all n. Thus the lemma follows from 1.6.1. □

Lemma 1.6.4 [**associative and nilpotent**] *Let A be an associative $\mathbb{K}$-algebra.*

(a) [**a**] *Let $D, E \leq A$ be nilpotent with $[E, D] \leq D$. Then $D + E$ is nilpotent.*

(b) [**b**] *Let $D \leq A$ be nilpotent. Then D acts nilpotently on $\mathfrak{l}(A)$ by adjoint action.*

Proof: (a) By 1.6.3 $D + E$ acts nilpotently on A and so is nilpotent.

(b) Since $D^n = 0$ we have $\mathrm{l}(D)^n = 0$ and $\mathrm{r}(D)^n = 0$. Also since A is associative $\mathrm{l}(D)^n$ and $\mathrm{r}(D)^n$ commute. Thus (a) implies that $\mathrm{l}(D) + \mathrm{r}(D)$ is nilpotent. Since $\mathrm{ad}\,(a) = \mathrm{l}(a) - \mathrm{r}(a)$ we have $\mathrm{ad}\,(D) \leq \mathrm{l}(D) + \mathrm{r}(D)$ and so $\mathrm{ad}\,(D)$ is nilpotent in $\mathrm{End}(A)$ and so D acts nilpotently $\mathfrak{l}(A)$. □

Corollary 1.6.5 [**nil on V and in L**] *Suppose that L acts faithfully on V and that $X \subseteq L$ acts nilpotently on V. Then X acts nilpotently on L.*

Proof: Let $\Phi : L \to \mathfrak{gl}(V)$ be the corresponding representation. Then by the definition of nilpotent action, $\Phi(X)$ is nilpotent in $\mathrm{End}(V)$. From 1.6.4 the adjoint action of $\Phi(X)$ on $\mathfrak{gl}(V)$ is nilpotent. Thus $\Phi(X)$ acts nilpotently on $\Phi(L)$ and as Φ is one to one, X acts nilpotently on L. $\square$

Lemma 1.6.6 [**normalizer of nilpotent**] *Suppose L acts nilpotenly on V and $W \subset V$ with $0 \in W \neq V$. Then there exists $v \in V \setminus W$ with $Lv \leq W$.*

Proof: Since L acts nilpotently on V and $0 \in W$, we can choose $n \in \mathbb{N}$ minimal with $L^n V \subseteq W$. Since $W \neq V$, $n \neq 0$. By minmality of n, $L^{n-1}V \not\leq W$. Pick $v \in L^{n-1}V \setminus W$. Then $Lv \leq L(L^{n-1}V) = L^n V \leq W$. $\square$

For a subalgebra $A \leq L$ put $N_L(A) = \{l \in L \mid [l, A] \leq A\}$. Note that $N_L(A)$ is subalgebra of L and that A is an ideal in $N_L(A)$.

Corollary 1.6.7 [**normalizer of nilpotent II**] *Suppose that M is a subalgebra of L acting nilpotenly on L. If $M \neq L$, then $M \lneq N_L(M)$.*

Proof: By 1.6.6 (applied with (M, L, M) in the roles of (L, V, W)) there exists $d \in L \setminus M$ with $[M, d] \leq M$. Then $d \in N_L(M)$. $\square$

Definition 1.6.8 [**def:subideal**] *Let A be a $\mathbb{K}$-algebra and $I \subseteq A$. We write $I \trianglelefteq A$ if I is an ideal in A. We say that I is a* subideal *in A and write $I \trianglelefteq\trianglelefteq A$ if there exists chain $I = I_0 \trianglelefteq I_1 \trianglelefteq \ldots \trianglelefteq I_n \trianglelefteq I_n = A$.*

Lemma 1.6.9 [**subideals in nilpotent**] *Suppose L is nilpotent. Then every subalgebra in L is an subideal in L.*

Proof: Let n be minimal with $L^n * L = 0$ and $A \leq L$. Let $Z = L^{n-1} * L$. Then $L * Z = 0$, that is $Z \leq Z(L)$. Thus $[A, Z + A] = [A, A] \leq A$ and $A \trianglelefteq Z + A$. Put $\overline{L} = L/Z$. Since $L^{n-1} * L \leq Z$, $\overline{L}^{n-1} * \overline{L} = 0$. By induction on n we may assume $Z + A/Z \trianglelefteq\trianglelefteq L/Z$. Thus $Z + A \trianglelefteq\trianglelefteq L$ and so $A \trianglelefteq\trianglelefteq L$ since $A \trianglelefteq Z + A \trianglelefteq\trianglelefteq L$ $\square$

Theorem 1.6.10 [**elementwise nilpotent**] *Let L be a finite dimensional Lie algebra and V a L-module. If all elements of L act nilpotently on V, then L acts nilpotently on V.*

Proof: We may assume without loss that L is faithful on V. The proof is by induction on $\dim V$. Let M be a maximal subalgebra of L. By induction M acts nilpotently on V. So by 1.6.5 M acts nilpotenly on L. 1.6.7 implies that there exists $d \in N_L(M) \setminus M$. Note that $\mathbb{K}d$ is a subalgebra and $M + \mathbb{K}d$ are subalgebras of L. By maximality of M, $L = M + \mathbb{K}d \leq N_L(M)$. As d is nilpotent on V, $\mathbb{K}d$ is nilpotent on V as well. Thus 1.6.3 implies that L is nilpotent on V $\square$.

Corollary 1.6.11 (Engel) [**engel**] *Let L be a finite dimensional Lie algebra all of whose elements act nilpotently on L. Then L is nilpotent.*

Proof: Apply 1.6.10 to the adjoint module. □

1.7 Finite Dimensional Modules

Definition 1.7.1 [**series**] *Let A be Lie or an associate $\mathbb{K}$-algebra and V an A-module.*

(a) [**a**] *V is called* simple *if V has no proper A-submodules. (that is O and V are the only A-submodules. V is* semisimple *if its the direct sum of simple modules and its* homogeneous *if its the direct sum of isomorphic simple modules.*

(b) [**b**] *A* series *for A on V is a chain $\mathcal{S}$ of A-submodules of V such that*

  (a) [**a**] *$0 \in \mathcal{S}$ and $V \in \mathcal{S}$.*

  (b) [**b**] *$\mathcal{S}$ is closed under intersections and unions, that is for every nonempty $\mathcal{D} \subset \mathcal{S}$, $\bigcap \mathcal{D} \in \mathcal{S}$ and $\bigcup \mathcal{D} \in \mathcal{S}$.*

  (Here a chain is a set of sets which is totally ordered with respect inclusion)

(c) [**c**] *Let $\mathcal{S}$ be an A-series. A* jump *of $\mathcal{S}$ is pair $(D, E))$ such that $D, E \in \mathcal{S}$, $D < E$ and $C \in \mathcal{S}$ with $D \leq C \leq E$ implies $C = D$ or $C = E$. In this case E/D is called a* factor *of $\mathcal{S}$.*

(d) [**d**] *A* composition series *for A on V is a series all of whose factors are simple A-modules.*

(e) [**e**] *Let $\mathcal{S}$ and $\mathcal{T}$ be A-series on V. We say that $\mathcal{S}$ and $\mathcal{T}$ have isomorphic factors if there exists a bijection Φ between the sets of factors of $\mathcal{S}$ and $\mathcal{T}$ such that for each factor F of $\mathcal{S}$, F and ΦF are isomorphic A-modules. Such a Φ is called an isomorphism of the sets of factor.*

(f) [**f**] *Let V and W be A-modules and $\phi \in \mathrm{Hom}(V, W)$. Then ϕ is called A-invariant if $\phi(av) = a\phi(v)$ for all $a \in A$, $v \in V$. $\mathrm{Hom}_A(V, W)$ denotes the set of such ϕ.*

(g) [**g**] *If X and Y are A-submodules of V with $X \leq Y$, then Y/X is called an A-section of V.*

Lemma 1.7.2 [**lifting series**] *Let V be an L-module and W an L-submodule of V. Let $\mathcal{S}$ be a L-series on W and $\overline{\mathcal{T}}$ a L-series on V/W. Let $\mathcal{T}$ be the inverse image of $\overline{\mathcal{T}}$ in V (so $\overline{\mathcal{T}} = \{T/W \mid T \in \mathcal{T}\}$). Then $\mathcal{S} \cup \mathcal{T}$ is a series for L on V. The factors of $\mathcal{S} \cup \mathcal{T}$ are the factors of $\mathcal{S}$ and $\overline{\mathcal{T}}$. In particular, $\mathcal{S} \cup \mathcal{T}$ is an L-composition seres if and only if both $\mathcal{S}$ and $\overline{\mathcal{T}}$ are L-composition series.*

Proof: This follows readily from the definition. We leave the details as an exercise. □

Lemma 1.7.3 (Jordan Hölder) [**jordan hoelder**] *Let A be a Lie or an associative $\mathbb{K}$-algebra. Suppose that there exists a finite composition series for A on V. Then any two composition series for A on V have isomorphic factors.*

Proof: Let $\mathcal{S}$ be a finite composition series for A on V and $\mathcal{T}$ any compostion series. For a jump (B, C) of $\mathcal{T}$ choose $D \in \mathcal{S}$ maximal with $C \not\leq B + D$. Let E be minimal in $\mathcal{S}$ with $D < E$. Then E/D is a factor of $\mathcal{T}$, C/B is a factor of $\mathcal{S}$ and we will show that map $B/C \to E/D$ is an isomorphism of the sets of factor.

By maximality of D we have

$$C \leq B + E.$$

Thus $C = C \cap (B + E) = B + (C \cap E)$ and so

$$C/B \cong C \cap E/C \cap E \cap B = C \cap E/B \cap E.$$

Since $C \not\leq B + D$, $C \cap E \not\leq D$ and since E/D is simple, $E = D + (C \cap E)$. Thus

$$E/D \cong C \cap E/C \cap D.$$

If $B \cap E \not\leq D$, then $E = (B \cap E) + D \leq B + D$ and so $C \leq B + E \leq B + D$, contrary to our choice of D. Thus $B \cap E \leq D$ and so $B \cap E = B \cap D$. Suppose that $C \cap D \not\leq B$. Then $C = (C \cap D) + B \leq B + D$, again a contradiction. Thus $C \cap D = B \cap D = B \cap E$ and so

$$C/B \cong C \cap E/B \cap D = C \cap E/B \cap E \cong E/D.$$

It remains to show that our map between the factor sets is a bijection. Let (B', C') be a jump other than (B, C) and say $C' \leq B$. Then $C' \cap E \leq B \cap E = B \cap D \leq D$ and so (B', C') is not mapped to E/D. So our map is one to one.

Since $\mathcal{S}$ is finite we conclude that, $\mathcal{T}$ has finitely many jumps and so also $\mathcal{T}$ is finite and $|\mathcal{T}| \leq |\mathcal{S}|$. But now the situation is symmetric in $\mathcal{T}$ and $\mathcal{S}$. Thus $|\mathcal{S}| \leq |\mathcal{T}|$, $|\mathcal{S}| = |\mathcal{T}|$ and our map is a bijection. □

Lemma 1.7.4 [**submodules for ideals**] *Let L be a Lie algebra, V an L-module, I an ideal in L, W an I-submodule in V and $l \in L$. Let X be an I submodule of V containing $[I, l]W$*

(a) [**a**] *The map $W \to V/X, w \to lw + X$ is I-invariant.*

(b) [**b**] *The map $W \to V/W, w \to lw + W$ is I-invariant.*

(c) [**c**] *If $[I, l] = 0$ then the map $W \to V, w \to lw$ is I-invariant.*

Proof: (a) Let ϕ be the map in question. Let $i \in I$ and $w \in W$. Then $ilw = liw + [i,l]w \in liw + X$ and so $i\phi(w) = \phi(iw)$.

(b) Since W is an I-submodule and I is an ideal, $[I,l]W \leq W$ and so we can apply (a) with $X = W$.

(c) Apply (b) with $X = 0$.

Definition 1.7.5 [**def:nil v**] *Let V be a finite dimensional L-module.*

(a) [**a**] $\mathrm{Comp}_V(L)$ *is the set of factors of some L-compositions series on V. (Note by the Jordan Hölder Theorem,* $\mathrm{Comp}_V(L)$ *is essentially independent from the choice of the composition series)*

(b) [**b**] $\mathrm{Nil}_L(V) = \bigcap\{C_L(W) \mid W \in \mathrm{Comp}_V(L)\}$

Lemma 1.7.6 [**nil V**] *Let V be a finite dimensional L-module. Then* $\mathrm{Nil}_L(V)$ *is the unique maximal ideal of L acting nilpotently on V.*

Proof: $\mathrm{Nil}_L(V)$ is the intersection of ideals and so an ideal in L. By 1.6.1(b), $\mathrm{Nil}_L(V)$ is nilpotent on V. Now let I be an ideal of L acting nilpotently on V. Also let W be a composition factor for L on V. Then $0 \neq C_W(I)$ is an L-submodule of W and so $C_W(I) = W$, $I \leq C_L(W)$ and $I \leq \mathrm{Nil}_L(V)$. □

Corollary 1.7.7 [**Nil L**] *Let L be finite dimensional. Then L has a unique maximal nilpotent ideal* $\mathrm{Nil}(L)$*.*

Proof: An ideal in L is nilpotent if and only if its acts nilpotently on L. So the lemma follows from 1.7.6 applied to the adjoint module. □

We remark that there may not exist a unique largest nilpotenly acting subideal in L. For example consider $L = \mathfrak{sl}(\mathbb{K}^2)$ and let $V = \mathbb{K}^2$. Let

$$x = E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h = E_{11} - E_{22} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then $[h,x] = 2x$, $[y,h] = 2y$ and $[x,y] = h$.

If $\operatorname{char}\mathbb{K} = 2$ we conclude that $\mathbb{K}x + \mathbb{K}h$ is an ideal in $\mathfrak{sl}(\mathbb{K}^2)$ and $\mathbb{K}x$ is an ideal in in $\mathbb{K}x + \mathbb{K}h$. Thus $\mathbb{K}x$ is a subideal acting nilpotently on $\mathbb{K}^2$. The same holds for $\mathbb{K}y$. But $\mathfrak{sl}(\mathbb{K}^2)$ is the subalgebra generated by x and y. Since $\mathfrak{sl}(\mathbb{K}^2)V = V$, $\mathfrak{sl}(\mathbb{K}^2)$ does not act nilpotently on V and so $\mathbb{K}x$ and $\mathbb{K}y$ are not contained in common nilpotently acting subideal of L.

Definition 1.7.8 [**def:vd**] *Let V be finite dimesional L-module.*

(a) [**a**] $\mathrm{Sim}(L)$ *is the set of all isomorphism classes of finite dimensional simple L-modules.*

(b) [**b**] $\mathrm{Sim}_V = \mathrm{Sim}_V(L)$ *is the set of the isomorphism classes of the L-composition factors of V.*

(c) [**c**] *Let $\mathcal{D} \subseteq \mathrm{Sim}(L)$. A $\mathcal{D}$-module is an L-module W with $\mathrm{Sim}_W \subseteq \mathcal{D}$ (If W is simple this means that the isomorphism class of W is in $\mathcal{D}$.*

(d) [**d**] *$V_{\mathcal{D}}$ is the sum of all the simple $\mathcal{D}$-submodules in V*

(e) [**e**] *$V_{\mathcal{D}}(0) = 0$ and inductively define the submodule $V_{\mathcal{D}}(n+1)$ of L in V by*

$$V_{\mathcal{D}}(n+1)/V_{\mathcal{D}}(n) = (V/V_{\mathcal{D}}(n))_{\mathcal{D}}.$$

(f) [**f**] *$V^c_{\mathcal{D}} = \bigcup_{i=0}^{\infty} V_{\mathcal{D}}(i)$.*

(g) [**g**] *Let $A \leq L$ and $\mathcal{A} \subseteq \mathrm{Sim}(A)$. Then $\mathcal{A} \mid^L$ is the set of isomorphism classes of the finite dimensional simple L-modules which are $\mathcal{A}$-modules.*

To digest the preceeding definitions we consider an example. Let L be the subalgebra of $\mathfrak{gl}(\mathbb{K}^3)$ consisting of all 3×3 matrices of the form

$$\begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $V = \mathbb{K}^3$ viewed as an L-module via left multiplication. Let (e_1, e_2, e_3) be the standard basis for $\mathbb{K}^3$. Let $V_i = \sum_{j=0}^{i} \mathbb{K}e_j, i = 0, 1, 2, 3$ and $e_0 = 0$. Then

$$0 = V_0 < V_1 < V_2 < V_3 = V$$

is a composition series for L on V. Put $I_k = V_k/V_{k-1}$. Then I_k is a simple 1-dimensional L-module. Note that $LI_1 = 0$ and $LI_3 = 0$ while $LI_2 \neq 0$. So $I_1 \cong I_3$ but $I_1 \ncong I_2$ as L-module. For an L-module W let $[W]$ be the isomorphism class of W(that is the class of L-modules isomorphic to W. Then $\mathrm{Sim}_V = \{[I_1], [I_2]\}$. For $k = 1, 2$ let $\mathcal{D}_k = \{[I_k]\}$. Also put $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 = \mathrm{Sim}_V$. Observe that any L-submodule of V is one of the V_i.

By definition $V_{\mathcal{D}_1}$ is the sum of all the simple L-submodule of V isomorphic to I_1. V_1 is the only simple L-submodule of V and $V_1 \cong I_1$ so $V_{\mathcal{D}_1} = V_1$. To compute $V_{\mathcal{D}_1}$, put $\overline{V} = V/V_1$. The only simple submodule of $\overline{V}$ is $I_2 = V_2/V_1$. Since $I_1 \ncong I_2$ we get $\overline{V}_{\mathcal{D}_1} = 0$. Thus $V_{\mathcal{D}_1}(2) = V_1$. It follows that $V_{\mathcal{D}_1}(j) = V_1$ for all $j \geq 1$ and so also $V^c_{\mathcal{D}_1} = V_1$.

No submodule of V is isomorphic to I_2 and hence $V_{\mathcal{D}_2} = 0$. Thus $V^c_{\mathcal{D}_2} = V_{\mathcal{D}_2}(j) = 0$ for all $j \geq 0$.

V_1 is the only submodule of V isomorphic to I_1 or I_2 and so $V_{\mathcal{D}} = V_1$. V_2/V_1 is the only submodule of V/V_1 isomorphic to I_1 or I_2. So $(V/V_1)_{\mathcal{D}} = V_2/V_1$ and $V_{\mathcal{D}}(2) = V_2$. V/V_2 is isomorphic to I_1 and so $V = V_{\mathcal{D}}(3) = V^c_{\mathcal{D}}$.

Definition 1.7.9 [**def:linear indep**] *Let V be $\mathbb{K}$-space and $\mathcal{V}$ a set of $\mathbb{K}$-subspaces of V. We say that $\mathcal{V}$ is linearly independent if $\sum \mathcal{V} = \bigoplus \mathcal{V}$.*

Lemma 1.7.10 [**basic semisimple**] *Let V be an L-modules and $\mathcal{V}$ a set of simple L-submodules in V. Suppose that $V = \sum \mathcal{V}$.*

(a) [**a**] *Let W be an L-submodule of V, Then there exists $\mathcal{W} \subseteq \mathcal{V}$ such that $V = W \oplus \bigoplus \mathcal{W}$.*

(b) [**b**] *Let $X \leq Y$ be L-submodules. Then there exists $\mathcal{W} \subseteq \mathcal{V}$ with $Y/X \cong \bigoplus \mathcal{W}$ as L-modules.*

(c) [**c**] *Every L-section of V is semisimple. In particular, V is semisimple.*

(d) [**d**] *Every composition factor of V is isomorphic to some member of $\mathcal{V}$.*

Proof: (a) Let $\mathcal{C}$ be the set of linearly independet subsets $\mathcal{W}$ of $\mathcal{V}$ with $W \cap \sum \mathcal{W} = 0$. Order $\mathcal{C}$ by inclusion. If $\mathcal{D}$ is a chain in $\mathcal{C}$, then it is easy to verify that $\bigcup \mathcal{D} \in \mathcal{C}$. So every chain in $\mathcal{C}$ has an upper bound. By Zorn's Lemma, $\mathcal{C}$ has a maximal elements $\mathcal{W}$. Suppose that $V \neq W + \sum \mathcal{W}$. Then there exists $U \in \mathcal{V}$ with $U \not\leq W + \sum \mathcal{W}$. Since U is simple, $U \cap (W + \sum \mathcal{W}) = 0$. But then $\mathcal{W} \cup \{U\} \in \mathcal{C}$, contradicting the maximality of $\mathcal{W}$. Thus $V = W + \sum \mathcal{W}$ and the definition of $\mathcal{C}$ implies that $V = W \oplus \bigoplus \mathcal{W}$.

(b) By (a) there exists an L-submodule Z of V with $V = Y \oplus Z$. Put $\overline{V} = V/Z$. Then $Y \cong \overline{V}$. Let $W \in \mathcal{V}$ with $W \not\leq Z$. Then $W \cap Z = 0$ and $\overline{W} = W + Z/Z \cong W$. Let $\overline{\mathcal{V}} = \{\overline{W} \mid W \in \mathcal{V}, W \not\leq Z$. Then $\overline{V} = \sum \overline{\mathcal{V}}$. By (a) applied to $\overline{X} \leq \overline{V}$ there exists $\overline{\mathcal{W}} \subseteq \overline{\mathcal{V}}$ with $\overline{V} = \overline{X} \oplus \bigoplus \overline{\mathcal{W}}$. Hence $Y/X \cong \overline{Y}/\overline{X} = \overline{V}/\overline{X} = \bigoplus \overline{\mathcal{W}}$ and so (b) holds.

(c) and (d) follow directly from (b). □

Lemma 1.7.11 [**basic vd**] *Let V be a finite dimensional L-module and $\mathcal{D} \subseteq \mathrm{Sim}(L)$.*

(a) [**z**] *Let $A \leq L$ and $\mathcal{A} \subseteq \mathrm{Sim}(A)$. Then V is an $\mathcal{A}$-module if and only if V is an $\mathcal{A} \mid^L$-module.*

(b) [**a**] *Let $A \leq B \leq C \leq L$ and $\mathcal{A} \subseteq \mathrm{Sim}(A)$. Then $\mathcal{A} \mid^B \mid^C = \mathcal{A} \mid^C$.*

(c) [**b**] *Let W be L-submodule of V. Then V is an $\mathcal{D}$-module if and only if W and V/W are $\mathcal{D}$-modules.*

(d) [**y**] *$V_{\mathcal{D}}$ is the unique maximal semisimple $\mathcal{D}$-submodule in V.*

(e) [**c**] *$V^c_{\mathcal{D}}$ is the unique maximal $\mathcal{D}$-submodule of V.*

(f) [**x**] *Let $\mathcal{E} \subseteq \mathrm{Sim}(L)$. Then $V^c_{\mathcal{D}} \cap V^c_{\mathcal{E}} = V^c_{\mathcal{D} \cap \mathcal{E}}$ and $V^c_{\mathcal{D}} + V^c_{\mathcal{E}} \leq V^c_{\mathcal{D} \cup \mathcal{E}}$.*

(g) [**d**] *Let $I \trianglelefteq \trianglelefteq L$ and $\mathcal{I} \subseteq \mathrm{Sim}(I)$. Put $\mathcal{L} = \mathcal{I} \mid^L$. Then $V^c_{\mathcal{I}}$ is an L submodule and $V^c_{\mathcal{I}} = V^c_{\mathcal{L}}$.*

(h) [**e**] *Suppose $I \leq Z(L)$, $\mathcal{I} \subseteq \mathrm{Sim}(I)$ and $i \in \mathbb{N}$. Then $V_{\mathcal{I}}(i)$ is an L-submodule of V.*

Proof:

(a) Let $\mathcal{S}$ a composition series for L on V and choose a composition series $\mathcal{R}$ for A on V with $\mathcal{S} \subseteq \mathcal{R}$. Then a factor A/B of $\mathcal{S}$ is an $\mathcal{A} \mid^L$-module iff all the factors C/D of $\mathcal{R}$ with $B \leq C < D \leq A$ are $\mathcal{A}$-modules. Thus V is an $\mathcal{A} \mid^L$-module iff each factor T of $\mathcal{S}$ is an $\mathcal{A} \mid^L$-module iff each factor of $\mathcal{R}$ is an $\mathcal{A}$-module iff V is an $\mathcal{A}$-module.

(b) Let X be finite dimensional C-module. Then by (b), the following are equivalent.

X is an $\mathcal{A} \mid^B\mid^C$-module, X is an $\mathcal{A} \mid^B$-module, X is an $\mathcal{A}$-module, X is an $\mathcal{A} \mid^C$-module.

(c) follows from 1.7.2

(d) By 1.7.10(c), $V_{\mathcal{D}}$ is semisimple and by 1.7.10(d), $V_{\mathcal{D}}$ is a $\mathcal{D}$-module. Conversely every semisimple $\mathcal{D}$-module is a sum of simple $\mathcal{D}$-modules and so contained in $V_{\mathcal{D}}$.

(e) Any composition factor of $V_{\mathcal{D}}^c$ is isomorphic to a compostion factor of some $V_{\mathcal{D}}(n+1)/V_{\mathcal{D}}(n)$ and so (by (d)) is a $\mathcal{D}$-module. So $V_{\mathcal{D}}^c$ is a $\mathcal{D}$-module. Conversely let W be a $\mathcal{D}$-submodule and $0 = W_0 < W_1 < \ldots < W_n = W$ an L-composition series on W.

We show by induction on i that $W_i \leq V_{\mathcal{D}}(i)$. For $i = 0$ this is obvious. So suppose $W_i \leq V_{\mathcal{D}}(i)$. Since W_i is a maximal submodule of W_{i+1} we either have $W_{i+1} \cap V_{\mathcal{D}}(i) = W_i$ or W_{i+1}. In the latter case, $W_{i+1} \leq V_{\mathcal{D}}(i+1)$. In the former put $\overline{V} = V/V_{\mathcal{D}}(i)$ and note that $\overline{W_{i+1}} \cong W_{i+1}/W_i$ is a simple $\mathcal{D}$-module. Hence $\overline{W_{i+1}} \leq \overline{V}_{\mathcal{D}}$. Hence the defintion of $V_{\mathcal{D}}(i+1)$ implies $W_{i+1} \leq V_{\mathcal{D}}(i+1)$.

In particular, $W_n \leq V_{\mathcal{D}}(n) \leq V_{\mathcal{D}}^c$ and (e) is proved.

(f) This follow easily from (e). We leave the details to the reader.

(g) Suppose first that I is an ideal in L. Let $W = V_{\mathcal{I}}^c$. We claim that W is a L-submodule. Let $l \in L$. Then by 1.7.4(b), $\phi : W \to V/W, w \to lw + W$ is I-invariant. Hence $\phi(W) \cong W/\ker\phi$ and so by (c), $\phi(W)$ is an $\mathcal{I}$-submodule. Now $\phi(W) = lW + W/W$ and so by (c), $lW + W$ is an $\mathcal{I}$-submodule. According to (e), W is a maximal $\mathcal{I}$-submodule. Thus $lW + W = W$, $lW \leq W$ and W is an L-submodule. By (a) W is an $\mathcal{L}$-submodule. Thus by (e), $W \leq V_{\mathcal{L}}^c$. Also by (a), $V_{\mathcal{L}}^c$ is an $\mathcal{I}$-submodule and thus by (e), $V_{\mathcal{L}}^c \leq W$.

So (g) holds if I is an ideal. In the general can choose $I \trianglelefteq I_1 \trianglelefteq \ldots \trianglelefteq I_{n-1} \trianglelefteq I$. We prove (g) by induction. Let $\mathcal{A} = \mathcal{I} \mid^{I_1}$, since $I \trianglelefteq I_1$, we have $V_{\mathcal{I}}^c$ is an I_1-module and $V_{\mathcal{I}}^c = V_{\mathcal{A}}^c$. Put $J = I_{n-1}$ and $\mathcal{J} = \mathcal{I} \mid^J$. By induction assumption, $V_{\mathcal{I}}^c = V_{\mathcal{J}}^c$. By (b), $\mathcal{L} = \mathcal{I} \mid^L = \mathcal{I} \mid^J\mid^L = \mathcal{J} \mid^L$ and so by the ideal case $V_{\mathcal{J}}^c = V_{\mathcal{L}}^c$. Thus $V_{\mathcal{I}}^c = V_{\mathcal{L}}^c$. As the latter is an L-submodule,so is $V_{\mathcal{I}}^c$ and (g) is proved.

(h) Let $l \in L$. Note that I is and ideal of L. By 1.7.4(c), $lV_{\mathcal{I}}$ is a sum of simple $\mathcal{I}$-modules. So $lV_{\mathcal{I}} \leq V_{\mathcal{I}}$. Thus $V_{\mathcal{I}}$ is an L-submodule. The definition of $V_{\mathcal{I}}(n+1)$ and induction on n now shows that (h) holds. □

Proposition 1.7.12 [**clifford**] *Let I a subideal in L and V a finite dimensional simple L-module. Then any two composition factors for I on V are isomorphic. If in addition $I \leq Z(L)$, then V is an homogenous I-module.*

Proof: Let W be as simple I-submodule in V and $\mathcal{I}$ the isomorphism class of W. Then by 1.7.11 $V_{\mathcal{I}}^c$ is a non-trival L-submodule of V. Since V is simple, $V = V_{\mathcal{I}}^c$. Similarly if $I \leq Z(L)$, $V = V_{\mathcal{I}}$ □

Lemma 1.7.13 (Schur) [**schur**] *Let V be a simple L-module. Then* $\mathrm{End}_L(V)$ *is a skew-field. If* $\mathbb{K}$ *is algebraicly closed and V is finite dimensional, then* $\mathrm{End}_L(V) = K^* = \mathbb{K}\mathrm{id}_V$*, where K^* is the image of* $\mathbb{K}$ *in* $\mathrm{End}_{\mathbb{K}}(V)$*.*

Proof: Let $0 \neq f \in \mathrm{End}_L(V)$. Then $V \neq C_V(f)$ is L-submodule of V and since V is simple, $C_V(f) = 0$. So f is $1-1$. Similarly $V = fV$ and so f is onto. Simple calculations show that $f^{-1} \in \mathrm{End}_L(V)$ and so $\mathrm{End}_L(V)$ is a skew-field. Suppose now that V is finite dimensional and $\mathbb{K}$ is algebraicly closed. Then $\mathrm{End}_L(V)$ is a finite field extension of K^* and so $\mathrm{End}_L(V) = K^*$. □

Lemma 1.7.14 [**simple for abelian**] *Let L be an abelian Lie algebra and V a simple L-module. Put* $\mathbb{D} = \mathrm{End}_L(V)$*. Then $\mathbb{D}$ is a field, V is 1-dimensional over $\mathbb{D}$ and* $\mathbb{D} = K^*(L^*)$*, where K^* and L^* are the images of $\mathbb{K}$ and L in* $\mathrm{End}(V)$*. If $\mathbb{K}$ is algebrilcy closed and V is finite dimensional, then $K^* = \mathbb{D}$ and V is 1-dimensional over $\mathbb{K}$.*

Proof: Note that L^* is abelian and $L^* \leq Z(\mathbb{D})$. Let $\mathbb{E}$ be the subfield of $Z(\mathbb{D})$ generated by K^* and L^*. Let $0 \neq v \in V$. Then $\mathbb{E}v$ is an L-submodule and since V is simple we get $V = \mathbb{E}v$. Hence V is 1-dimensional over $\mathbb{E}$. Moreover, if $d \in \mathbb{D}$, then $dv = ev$ for some $e \in \mathbb{E}$. Then $(d-e)v = 0$, $d = e$, $\mathbb{E} = \mathbb{D}$ and the $\mathbb{E} = \mathbb{D}$.

Suppose in addition that $\mathbb{K}$ is alegbraicly closed and V is finite dimensional. Then $\mathbb{D}$ is a finite extension of K^* and so $\mathbb{D} = K^*$. □

Lemma 1.7.15 [**independence of d spaces**] *Let V be finite dimensional L-module and Δ a partition of* Sim_V*. Then* $(V^c_{\mathcal{D}} \mid \mathcal{D} \in \Delta)$ *is linearly independent, that is*

$$\sum\{V^c_{\mathcal{D}} \mid \mathcal{D} \in \Delta\} = \bigoplus\{V^c_{\mathcal{D}} \mid \mathcal{D} \in \Delta\}.$$

Proof: Let $\mathcal{D} \in \Delta$ and $W = \sum\{V^c_{\mathcal{A}} \mid \mathcal{D} \neq \mathcal{A} \in \Delta\}$. We need to show that $V^c_{\mathcal{D}} \cap W = 0$.

For this put $\mathcal{E} = \bigcup \Delta \setminus \{\mathcal{D}\}$. Then $V^c_{\mathcal{D}}$ is a $\mathcal{D}$-module, W is an $\mathcal{E}$-module and so $V^c_{\mathcal{D}} \cap W$ is an $\mathcal{D} \cap \mathcal{E}$-module. As Δ was a partition, $\mathcal{D} \cap \mathcal{E} = \emptyset$. Hence $V^c_{\mathcal{D}} \cap W = 0$. □

Definition 1.7.16 [**def:trace**] *Let V be a finite dimensional L-module, $\phi : L \to \mathfrak{U}$ be the univeral enveloping algebra and $u \in \mathfrak{U}$. Then* $\mathrm{tr}_V(u) = \mathrm{tr}(u^*)$*, where u^* is the image of u in* $\mathrm{End}(V)$*.* tr_V *denotes the corresponding function* $\mathfrak{U} \to \mathbb{K}, u \to t_V(u)$*. tr^L_V denotes the restriction of tr_V to L.*

Lemma 1.7.17 [**trace and series**] *Let V be a finite dimensional L-module.*

(a) [**a**] tr_V *is $\mathbb{K}$-linear and* $\mathrm{tr}_V(ab) = \mathrm{tr}_V(ba)$ *for all $a, b \in \mathfrak{U}$.*

(b) [**b**] $\mathrm{tr}_V(l) = 0$ *for all $l \in [L, L]$*

(c) [**c**] *Let $\mathcal{W}$ the set of factors of some L-series on V. Then*

$$\mathrm{tr}_V = \sum_{W \in \mathcal{W}} \mathrm{tr}_W$$

(d) [**d**] *If W is an L-module isomorphic to V, then* $\mathrm{tr}_V = \mathrm{tr}_W$.

This follows from elementray facts about traces of linear maps. $\square$

Chapter 2

The Structure Of Standard Lie Algebras

2.1 Solvable Lie Algebras

Put $L^{(0)} = L$ and inductively, $L^{(n+1)} = [L^{(n)}, L^{(n)}]$. We say that L is solvable if $L^{(k)} = 0$ for some $k < \infty$.

Lemma 2.1.1 [**basic solvable**]

(a) [**a**] *Let $I \trianglelefteq L$. Then L is solvable if and only if I and L/I are solvable.*

(b) [**b**] *Let $A, B \leq L$ with $A \leq N_L(B)$. Then $A + B$ is solvable if and only if A and B are solvable.*

(c) [**c**] *Suppose that L is finite dimensional. Then L has a unique maximal solvable ideal $Sol(L)$.*

Proof: (a) If $L^{(k)} = 0$, then $I^{(k)} = 0$ and $(L/I)^{(k)} = 0$. If $I^{(n)} = 0$ and $(L/I)^{(m)} = 0$, then $L^{(m)} \leq I$ and $L^{(m+k)} = L^{(m)(k)} = 0$.

(b) Suppose A and B are solvable. Since $A \leq N_L(B)$, $B \trianglelefteq A + B$. Now B and $A + B/B \cong A/A \cap B$ are solvable and so by (a) $A + B$ is solvable.

(c) Since L is finite dimensional, there exists a maximal solvable ideal B in L. Let A be any solvable ideal in L. Then by (b), $A + B$ is solvable ideal and so by maximality of B, $A \leq B$. □

Lemma 2.1.2 [**nilpotent is solvable**]

(a) [**a**] $L^{(k+1)} \leq L^k * L$.

(b) [**b**] *Any nilpotent Lie algebra is solvable.*

(c) [**c**] *If* $\mathrm{Nil}(L) = 0$*, then* $\mathrm{Sol}(L) = 0$.

Proof: (a) By induction on k: The statement is obviously true for $k = 1$. Suppose $L^{(k)} \leq L^{k-1} * L$. Then

$$\begin{aligned} L^{(k+1)} &= [L^{(k)}, L^{(k)}] \\ &\leq [L, L^{k-1} * L] \\ &= L * (L^{k-1} * L) \\ &= L^k * L \end{aligned}$$

(b) follows from (a).

(c) If $\mathrm{Sol}(L) \neq 0$ the last non-trival term of the derived series of L is an abelian and so nilpotent ideal in L. □

Write $L' = [L, L] = L^{(1)}$. We say that L is perfect if $L = L'$. Let $L^{(\infty)}$ be the sum of the perfect ideals in L. Then $L^{(\infty)}$ is perfect and so the unique maximal perfect ideal in L.

If L is finite dimensional there exists $k \in \mathbb{N}$ with $L^{(k)} = L^{(k+1)}$. It follows that $L^{(\infty)} = L^{(k)}$, $L/L^{(\infty)}$ is solvable, $L^{(\infty)}$ is the unique ideal minimal such that $L/L^{(\infty)}$ is solvable and $L^{(\infty)}$ is the unique maximal perfect subalgebra in L.

Definition 2.1.3 [**standard**] *We say* $\mathbb{K}$ *is standard if* $\mathrm{char}\, K = 0$ *and* $\mathbb{K}$ *is algebraicly closed. We say that* L *is standard if* $\mathbb{K}$ *is standard and* L *is finite dimensional. We say that the* L*-module* V *is standard if* L *is standard and* V *is finite dimensional.*

Proposition 2.1.4 [**sol and simple**] *Let* V *be a simple, standard* L*-module.*

(a) [**a**] $[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Sol}(L) \cap \ker \mathrm{tr}_V = \mathrm{Sol}(L) \cap C_L(V)$.

(b) [**b**] *The elements of* $\mathrm{Sol}(L)$ *act as scalars on* V*.*

Proof:

(a) Let $I = \mathrm{Sol}(L) \cap \ker \mathrm{tr}_V$. Obviously $[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L'$ and $\mathrm{Sol}(L) \cap C_L(V) \leq I$. By 1.7.17(b), $L' \leq \ker \mathrm{tr}_V$, so we also have $\mathrm{Sol}(L) \cap L' \leq I$. It remains to show to show that $I \leq \mathrm{Sol}(L) \cap C_L(V)$. We now let $\bar{L} = L/C_L(V)$ and consider $\bar{V} = V$ as an $\bar{L}$-module. Then $\bar{V}$ is a faithful, simple, standard $\bar{L}$-module. Note that $\bar{V}$ and V has the same set. Now we want to show that $\bar{I} := \mathrm{Sol}(\bar{L}) \cap \ker \mathrm{tr}_{\bar{V}} = 0$ If not, let k be the derived length of $\bar{I}$ and put $J = \bar{I}^{(k-1)}$. Then J is a non-trivial abelian ideal in $\bar{L}$ and $\mathrm{tr}_{\bar{V}}(J) = 0$. Let $0 \neq j \in J$ and let Z be a simple J-submodule in $\bar{V}$. Since $\mathbb{K}$ is algebraicly closed, 1.7.14 implies that Z is 1-dimensional over $\mathbb{K}$. Hence there exists $k \in \mathbb{K}$ with $jz = kz$ for all $z \in Z$. By 1.7.12 all composition factors for J on $\bar{V}$ are isomorphic and so 1.7.17 implies that $0 = \mathrm{tr}_{\bar{V}}(j) = \dim \bar{V} \cdot k$. Since $\mathrm{char}\, \mathbb{K} = 0$ we get $k = 0$. Thus $J \leq \mathrm{Nil}_{\bar{L}}(\bar{V}) \leq C_{\bar{L}}(\bar{V}) = 0$. Now note that $\bar{I} := \mathrm{Sol}(\bar{L}) \cap \ker \mathrm{tr}_{\bar{V}} = (\mathrm{Sol}(L)/C_L(V)) \cap (\ker \mathrm{tr}_V/C_L(V)) = I/C_L(V)$, so $\bar{I} = 0$ implies $I \leq C_L(V)$. Hence $I \leq \mathrm{Sol}(V) \cap C_L(V)$ and (a) is proved.

(b) Contiune to using the notations $\bar{L}$ and $\bar{V}$ as above. By (a), $[\mathrm{Sol}(\bar{L}), \bar{L}] \leq C_{\bar{L}}(\bar{V}) = 0$. Thus $\mathrm{Sol}(\bar{L}) \leq Z(\bar{L})$. Hence by 1.7.12 $\bar{V}$ is a homogeneous $\mathrm{Sol}(\bar{L})$-module. Now $\mathrm{Sol}(\bar{L})$ is abelian and by 1.7.14 all the simple $\mathrm{Sol}(\bar{L})$ submodules in $\bar{V}$ are 1-dimensional. Therefore elements of $\mathrm{Sol}(\bar{L})$ act as scalars on $\bar{V}$. Since $\mathrm{Sol}(\bar{L}) = \mathrm{Sol}(L)/C_L(V)$ and $\bar{V} = V$ as a $\mathbb{K}$-space, elements $\mathrm{Sol}(L)$ also act as scalars on V. Hence (b) holds.

Theorem 2.1.5 (Lie) [**lie**] *Let V be a standard L-module.*

$$[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V).$$

Proof: Let W be a composition factor for L on V. By 2.1.4 $\mathrm{Sol}(L) \cap L' \leq C_L(W)$ and so $\mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V)$. □

Corollary 2.1.6 [**solvable and flags**] *Suppose that L is solvable and V is a standard L-module. Then*

(a) [**a**] $L' \leq \mathrm{Nil}_L(V)$.

(b) [**b**] *If V is simple, then V is 1-dimensional.*

(c) [**c**] *There exists a series of L-submodules $0 = V_0 \leq V_1 \leq \ldots \leq V_n = V$ with $\dim V_i = i$.*

(d) [**d**] $\mathrm{Nil}_L(V) = \{l \in L \mid l$ *acts nilpotently on* $V\}$.

Proof: By 2.1.6(b) L acts as scalars on any composition factor for L on V. Thus (a)-(c) holds.

(d) Clearly each elements of $\mathrm{Nil}_L(V)$ acts nilpotently on V. Now let $l \in L$ act nilpotently on V. Then l also acts nilpotently any every composition factor W of L on V. (b) implies that l centralizes W and so $l \in \mathrm{Nil}_L(V)$.

Corollary 2.1.7 [**[sol l, l] nilpotent**] *Let L be standard. Then*

(a) [**a**] $[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}(L)$.

(b) [**b**] *If L is solvable then L' is nilpotent and there exists a series of ideals $0 = L_0 \leq L_1 \leq \ldots \leq L_n = L$ in L with $\dim L_i = i$.*

Proof: Apply 2.1.5 and 2.1.6 to V being the adjoint module L.

2.2 Tensor products and invariant maps

Let V, W and Z be L-module. Then L acts on $V \otimes W$ by

$$l(v \otimes w) = (lv) \otimes w + v \otimes (lw)$$

and L acts on $\mathrm{Hom}(V, W)$ by

$$(l\phi)(v) = l(\phi(v)) - \phi(l(v)).$$

In particular, if we view $\mathbb{K}$ as a trivial L-module, L acts on $V^* := \mathrm{Hom}(V, \mathbb{K})$ by

$$(l\phi)(v) = -\phi(lv).$$

Let $X \subseteq L$ and $\phi \in \mathrm{Hom}(V, W)$. We say that ϕ is X-invariant if $\phi(lv) = l(\phi(v))$ for all $v \in V$ and $l \in X$. Note that this is the case if and only if $l\phi = 0$ for all $l \in X$. $\mathrm{Hom}_X(V, W)$ denotes all the X-invariant $\mathbb{K}$-linear maps from V to W. So $\mathrm{Hom}_X(V, W)$ is just the centralizer of X in $\mathrm{Hom}(V, W)$. Let $f : V \times W \to Z$ be $\mathbb{K}$-bilinear. Then f gives rise to a unique $\mathbb{K}$-linear map $\widetilde{f} : V \otimes W \to Z$ with $\widetilde{f}(v \otimes w) = f(v, w)$. We say that f is X invariant if $\widetilde{f}$ is X invariant. So f is X-invariant if and only if

$$f(lv, w) + f(v, lw) = l(f(v, w))$$

for all $l \in X, v \in V$ and $w \in W$. In the special case that Z is a trivial L-module we see that f is X-invariant if and only if

$$f(lv, w) = -f(v, lw)$$

for all $l \in X, v \in V$ and $w \in W$.

Note that the sets of all l in L which leave f invariant (that f is l-invariant) is equal to $C_L(\widetilde{f})$ and so forms a subalgebra of L.

Let $f : V \times W \to Z$ be $\mathbb{K}$-bilinear. For $X \subset V$ define

$$X^{\perp} = \{w \in W \mid f(x, w) = 0 \,\forall x \in X\}.$$

Similarly for $Y \subseteq W$ define

$$^{\perp}Y := \{v \in V \mid f(v, y) = 0 \,\forall y \in Y\}.$$

f is called non-degenerate, if $V^{\perp} = {}^{\perp}W = 0$.

Consider now the case where $V = W$. We say that f is symmetric if (for all $v, w \in W$) $f(v, w) = f(w, v)$, f is alternating if $f(v, w) = -f(w, v)$ and f is sympletic if $f(v, v) = 0$. Note that if f is symplectic then f is alternating. We say that f is $\perp$-symmetric, provided that $f(v, w) = 0$ if and only if $f(w, v) = 0$. Observe that if f is symmetric or alternating, then f is $\perp$-symmetric.

If f is $\perp$-symmetric then $V^{\perp} = {}^{\perp}V$ and we define $\mathrm{rad}(f) = V^{\perp}$.

Lemma 2.2.1 [**basic bilinear**] *$f : V \times W \to Z$ a L-invariant and $\mathbb{K}$-bilinear. Let X be a L-submodule of V then $X^\perp$ is L-submodule of W.*

Proof: Let $w \in X^\perp$, $l \in L$ and $x \in X$. Then $lx \in X$ and so

$$f(x, lw) = lf(x, w) - f(lx, w) = l0 - 0 = 0.$$

Thus $lw \in X^\perp$ and $X^\perp$ is a submodule of W. □

Lemma 2.2.2 [**multiplications are invariant**]

(a) [**a**] *Let V be an L-module. Then the map $\mathfrak{l}(\mathfrak{U}) \times V \to V, (u, v) \to uv$ is L-invariant.(Here we view $\mathfrak{l}(\mathfrak{U})$ as an L-module via the adjoint representation.)*

(b) [**b**] *$L \times L \to L, (a, b) \to [a, b]$ is L-invariant.*

(c) [**c**] *$L \times \mathfrak{U} \to \mathfrak{U}, (a, u) \to au$ is L-invariant. (Here we view $\mathfrak{U}$ as an L-module via left multiplication.)*

(d) [**d**] *$L \times L \to \mathfrak{l}(\mathfrak{U}), (a, b) \to ab$ is L-invariant. (Here we view $\mathfrak{l}(\mathfrak{U})$ as an L-module via the adjoint representation.)*

Proof: (a) Let $a \in L$, $u \in U$ and $v \in V$. Define $f(u, v :) = uv$. Then

$$f(a * u, v) + f(u, av) = [a, u]v + u(av) = a(uv) = af(u, v).$$

(b) and (c) are special cases of (a).
(d) Let a, b, c in L and define $f(b, c) := bc$. Then using 1.3.3 b

$$f(a * b, c) + f(b, a * c) = [a, b]c + b[a, c] = [a, bc] = a * f(b, c).$$

□

2.3 A first look at weights

Definition 2.3.1 [**def:weights**] *A weight for L is a Lie-algebra homomorphism $\lambda : L \to \mathfrak{l}(K)$. $\Lambda(L) = \mathrm{Hom}_{Lie}(L, \mathfrak{l}(\mathbb{K}))$ is the set of all weights of L.*

Note that a weight for Λ is nothing else as $\mathbb{K}$-linear map $\lambda : L \to \mathbb{K}$ with $L' \leq \ker \lambda$. Thus $\Lambda(L) \cong \Lambda(L/L') = (L/L')^*$. For a weight λ we denote by $\mathbb{K}_\lambda$ the L-module with action $L \times \mathbb{K} \to \mathbb{K}, (l, k) \to \lambda(l)k$.

Lemma 2.3.2 [**weights and simple**] *The map $\lambda \to \mathbb{K}_\lambda$ is a one to one correspondence between weights of L and isomorphism classes of 1-dimensional L-modules.*

Proof: Let V be a 1-dimensional L-module. Then $lv = \mathrm{tr}_V(l)v$ for all $l \in L, v \in V$ and so tr_V is a weight and $V \cong \mathbb{K}_{\mathrm{tr}_V}$. Clearly two 1-dimensional L-modules are isomorphic if and only their trace functions are equal. $\square$

Corollary 2.3.3 [**simple for solvable**] *Let L be standard and solvable. Then the map $\lambda \to K_\lambda$ is one to one correspondence between the weights of L and finite dimensional simple L-modules.*

Proof: 2.1.6(b) and 2.3.2. $\square$

Let λ be a weight for L. Since λ corresponds to an isomorphism class of simple L-modules we obtain from Definition 1.7.8 the notations V_λ, $V_\lambda(i)$ and V_λ^c. V_λ is called the weight space for λ on V.

A weight λ for L on V is a weight with $V_\lambda \neq 0$. $\Lambda_V = \Lambda_V(L)$ is the set of weights for L on V. V_λ^c is called the generalized weight space for λ on V. We also will write $V_\lambda(\infty)$ for V_λ^c

Lemma 2.3.4 [**weights and eigenspaces**] *Suppose that $L = \mathbb{K}l$ is 1-dimensional, λ a weight of for L, $k = \lambda(l)$ and $n \in \mathbb{N}$.*

(a) [**a**] *V_λ is the eigenspace for l on V corresponding to k,*

(b) [**b**] *$V_\lambda(n) = C_V((k-l)^n)$.*

(c) [**c**] *V_λ^c is the generalized eigenspace for l on V corresponding to k.*

Proof: (a) By definition V_λ is the sum of all L-submodules isomorphic to $\mathbb{K}_\lambda$. Since $(k-l)\mathbb{K}_\lambda = 0$, $V_\lambda \leq C_V(k-l)$. Clearly $C_V(k-l)$ is the sum of submodules isomorphic to $\mathbb{K}_\lambda$. For if $\beta = \{v_1, ..., v_n\}$ is a $\mathbb{K}$-basis for $C_V(k-l)$, $\mathbb{K}v_i \cong \mathbb{K}_\lambda$ for all $i = 1, ..., n$. So $V_\lambda = C_V(k-l)$.

(b) For $n = 0$ both sides are 0. By induction we may assume $W = V_\lambda(n-1) = C_V((k-l)^{n-1})$. Applying (a) to $\bar{V} = V/W$ we get

$$V_\lambda(n)/W = \bar{V}_\lambda = C_{\bar{V}}(k-l) = C_V((k-l)^n)/W.$$

So (b) holds.

(c) follows from (b). $\square$

Lemma 2.3.5 [**weights and invariant maps**] *Let $f : V \times W \to Z$ be L-invariant, $\mathbb{K}$-bilinear map of L-modules. Let λ and μ be weights of L and $i, j \in \mathbb{N} \cup \{\infty\}$. Then*

$$f\left(V_\lambda(i), W_\mu(j)\right) \leq Z_{\lambda+\mu}(i+j-1).$$

Proof: We first consider the case $i = j = 1$. Let $l \in L$,$v \in V_\lambda$ and $w \in W_\mu$. Then

$$\begin{aligned} lf(v,w) = {} & f(lv,w) + f(v,lw) && = f(\lambda(l)v,w) + f(v,\mu(l)w) \\ = {} & \lambda(l)f(v,w) + \mu(l)f(v,w) && = (\lambda+\mu)(l)f(v,w). \end{aligned}$$

So the lemma holds in this case.

Also the lemma is obviously true for $i = 0$ or $j = 0$. If the lemma holds for all finite i and j it also holds for $i = \infty$ or $j = \infty$.

So assume $1 \le i < \infty$ and $1 \le j < \infty$. By induction on $i + j$ we also may assume that

$$f(V_\lambda(i-1), W_\mu(j)) \le Z_{\lambda+\mu}(i+j-2) \text{ and } f(V_\lambda(i), W_\mu(j-1)) \le Z_{\lambda+\mu}(i+j-2).$$

Put $\bar{X} = V_\lambda(i)/V_\lambda(i-1)$, $\bar{Y} = W_\mu(j)/W_\mu(j-1)$ and $\bar{Z} = Z/Z_{\lambda+\mu}(i+j-2)$. Then we obtain a well defined L-invariant map $\bar{f} : \bar{X} \times \bar{Y} \to \bar{Z}$ with $\bar{f}(\bar{v},\bar{w}) = \overline{f(v,w)}$ for all $v \in V_\lambda(i)$ and $w \in W_\mu(j)$. Note that $\bar{X} = \bar{X}_\lambda$ and $\bar{Y} = \bar{Y}_\mu$. So by the "$i = j = 1$"-case we get that $\bar{f}(\bar{X},\bar{Y}) \le \bar{Z}_{\lambda+\mu}$. Taking inverse images in V, W and Z we see that the lemma holds. □

Corollary 2.3.6 [**weight formula**] *Let V be an L modules, $A \le L$, λ and μ weights for A and $i, j \in \mathbb{N} \cup \{\infty\}$*

(a) [**a**] $L_\lambda(i)V_\mu(j) \le V_{\lambda+\mu}(i+j-1)$

(b) [**b**] $[L_\lambda(i), L_\mu(j)] \le L_{\lambda+\mu}(i+j-1)$.

Proof: By 2.2.2 the map $(l, v) \to v$ is L- and so also A-invariant. Hence (a) follows from 2.3.5. (b) is just a special case of (a). □

2.4 Minimal non-solvable Lie algebras

Proposition 2.4.1 [**minimal non solvable**] *Let L be a standard Lie algebra such that all proper subalgebras are solvable but L is not solvable. Then*

(a) [**a**] $L = L'$.

(b) [**b**] $\mathrm{Sol}(L)$ *is the unique maximal ideal in* L.

(c) [**c**] $L/\mathrm{Sol}(L)$ *is simple.*

(d) [**d**] $\mathrm{Sol}(L) = C_L(W)$, *where W is any non-trivial, finite dimensional simple L-module.*

(e) [**e**] $\mathrm{Sol}(L) = \mathrm{Nil}_L(V)$, *where V is any non-trivial, finite dimensional L-module.*

(f) [**f**] $\mathrm{Sol}(L) = \mathrm{Nil}(L)$.

Proof: (a) If $L' \neq L$, then both L' and L/L' are solvable. Thus L is solvable, a contradiction.

(b) Let I be any proper ideal in L. Then I is solvable and so $I \leq Sol(L)$.

(c) By (b), $L/\mathrm{Sol}(L)$ has no proper ideals.

(d). Since W is non-trivial, $C_L(W) \neq L$. Thus $C_L(W) \leq \mathrm{Sol}(L)$. By (a), $\mathrm{Sol}(L) \leq L'$ and so by 2.1.4, $\mathrm{Sol}(L) \leq C_L(W)$.

(e) Note that by (a) $L/C_L(V)$ is perfect. If L acts nilpotently on V, then 1.6.2 implies that $L/C_L(V)$ is nilpotent and perfect, and so trivial. This contradictions shows that $L \neq \mathrm{Nil}_L(V)$. By (d) $\mathrm{Sol}(L) \leq \mathrm{Nil}_L(V)$ since $\mathrm{Nil}_L(V) = \cap\{C_L(W) \mid W \text{ is a factor of } V\}$ and so (b) implies $\mathrm{Sol}(L) = \mathrm{Nil}_L(V)$.

(f) Apply (e) to $V = L$. □

Theorem 2.4.2 [**minimal simple**] *Let L be a non-solvable, standard simple Lie-algebra all of whose proper subalgebra are solvable. Then $L \cong \mathfrak{sl}(\mathbb{K}^2)$.*

Proof: For $X \leq L$ let $\widetilde{X} = \mathrm{Nil}_X(L)$. Also let $\mathcal{N}$ be the set of elements in L acting nilpotently on L.

1° [**1**] *Let $X \lneq L$, then $X' \leq \widetilde{X} = X \cap \mathcal{N}$. and $\widetilde{X}$ is a nilpotent ideal in X.*

Since $X \neq L$, X is solvable by assumption. Thus 2.1.5 $X' \leq \mathrm{Nil}_X(L)$ and by 2.1.6(d), $\widetilde{X} = X \cap \mathcal{N}$. Since L is non-abelian, $L \neq Z(L)$ and since L is simple, $Z(L) = 0$. Thus L is a faithful L-module and so by 1.6.2 $\widetilde{X}$ is nilpotent.

Let A and B be distinct maximal subalgebras of L and $D = A \cap B$

2° [**2**] $L = A + B$ *and* $\dim L/A = 1 = \dim A/D = \dim B/D = \dim L/B$.

Since A is solvable 2.1.6 applied to $V = L/A$ implies that there exists a 1-dimensional A submodule W/A in L/A. Let $w \in W \setminus A$. Then $[A, W] \leq W$ as W is a A-submodule of L. Also $W = A + \mathbb{K}w$ and so $[w, W] = [w, A] \leq W$. Thus $[W, W] \leq W$, that is W is a subalgebra of L. The maximality of A implies $L = W$. So $\dim L/A = 1$. By symmetry $\dim L/B = 1$. Since $A \neq B$, $L = A + B$. Thus $A/D = A/A \cap B \cong A + B/B = L/B$ and (2°) holds.

3° [**3**] *$\widetilde{D}$ is an ideal in A.*

If $\widetilde{D} = \widetilde{A}$, this is obvious. So suppose that $\widetilde{A} \neq \widetilde{D}$. Since $\widetilde{D}$ acts nilpotenly on $\widetilde{A}$ we get from 1.6.7 that $\widetilde{D} \lneq N_{\widetilde{A}}(\widetilde{D})$. By (1°), $\widetilde{A} \cap D = (A \cap \mathcal{N}) \cap D = \widetilde{D}$ and so $N_{\widetilde{A}}(\widetilde{D}) \not\leq D$. For if $N_{\widetilde{A}}(\widetilde{D}) \leq D$, then $N_{\widetilde{A}}(\widetilde{D}) \leq \widetilde{A} \cap D = \widetilde{D}$ which leads to a contradiction. By (2°) A/D is 1-dimensioanal and so

$$A = N_{\widetilde{A}}(\widetilde{D}) + D \leq N_A(\widetilde{D}).$$

Thus (3°) holds.

4° [**4**] $A' \cap D = B' \cap D = \widetilde{D} = 0.$

By (3°), $\widetilde{D}$ is an ideal in A and by symmetry also in B. Thus $\widetilde{D}$ is an ideal in $L = A+B$, and as L is simple, $\widetilde{D} = 0$. By (1°) $A' \cap D \le \widetilde{A} \cap D = \widetilde{D}$. Thus (4°) holds.

5° [**5**] *D is abelian and A' is at most 1-dimensional.*

By (4°) $D' \le A' \cap D = 0$. Also $A' \cong A'/A' \cap D \cong A' + D/D \le A/D$ and so by (2°), A' is at most 1-dimensional.

Let $a \in A \setminus D$ with $a \in A'$ if possible. If $A' = 0$ the $[a, A] = 0$ and if $A' \neq 0$, then by (5°), $A' = \mathbb{K}a$. In any case $\mathbb{K}a$ is an ideal in A. Let $\lambda_A = \mathrm{tr}^D_{\mathbb{K}a}$. Similarly, define $b \in B$ and λ_B.

6° [**6**] $L = \mathbb{K}a \oplus D \oplus \mathbb{K}b$ *and* $\lambda_A = -\lambda_B$.

The first statement follows immediately from (2°). In particular $\mathrm{tr}^D_L = \lambda_A + \mathrm{tr}^D_D + \lambda_B$. Since D is abelian, $\mathrm{tr}^D_D = 0$. Since $L = [L, L]$, $\mathrm{tr}^D_L = 0$. Thus (6°) holds.

7° [**7**] $\ker \lambda_A = \ker \lambda_B = 0$ *and D is one-dimensional.*

Note that $[\ker \lambda_A, \mathbb{K}a + D] = 0$ and since $A = \mathbb{K}a + D$ we get $\ker \lambda_A \le Z(A)$. By (6°), $\ker \lambda_A = \ker \lambda_B$ and so $\ker \lambda_A \le Z(A) \cap Z(B) \le Z(L) = 0$. If $D = 0$, $\dim L = 2$ and L is solvable by 1.2.1, a contradiction. Thus $D \neq 0$. Since $\dim D = \dim(D/\ker \lambda_A) = \dim \lambda_A(D) \le \dim \mathbb{K} = 1$ we conclude that D is 1-dimensional.

In particular, we have $\lambda_A \neq 0$ and so λ_A is onto and there exists $d \in D$ with $\lambda_A(d) = 1$.

Also note that $0, \lambda_A$ and λ_B are the weights of D on V and are pairwise distinct. Also $\mathbb{K}a \le L_{\lambda_A}(D), \mathbb{K}b \le L_{\lambda_B}(D)$ and $D \le L_0(D)$. Thus 1.7.15 implies that $L_{\lambda_A}(D) = \mathbb{K}a$, $L_{\lambda_B}(D) = \mathbb{K}b$ and $L_0(D) = D$. Now 2.3.6 shows that $[a, b] \in D$. Suppose that $[a, b] = 0$. Then A is an ideal in L, a contradiction. Thus $[a, b] = kd$ for some non-zero $k \in \mathbb{K}$. Replacing b by $k^{-1}b$ we may assume $[a, b] = d$. Also from $[d, a] = a$ and $\lambda_B = -\lambda_A$ we have $[d, b] = -b$. Thus

8° [**8**] *a, b, d is a basis for L,* $[a, b] = d$, $[d, a] = a$ *and* $[b, d] = b$.

From (8°) we see that L is unique up to isomorphism. Since $\mathfrak{sl}(\mathbb{K}^2)$ fullfils the assumptions of the theorem we get $L \cong \mathfrak{sl}(\mathbb{K}^2)$. □

2.5 The simple modules for $\mathfrak{sl}(\mathbb{K}^2)$

In this section $L = \mathfrak{sl}(\mathbb{K}^2)$. Let $x = E_{12}, y = E_{21}$ and $h = E_{11} - E_{22}$. Then (x, y, h) is basis for L with $[h, x] = 2x$, $[y, h] = 2y$ and $[x, y] = h$. We call (x, y, h) the Chevalley basis for L.

Lemma 2.5.1 [**autos for sl2**] *Let $L = \mathfrak{sl}(\mathbb{K}^2)$ with Chevalley basis (x, y, h).*

(a) [**a**] *Let $\Phi : L \to L$ be the $\mathbb{K}$-linear map with $\Phi(x) = x, \Phi(y) = y$ and $\Phi(h) = -h$. Then Φ is an anti-automorphism of L.*

(b) [**b**] *Let $\Phi : L \to L$ be the $\mathbb{K}$-linear map with $\Phi(x) = y, \Phi(y) = x$ and $\Phi(h) = -h$. Then Φ is an automorphism of L.*

(c) [**c**] *Let $\Phi : L \to L$ be the $\mathbb{K}$-linear map with $\Phi(x) = y, \Phi(y) = x$ and $\Phi(h) = h$. Then Φ is an anti-automorphism of L.*

Proof: Readily verified from commutator relations of (x, y, h). □

Lemma 2.5.2 [**u for sl2**] *Let $L = \mathfrak{sl}(\mathbb{K}^2)$ with Chevalley basis (x, y, h) and let $i \in \mathbb{Z}_+$. Then the following holds in $\mathfrak{U}$.*

(a) [**a**] $hy^i = y^i(h - 2i)$

(b) [**b**] $xy^i = y^ix + iy^{i-1}(h - (i - 1))$.

(c) [**c**] $yx^i = xy^i - ix^{i-1}(h + i - 1)$

Proof: Readily verified using the commutator relations and induction on i. □

Corollary 2.5.3 [**u for sl2 in char 0**] *Let $L = \mathfrak{sl}(\mathbb{K}^2)$ with Chevalley basis (x, y, h). Suppose* $\operatorname{char}\mathbb{K} = 0$ *and define $x^{(i)} = \frac{1}{i!}x^i$ and $y^{(i)} = \frac{1}{i!}y^i$. Let $i \in \mathbb{Z}_+$. Then*

(a) [**a**] $hy^{(i)} = y^{(i)}(h - 2i)$.

(b) [**b**] $xy^{(i)} = y^{(i)}x + y^{(i-1)}(h - (i - 1))$.

This follows immediately from 2.5.2 □

Theorem 2.5.4 [**modules for sl2**] *Suppose $\mathbb{K}$ is standard, $L = \mathfrak{sl}(\mathbb{K}^2)$, V is a an L-module and (x, y, h) is the Chevalley basis for L. Let $k \in \mathbb{K}$ and $0 \neq v \in V$.*

(a) [**a**] *If V is finite dimensional, then there exists $0 \neq v \in V$ and $k \in \mathbb{K}$ with $xv = 0$ and $hv = kv$.*

(b) [**b**] *Suppose that there exist $0 \neq v \in V$ and $k \in \mathbb{K}$ with $xv = 0$ and $hv = kv$. Let $m \in \mathbb{N}$ be minimal with $y^{m+1}v = 0$, if such an m exists and $m = \infty$ otherwise. Also let $W = \mathfrak{U}v$ be the smallest L-submodule of V containing v. Put $v_i = \frac{y^i}{i!}v = y^{(i)}v$. Then*

(a) [**z**] *$(v_i \mid i \in \mathbb{N}, i \leq m)$ is a basis for W.*

(b) [**a**] *$yv_i = (i+1)v_{i+1}$.*

(c) [**b**] *$hv_i = (k-2i)v_i$.*

(d) [**c**] *$xv_i = (k-(i-1))v_{i-1}$, where $v_{-1} = 0$.*

(e) [**d**] *$xyv_i = (i+1)(k-i)v_i$*

(f) [**e**] *If $m < \infty$, then $m = k = \dim W - 1$.*

Proof: (a) Let $A = \mathbb{K}x + \mathbb{K}h$. Then A is solvable and $A' = \mathbb{K}x$. Let V_0 be a simple A-submodule in V. Then by 2.1.6 V_0 is 1-dimensional. Let $0 \neq v_0 \in V_0$. Then since $A' = \mathbb{K}x$, $xv_0 = 0$ and $hv_0 = kv_0$ for some $k \in \mathbb{K}$.

(b) $yv_i = yy^{(i)}v_0 = (i+1)y^{(i+1)}v_0 = (i+1)v_{i+1}$ and so (b:b) holds.

From 2.5.3(a) we have

$$hv_i = hy^{(i)}v_0 = y^{(i)}(h-2i)v_0 = y^{(i)}(k-2i)v_0 = (k-2i)v_i$$

and so (b:c) holds. From 2.5.3(b)

$$xv_i = xy^{(i)}v_0 = (y^{(i)}x + y^{(i-1)}(h-(i-1))v_0 = 0 + y^{(i-1)}(k-(i-1))v_0 = (k-(i-1))v_{i-1}$$

and (b:d) holds. (b:e) follows from (b:b) and (b:d).

By (b:c) v_i is an eigenvector with eigenvalue $k-2i$ for h. Thus the non-zero v_i's are linearly independent. From (b:b),(b:c) and (b:d) the $\mathbb{K}$-space spanned by $v_i's$ invariant under L and so is equal to W, which means it is an L-module by the property of W. Thus (b:a) holds.

Suppse now that $m < \infty$. By (b:d) with $i = m+1$ we get

$$0 = x0 = xv_{m+1} = (k-m)v_m$$

As $v_m \neq 0$ and $\mathbb{K}$ is a field, $k = m$. Thus (b:f) holds. □

2.6 Non-degenerate Bilinear Forms

In this section we establish some basic facts about non-degenerate bilinear forms that will be of use later on.

Lemma 2.6.1 [**basic non-deg bilinear**] *Let V and W be finite dimensional K-spaces and $f : V \times W \to K$ be non-degenerate and $\mathbb{K}$-bilinear.*

(a) [**a**] *There exists a unique $\mathbb{K}$-isomorphism $t : W^* \to V, \alpha \to t_\alpha$ with $\alpha(w) = f(t_\alpha, w)$ for all $\alpha \in W^*, w \in W$. In particular,* $\dim V = \dim W$.

(b) [**b**] *Let $(w_i \mid i \in I)$ be a basis for W. Then there a unique basis $(v_i \mid i \in I)$ of V with $f(v_i, w_j) = \delta_{ij}$ for all $i, j \in I$.*

(c) [**c**] *Let X be a subspace of V. Then* $\dim X + \dim X^\perp = \dim V$. *In particular, $X = V$ if and only if $X^\perp = 0$.*

Proof: (a) Note first that if we define $\phi_i \in W^*$ by $\phi_i(w_j) = \delta_{ij}$, then $(\phi_i \mid i \in I)$ is a basis for W^*. In particular, $\dim W = \dim W^*$. Since f is non-degenerate the map $\Psi : V \to W^*$ with $\Psi(v)(w) = f(v, w)$ is one to one. Thus $\dim V =\leq \dim W^* = \dim W,$. By symmetry $\dim W \leq \dim V$. So $\dim V = \dim W$ and Ψ is an isomorphism. Putting $t = \Psi^{-1}$ we see that (a) holds

(b) Just put $w_i = t_{\phi_i}$.

(c) The form $X \times W/X^\perp, (x, w + X^\perp) \to f(x, y)$ is well defined and non-degenerate. Thus by (a) $\dim X = \dim W/X^\perp$ and so (c) holds. □

Definition 2.6.2 [**def:omega**] *A quadratic form on the $\mathbb{K}$-space V is a map $q : V \to \mathbb{K}$ such that $q(kv) = k^2 q(v)$ for all $k \in \mathbb{K}, v \in V$ and such that the function $s : V \times V \to \mathbb{K}, (v, w) \to q(v + w) - q(v) - q(w)$ is $\mathbb{K}$-bilinear. Note that s is symmetric. We call s the bilinear form associated to q. Let $u \in V$ with $q(u) \neq 0$. Define $\breve{u} = q(u)^{-1}u$ and*

$$\omega_u : V \to V, v \to v - s(v, \breve{u})u = v - \frac{s(v, u)}{q(u)}u.$$

Lemma 2.6.3 [**omega u**] *Let V be a $\mathbb{K}$-space, $q : V \to \mathbb{K}$ a quadratic form with associated bilinear form s, $u \in V$ with $q(u) \neq 0$.*

(a) [**c**] *$s(v, v) = 2q(v)$ for all $v \in V$.*

(b) [**d**] *$q(\breve{u}) = q(u)^{-1} \neq 0$, $\breve{\breve{u}} = u$, $s(u, \breve{u}) = 2$ and $\omega_u(u) = -u$.*

(c) [**a**] *Let $0 \neq k \in \mathbb{K}$. Then $\breve{ku} = k^{-1}\breve{u}$ and $\omega_{ku} = \omega_u$. In particular, $\omega_u = \omega_{\breve{u}}$.*

(d) [**e**] *ω_u is an isometry of q.*

(e) [**f**] *Let σ be an isometry of q. Then $\sigma(\breve{u}) = \breve{\sigma(u)}$ and $\sigma\omega_u\sigma^{-1} = \omega_{\sigma(u)}$.*

Proof:

(a) We have $q(2v) = 4q(v)$ and $q(v + v) = q(v) + s(v, v) + q(v)$.

(b) $q(\breve{u}) = q(q(u)^{-1}u) = q(u)^{-2}q(u) = q(u)^{-1}$. So $\breve{\breve{u}} = (q(u)^{-1})^{-1}q(u)^{-1}u = u$. Also and $s(u, \breve{u}) = \frac{s(u,u)}{q(u)}$. So by (a), $s(u, \breve{u}) = 2$ and hence $\omega_u(u) = u - 2u = -u$.

(c) $\breve{ku} = q(ku)^{-1}ku = k^{-2}q(u)^{-1}ku = k^{-1}\breve{u}$ and

$$\omega_{ku}(v) = v - s(v, \breve{ku})ku = v - s(v, k^{-1}\breve{u})ku = v - s(v, \breve{u})u = \omega_u(v).$$

(d)

$$\begin{aligned} q(\omega_u(v)) &= q(v - s(v,\check{u})u) &&= q(v) - s(v,\check{u})s(v,u) + s(v,\check{u})^2 q(u) \\ &= q(v) - s(v,\check{u})(s(v,u) - s(v,q(u)^{-1}u)q(v)) &&= q(v). \end{aligned}$$

(e) $\check{\sigma(u)} = q(\sigma(u)^{-1}\sigma(u)) = q(u)^{-1}\sigma(u) = \sigma(q(u)^{-1}u) = \sigma(\check{u})$ and

$$(\sigma\omega_u\sigma^{-1})(v) = \sigma(\sigma^{-1}(v) - s(\sigma^{-1}(v)v, \check{u}) = v - s(v, \check{\sigma(u)})\sigma(u) = \omega_{\sigma u}(v).$$

□

Lemma 2.6.4 [**1/2 f**] *Suppose f is a non-degenerate symmetric form on the $\mathbb{K}$-space V and that* $\operatorname{char}\mathbb{K} \neq 2$. *Then $q(v) := \frac{1}{2}f(v,v)$ is a quadratic form and f is its associated bilinear form.*

Proof: $q(v+w) - q(v) - q(w) = \frac{1}{2}(f(v+w) - f(v,v) - f(w,w) = f(v,w)$ □

Lemma 2.6.5 [**f circ**] *Let V and W be finite dimensional $\mathbb{K}$-spaces and $f : V \times W \to \mathbb{K}$ a non-degenerate bilinear form. Define $\Phi : V \otimes W \to (V \otimes W)^*$ by $\Phi(v \otimes w)(v' \otimes w') = f(v,w')f(v',w)$ for all $v, v' \in V$ and $w, w' \in W$.*

(a) [**a**] *According to 2.6.1 choose bases $(v_i, i \in I)$ and $(w_i, i \in I)$ for V and W such that $f(v_i, w_j) = \delta_{ij}$. Then $(\Phi(v_i \otimes w_j))_{ij}$ is the dual of the basis $(v_j \otimes w_i)_{ij}$ of $V \otimes W$*

(b) [**b**] *Φ is an isomorphism.*

(c) [**c**] *Let $f^\circ = \Phi^{-1}(\widetilde{f})$. Then $f^\circ = \sum_{i \in I} v_i \otimes w_i$.*

(d) [**e**] $\widetilde{f}(f^\circ) = \dim V$

(e) [**d**] *Suppose that V and W are L-modules and f is L-invariant. Then Φ is L-invariant and $Lf^\circ = 0$.*

Proof: We compute

$$(*) \qquad \Phi(v_i \otimes w_j)(v_k \otimes w_l) = f(v_i, w_l)f(v_l, w_j) = \delta_{il}\delta_{jk}.$$

Thus (a) holds. (b) follows directly from (a).
(c) Let $t = \sum_{i \in I} v_i \otimes w_i$. Then by (*)

$$\Phi(t)(v_k \otimes w_l) = \sum_{i \in I} \Phi(v_i \otimes w_i)(v_k \otimes w_l) = \sum_{i \in I} \delta_{il}\delta_{ik} = \delta_{kl} = f(v_k, w_l) = \widetilde{f}(v_k \otimes w_l)$$

Thus $\Phi(t) = \widetilde{f}$ and so $t = \Phi^{-1}(\widetilde{f}) = f^\circ$

(d) From (c) we compute

$$\widetilde{f}(f^\circ) = \sum_{i \in I} \widetilde{f}(v_i \otimes w_i) = \sum_{i \in I} f(v_i, w_i) = \sum_{i \in I} 1 = |I| = \dim V.$$

(e) That Φ is L-invariant is readily verified. Since f is L-invariant, $L\widetilde{f} = 0$. Since Φ is an L-isomorphism, $Lf^\circ = 0$. $\square$

2.7 The Killing Form

For a finite dimensional L-module V we define $f_V : L \times L \to \mathbb{K}, (a, b) \to \mathrm{tr}_V(ab)$. f_V is called the killing form of L with respect to V. In the case of the adjoint module, f_L is just called the Killing form of L.

Lemma 2.7.1 [**basic killing**] *Let V be a finite dimension L-module.*

(a) [**a**] *f_V is a symmetric, L-invariant bilinear form on L.*

(b) [**b**] *If $I \trianglelefteq L$, then $I^\perp \trianglelefteq L$ and $[I, I^\perp] \leq \mathrm{rad}(f_V)$.*

(c) [**c**] *Let $\mathcal{W}$ be the set of factors for some L-series on V. Then*

$$f_V = \sum_{W \in \mathcal{W}} f_W.$$

(d) [**d**] *Let I be an ideal in L. Then $f_I \mid_{L \times I} = f_L \mid_{L \times I}$.*

(e) [**e**] $\mathrm{Nil}_L(V) \leq \mathrm{rad}(f_V)$.

(f) [**f**] *If L is finite dimensional, then* $\mathrm{Nil}(L) \leq \mathrm{rad}(f_L)$.

(a) Clearly f_V is K-bilinear. Let $a, b \in \mathfrak{U}$. Then $\mathrm{tr}_V(ab) = \mathrm{tr}_V(ba)$ so f_V is symmetric. Thus also shows that $\mathrm{tr}_V([a, b]) = \mathrm{tr}_V(ab - ba) = 0$ and so $\mathrm{tr}_V : \mathfrak{U} \to \mathbb{K}$ is L-invariant. By 2.2.2 the map $m_L : L \times L \to \mathfrak{U}, (a, b) \to ab$ is L-invariant. So also $f_V = \mathrm{tr}_V \circ \mathfrak{m}_L$ i L-invariant.

(b) The first statement follows from 2.2.1. For the second, let $i \in I, j \in I^\perp$ and $l \in L$. Then $[j, l] \in I^\perp$ and so since f_V is L-invariant:

$$f_V([i, j], l) = -f_V(i, [j, l]) = 0.$$

Thus $[i, j] \in \mathrm{rad}(f_V)$.

(c) Follows from 1.7.17(c).

(d) By (c), $f_L = f_I + f_{L/I}$. Then I and so also LI acts trivially on L/I. Thus $f_{I/L} \mid_{L \times I} = 0$ and (d) holds.

(e) Let W be composition factors for L on V. Then $\mathrm{Nil}_L(V)W = 0$ and so also $L\mathrm{Nil}_L(V)W = 0$. Thus for all $l \in L$ and $n \in \mathrm{Nil}_L(V)$ we have $f_W(l,n) = \mathrm{tr}_W(ln) = 0$. So by (c) $f_V(l,n) = 0$ and (e) holds.

(f) This is (e) applied to the adjoint module. □

Theorem 2.7.2 (Cartan's Solvabilty Criterion) [**cartan**] *Suppose L posseses a standard, faithful L-module with $f_V = 0$. Then L is solvable.*

Proof: Suppose L is a counter example with $\dim L$ minimal. Then all proper algebras of L are solvable, but L is not. Thus by 2.4.1 and 2.4.2, $\overline{L} := L/\mathrm{Nil}_L(V) \cong \mathfrak{sl}(\mathbb{K}^2)$. Let (x,y,h) be a Chevalley basis for $\overline{L}$ and choose $\widetilde{x}$ and $\widetilde{y}$ in L which are mapped onto x and y. Since $L \neq \mathrm{Nil}_L(V)$ there exists a non-trivial compostion factor W for L on V. For any such W we have $C_L(W) = Nil_L(V)$ and 2.5.4(b:e) implies that $\mathrm{tr}_W(\widetilde{x}\widetilde{y})$ is a positive integer. Hence 1.7.17(c) implies that $f_V(\widetilde{x},\widetilde{y}) = \mathrm{tr}_V(\widetilde{x}\widetilde{y})$ is a positive integer. This is contradiction to $f_V = 0$ and the theorem is proved. □

Proposition 2.7.3 [**rad=sol**] *Let V be standard,faithful L-module. Then*

$$[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V) \leq \mathrm{rad}(f_V) \leq \mathrm{Sol}(L)$$

In particular, if L is perfect, then $\mathrm{Sol}(L) = \mathrm{Nil}_L(V) = \mathrm{rad}(f_V)$.

Proof: By Lie's Theorem 2.1.5

$$[\mathrm{Sol}(L), L] \leq \mathrm{Sol}(L) \cap L' \leq \mathrm{Nil}_L(V).$$

By 2.7.1(d), $\mathrm{Nil}_L(V) \leq \mathrm{rad}(f_V)$. Finally, Cartan's Solvabilty Criterion 2.7.2 (applied to $\mathrm{rad}(f_V)$ in place of L), we have that $\mathrm{rad}(f_V)$ is solvable and so $\mathrm{rad}(f_V) \leq \mathrm{Sol}(L)$. □

Corollary 2.7.4 [**basic non-degenerate**] *Let V be a standard L-module with f_V non-degenerate. Then*

(a) [**a**] *V is faithful and $Nil_L(V) = 0$.*

(b) [**b**] $\mathrm{Sol}(L) = Z(L)$ *and* $\mathrm{Sol}(L) \cap L' = 0$.

(c) [**c**] *If L is solvable, then L is abelian.*

Proof: By 2.7.1(c), $C_L(V) \leq \mathrm{Nil}_L(V) \leq \mathrm{rad}(f_V) = 0$. So (a) holds. (b) now follows from 2.7.3. (c) follows from the first statement in (b). □

Corollary 2.7.5 [**faithful=non-degenerate**] *Suppose* $\mathrm{Sol}(L) = 0$ *and V is a standard L-module. Then f_V is non-degenerate if and only if V is faithful.*

Proof: If f_V is non-degenerate, then V is faithful by 2.7.4(a). Suppose now that V is faithful. Then by 2.7.3 $\mathrm{rad}(f_V) \leq \mathrm{Sol}(L) = 0$ and so f_V is non-degenerate. □

Lemma 2.7.6 [**non-degenerate implies semisimple**] *Suppose that L is finite dimensional and f_L is non-degenerate. Then* $\mathrm{Sol}(L) = 0$.

Proof: By 2.7.1(e), $\mathrm{Nil}(L) \leq \mathrm{rad}(f_L) = 0$. So by 2.1.2(c), $\mathrm{Sol}(L) = 0$. □

Corollary 2.7.7 [**semisimple=non-degenerate**] *Suppose L is standard. Then* $\mathrm{Sol}(L) = 0$ *if and only if f_L is non-degenerate.*

Proof: If f_L is non-degenerate, then by 2.7.6 $\mathrm{Sol}(L) = 0$. If $\mathrm{Sol}(L) = 0$, then also $Z(L) = 0$ and so the adjoint module is faithful. So by 2.7.5, f_L is non-degenerate. □

If f is a symmetric bilinear form on a vector space W, we write $W = W_1 \bigoplus W_2$ if W_i are subspaces of W with $W = W_1 \oplus W_2$ and $f(w_1, w_2) = 0$ for all $w_i \in W_i$. Note that in this case, W is non-degenerate if and only if $f\mid_{W_i}$ is non-degenerate for $i = 1$ and 2.

Proposition 2.7.8 [**decomposing l**] *Let V be a finite dimensional L module and suppose that f_V is non-degenerate. Let I be an ideal in L with $I \cap \mathrm{Sol}(L) = 0$. Then*

(a) [**a**] $[I, I^\perp] = 0$.

(b) [**b**] $L = I \bigoplus I^\perp$.

(c) [**c**] $I^\perp = C_L(I)$.

Proof: By 2.7.1(b), $[I, I^\perp] \leq \mathrm{rad}(f_V) = 0$. Thus (a) holds and

(1) $$I^\perp \leq C_L(I).$$

Since $I \cap C_L(I)$ is an abelian ideal of L and since $\mathrm{Sol}(L) \cap I = 0$, we get

(2) $$I \cap C_L(I) = 0.$$

From (1) and (2)

(3) $$I \cap I^\perp = 0.$$

From 2.6.1(c) we have $\dim I + \dim I^\perp = \dim L$ and so (3) implies that (b) holds. From (b), (1) and (2) we compute

$$C_L(I) = C_L(I) \cap L = C_L(I) \cap (I + I^\perp) = (C_L(I) \cap I) + I^\perp = I^\perp$$

So (c) holds. □

Theorem 2.7.9 [**composition of l**] *Let V be a finite dimensional L-module and suppose that* $\mathrm{Sol}(L) = 0$ *and f_V is non-degenerate. Then there exists perfect, simple ideals $L_1, L_2, \ldots, L_n$ in L such that*

$$L = L_1 \bigoplus L_2 \bigoplus \ldots \bigoplus L_n.$$

Proof: By induction on $\dim L$. If L is simple we can choose $n = 1$ and $L_1 = L$. So suppose that L is not simple and let I be proper ideal in L. Since $\mathrm{Sol}(L) = 0$ the assumptions of 2.7.8 are fulfilled. Hence $L = I \bigoplus I^\perp$ and $[I, I^\perp] = 0$. In particular, $f_V \mid_I$ and $f_V \mid_{I^\perp}$ are non-degenerate. Also any ideal in I or $I^\perp$ is an ideal in L. By induction we can decompose I and $I^\perp$ into an orthogonal sum of ideals. Thus the same is true for L. Since $\mathrm{Sol}(L) = 0$, the L_i are not abelian and so perfect. □

Corollary 2.7.10 [**decomposing standard**] *Let V be a standard L-module with f_V-non-degenerate. Then $L = L' \bigoplus Z(L)$ and L is semisimple*

Proof: By 2.7.4 $L' \cap \mathrm{Sol}(L) = 0$. So by 2.7.8, $L = L' \bigoplus L'^\perp$. In particular, $[L'^\perp, L] \leq L' \cap L'^\perp = 0$. Thus $L'^\perp = Z(L)$, $L = L^\perp \bigoplus Z(L)$. Thus $\mathrm{Sol}(L')$ is an ideal in L and hence $\mathrm{Sol}(L') \leq L' \cap \mathrm{Sol}(L) = 0$. By 2.7.9 L' is semisimple. Clearly also $Z(L)$ is semisimple and so L is semisimple. □

Corollary 2.7.11 [**standard semisimple**] *Suppose L is standard and* $\mathrm{Sol}(L) = 0$. *Then*

(a) [**a**] *f_L is non-degenerate.*

(b) [**b**] *There exists perfect,simple ideals $L_1, L_2 \ldots L_n$ such that*

$$L = L_1 \bigoplus L_2 \bigoplus \ldots \bigoplus L_n.$$

(c) [**c**] *$\{L_1, L_2, \ldots L_n\}$ is precisely the set of minimal ideals in L.*

(d) [**d**] *Every ideal in L is a sum of some of the L_i's.*

Proof: (a) follows from 2.7.7.

(b) By (a) we can apply 2.7.9 with V the adjoint module. Thus (b) holds.

(c) Let I be a minimal ideal in L. Since $\mathrm{Sol}(L) = 0$, $Z(L) = 0$ and so by (b), $[I, L_i] \neq 0$ for some i. As I is a minimal ideal and L_i is simple, $I = [I, L_i] = L_i$.

(d) Let I be an ideal in L. Let A be the sum of the L_i's with $L_i \leq I$ and B the sum of the remaining L_i's. Then $A \leq I$, $L = A + B$ and $[A, B] = 0$. Thus $I = I \cap (A + B) = A + (I \cap B)$.

Suppose that $I \cap B \neq 0$. Then $I \cap B$ contains a minimal ideal and so by (c), $L_i \leq I \cap B$ for some i. Since $L_i \leq I$, $L_i \leq A$. Since $L_i \leq B$ and $[A, B] = 0$ we conclude that $[L_i, L_i] = 0$, a contradiction since L_i is perfect.

Thus $I \cap B = 0$ and $I = A$. □

We say that L is semisimple if L is the direct sum of simple ideals. Note that this is the case if and only if the adjoint module is a semisimple L-module.

Corollary 2.7.12 [**sol l and semisimple**] *Let L be standard. Then* $\mathrm{Sol}(L) = 0$ *if and only if L is perfect and semisimple.*

Proof: One direction follows from 2.7.11 while the other is obvious. □

2.8 Non-split Extensions of Modules

In this section A is an associative algebra.

Definition 2.8.1 [**def:extension**]

(a) [**a**] *An extensions of A-modules is a pair of A-modules (W, V) with $W \leq V$.*

(b) [**b**] *An extension of A-modules (W, V) is called split if there exists a A-submodule X of V with $V = W \oplus X$.*

(c) [**c**] *Let B and T be A-modules and (W, V) an extension of A-modules. We say that (W, V) is an extension of B by T if $W \cong B$ and $V/W \cong T$ as A-modules.*

Lemma 2.8.2 [**basic split I**] *An extension (W, V) of A-modules. is split if and only if there exists $\phi \in \mathrm{Hom}_A(V, W)$ with $\phi \mid W = \mathrm{id}_W$.*

Suppose first that $V = W \oplus X$ for some A-submodule X of V. Let ϕ be the projection of V onto X. Then ϕ is A-invariant and $\phi_W = \mathrm{id}_W$.

Suppose next that $\phi : V \to W$ is A-invarinant with $\phi\,|_W = \mathrm{id}_W$. Let $X = \ker \phi$. Then X is submodule of V and $X \cap W = 0$. Let $v \in V$. Then $\phi(v) \in W$ and so $\phi(\phi(v)) = \phi(v)$.Hence $\phi(v - \phi(v)) == 0$. That is $v - \phi(v) \in \ker \phi = X$. So $v = \phi(v) + (v - \phi(v)) \in W + X$. Thus $V = W \oplus X$ and (W, V) splits. □

Lemma 2.8.3 [**basic split II**] *Let (W, V) be an extension of A-modules. Let*

$$\Phi : \mathrm{Hom}(V, W) \to \mathrm{Hom}(W, W)$$

be the restriction map. Then

(a) [**a**] *Φ is A-invariant and onto.*

(b) [**b**] $\ker \Phi \cong \mathrm{Hom}(V/W, W)$ *and* $\ker \Phi$ *is submodule of* $\mathrm{Hom}(V, W)$

(c) [**c**] *$S := \Phi^{-1}(\mathbb{K}\mathrm{id}_W)$ is an A-submodule of* $\mathrm{Hom}(V, W)$ *and* $S/\ker \Phi \cong \mathbb{K}$

(d) [**e**] *(W, V) is split if and only if* $(\ker \Phi, S)$ *is split.*

Proof:

(a) Clearly Φ is A-invariant. Note thate there exists K-subspace X of V with $V = W \oplus X$. Let $\alpha \in \mathrm{Hom}(W, W)$ and define $\tilde{a}(w + x) = \alpha(w)$. Then $\Phi(\tilde{\alpha}) = \alpha$. So Φ is onto.

(b) Since Φ is A-invarinat, $\ker \Phi$ is an A-submodule.

Let $\alpha \in \ker \Phi$. Define $\beta : V/W \to V, v + W \to \alpha(v)$. Conversely let $\beta \in \mathrm{Hom}(V/W, W)$ define $\alpha : V \to W, \alpha(v) = \beta(v + W)$. Then $\alpha \in \ker \Phi$.

(c) follows from (a).

(d) Suppose first that (W, V) splits. Let ϕ be as in 2.8.2 d. Then $S = \ker \Phi \oplus \mathbb{K}\phi$ and so $(\ker \Phi, S)$ splits.

Next suppose that $(\ker \Phi, S)$ splits and let Y be an A-submodule of S with $\S = \ker \Phi \oplus Y$. Then $\Phi \mid Y : Y \to \mathbb{K}\mathrm{id}_W, \phi \to \phi \mid_W$ is an A-invariant isomorphism. Hence Y is a trivial A-module, all $\phi \in Y$ are A-invariant and there exists $\phi \in Y$ with $\phi \mid_W = \mathrm{id}_W$. Thus by 2.8.2, (W, V) splits. □

Lemma 2.8.4 [**b simple**] *Let T be A-module and suppose there exists a non-split extension of a finite dimensional A-module by T. Then there exists on-split extension of finite dimensional simple A-module by T.*

Proof: Let (W, V) be a non-split extension with $V/W \cong T$ and W finite dimensional. Since W is finite dimensional we can choose a submodule Y of W maximal such that $(W/Y, V/Y)$ is non-split. Since $(V/Y)/(W/Y) \cong V/W \cong T$, $(W/Y, V/Y)$ has the same properties as (W, V). So we may assume that $Y = 0$. Let B be a simple A-submodule of W. The maximality of Y implies that $(W/B, V/B)$ is split. So $V/B = W/B \oplus X/B$ for some A-submodule X of V with $B \leq X$. Then $W \cap X = B$ and $W + X = V$. Thus

$$T \cong V/W = X + W/W \cong X/X \cap W = X/B.$$

Hence (B, X) is an extension of B by T. Suppose this extension is split. Then $X = B \oplus Y$ for some A-submodule Y of X. Thus $V = X + W = Y + B + W = Y + W$ and $Y \cap W \leq Y \cap (X \cap W) + Y \cap B = 0$. So $V = W \oplus Y$, contrary to the assumptions. Thus (B, X) is non-split and the lemma is proved. □

Corollary 2.8.5 [**splitting reduction**]

(a) [**a**] *a Suppose there exists a finite dimensional A-module which is not semisimple. Then there exists a non-split extension of finite dimensional A-modules.*

(b) [**b**] *Suppose there exists a non-split extension of finite dimensional A-modules. Then there non-split extension of finite dimensional simple A- module by $\mathbb{K}$.*

Proof:

(a) Let V be a finite dimensional A-module of minimal dimension with respect to not being semisimple. Let W be simple A-submodule of V. Suppose $V = W \oplus X$ for A-submodule X of W. Then by minimalty of V, X is semisimple. But then also V is semisimple.

(b) From 2.8.3 there exist a non-split extension of a finite dimensional module by $\mathbb{K}$. (b) now follows from 2.8.4 □

2.9 Casimir Elements and Weyl's Theorem

In this section will show that a standard module for a perfect, semisimple Lie algebra is semisimple.

Proposition 2.9.1 [**casimir**] *Suppose L is finite dimensional and $f : L \times L \to \mathbb{K}$ is a non-degenerate, L-invariant, $\mathbb{K}$-bilinear form. Define $\Psi : L \otimes L \to \mathfrak{U}$ by $\Psi(a \otimes b) = ab$. Let f° be as in 2.6.5 and put $c_f = \Psi(f^\circ)$.*

(a) [**a**] $c_f \in Z(\mathfrak{U}) \cap L^2$.

(b) [**b**] *Let $(v_i, i \in I)$ and $(w_i, \in I)$ be bases of L with $f(v_i, w_j) = \delta_{ij}$. Then*

$$c_f = \sum_{i \in I} v_i w_i$$

Proof: View $\mathfrak{U}$ as L-module via the adjoint representation. By 2.2.2, Ψ is L-invariant, By 2.6.5(e) $Lf^\circ = 0$ and so $[L, c_f] = 0$. Since $\mathfrak{U}$ is generated by L as an associative algebra, $[\mathfrak{U}, c_f] = 0$. Thus $c_f \in Z(U)$. Also $c_f \in \Psi(L \otimes L) = L^2$. So (a) holds. (b) follows immediately from 2.6.5(c). □

The elements c_f from the preceeding proposition is called the Casimir element of f.

Lemma 2.9.2 [**cv**] *Let V be a finite dimensional L-module and suppose that f_V is non-degenerate. Define $c_V = c_{f_V}$.*

(a) [**a**] $\mathrm{tr}_V(c_V) = \dim L$.

(b) [**b**] *Suppose $\mathbb{K}$ is algebraicly closed and V is simple. Then c_V acts as a scalar $k \in \mathbb{K}$ on V. Moreover one of the following holds:*

1. [**a**] $\mathrm{char}\, K \nmid \dim V$ *and* $k = \frac{\dim L}{\dim V}$.
2. [**b**] $\mathrm{char}\, K \mid \dim V$ *and* $\mathrm{char}\, K \mid \dim L$.

Proof: (a) Let Ψ be defined as in 2.9.1. Then by definiton of f_V, $\widetilde{f}_V = \mathrm{tr}_V \circ \Psi$. Thus

$$\mathrm{tr}_V(c_V) = \mathrm{tr}_V(\Psi(f^\circ)) = \widetilde{f}(f^\circ)$$

So (a) follows from 2.6.5(d).

(b) Since $c_V \leq Z(U)$, Schur's Lemma 1.7.13 applied to the image of c_V in $\mathrm{End}_L(V)$ gives that c_V acts as a scalar $k \in \mathbb{K}$. Thus $\mathrm{tr}_V(c_V) = k \dim V$ and so (b) holds. □

Theorem 2.9.3 (Weyl) [**weyl**] *Let L be standard, prefect and semisimple and V a finite dimensional L-module. Then V is semisimple.*

Proof: By 2.8.5 its suffices to show that any finite dimensional L-module extension (W, V) with W simple and $V/W \cong \mathbb{K}$ splits. Since $L/C_L(V)$ is also semisimple we may assume that V is faithful. By 2.7.5 f_V is non-degenerate. So by 2.9.1 $c := c_V \in Z(\mathfrak{U}) \cap L^2$. Since $LV \leq W$, $cV \leq W$. Since W is simple, Schur's lemma 1.7.13 applied to the image of c in $\mathrm{End}_L(V)$ gives that c acts as a scalar k on W. Then

$$\mathrm{tr}_V(c) = \mathrm{tr}_W(c) + \mathrm{tr}_{V/W}(c) = k \dim W + 0.$$

By 2.9.2(a) and since $\mathrm{char}\,\mathbb{K} = 0$, $\mathrm{tr}_V(c) \neq 0$ and so $k \neq 0$. Thus $k^{-1}cV \leq W$ and $k^{-1}c$ acts id_W on W. Thus $k^{-1}c$ induces an L-invarinat $\mathbb{K}$-linear map $\Phi : V \to W$ with $\phi\mid_W = \mathrm{id}_W$. Thus by 2.8.2 , (W, V) splits. □

2.10 Cartan Subalgebras and Cartan Decomposition

Definition 2.10.1 [**def:cartan**] *$H \leq L$ is called selfnormalizing if $H = N_L(H)$. A Cartan subalgebra of L is a nilpotent, selfnormalizing subalgebra of L.*

Lemma 2.10.2 [**existence of cartan**] *Suppose that L is finite dimensional and $|\mathbb{K}| \geq \dim L$. Then L has a Cartan subalgebra.*

Proof: Choose $d \in L$ with $H := L_0^c(\mathbb{K}d)$ minimal. Note that a simple module with weight 0 is a trivial module. So by 1.7.11(h), H is the largest $\mathbb{K}d$-submodule on which d acts nilpotently. In particular, $d \in H$. By 2.3.6 $[L_0^c(\mathbb{K}d), L_0^c(\mathbb{K}d)] \leq L_0^c(\mathbb{K}d)$ so H is a subalgebra. Let $V = L/H$. Then V is an H-module and $C_V(d) = 0$. In particular, the image d^* of d in $\mathrm{End}(V)$ is invertible. Also $N_L(H)/H \leq C_V(d) = 0$ and so $H = N_L(H)$. To complete the proof we may assume that H is not nilpotent and derive a contradiction. Let $D \leq H$ such that $d \in D$ and D is maximal with respect to acting nilpotently on H. Then $D \neq H$ and by 1.6.7 there exists $h \in N_H(D) \setminus D$. Since the number of eigenvalues for h^*d^{*-1} on V is at most $\dim V$ and since $|\mathbb{K}| \geq \dim L > \dim V$, there exists $k \in \mathbb{K}$ such that k is not an eigenvalue of h^*d^{*-1}. Then $h^*d^{-1} - k\mathrm{id}_V$ is invertible and so also $h^* - kd^*$ invertible. Put $l = h - kd$. Then $l \in N_H(D) \setminus D$ and $C_V(l) = 0$. Hence $V_0(^c)(\kappa L) = 0$. As

$L_0^c(\mathbb{K}l) + H/H \leq V_0^c(\mathbb{K}L) = 0$ we conclude $L_c^0(\kappa L) \leq H$. The minimality of $H = L_0^c(\mathbb{K}d)$ implies that $H = L_0^c(\mathbb{K}l)$. Thus l acts nilpotently on H. From 1.6.3 we conclude that $D+\mathbb{K}l$ acts nilpotently on H, contradicting the maximal choice of D. □

Lemma 2.10.3 [**cartan decomposition**] *Let V be a standard L-module and N a nilpotent subideal in L. Then*

$$V = \bigoplus_{\lambda \in \Lambda_V(N)} V_\lambda^c$$

Moreover, each of the V_λ^c are L-submodule.

Proof: By 1.7.11(g), the V_λ^c are L-submodules. So it remains to prove the first statement. If V is the direct sum of two proper N-submodules, we may by induction assume that the lemma holds for both summands. But then it also holds for V. So we may and do assume

(*) V is not the direct sum of proper N-submodules.

Let $l \in N$. Since N is nilpotent 1.6.7 implies that $\mathbb{K}l$ is subideal in N. The Jordan Canonical Form of l shows that V is the direct sum of the generalized eigenspaces of l. But the generalized eigenspaces are just the generalized weight spaces of $\mathbb{K}l$. Hence 1.7.11(g) shows that the generalized eigenspaces are N-submodules. So by (*), l has a unique eigenvalue λ_l on V.

Let W be any N-composition factor on V. Then by 2.3.3 $W \cong K_\lambda$ for some weight λ of N. Then $\lambda(l)$ is an eigenvalue for l on V and $\lambda(l) = \lambda_l$. As $l \in N$ was arbitrary, λ is independent of the choice W and so $V = V_\lambda^c$. □

2.11 Perfect semsisimple standard Lie algebras

In this section we will investigate the structure of the perfect semisimple standard Lie algebras. For this we fix the following

Notation 2.11.1 [**not:semisimple**]

(a) [**a**] *L is a perfect, semisimple, standard Lie algebra.*

(b) [**b**] *H is a Cartan subalgebra of L.*

(c) [**c**] *$\Lambda = \Lambda_L(H)$ is the set of weights for H on L.*

(d) [**d**] *$\Phi = \Lambda \setminus \{0\}$. The non-zero weights for H on L are called the roots of H.*

(e) [**e**] *$f = f_L$, and $\perp$ is always meant with respect to f.*

Lemma 2.11.2 [**root decomposition**] *(a)* [**a**] $L = \bigoplus_{\alpha\in\Lambda} L^c_\alpha$

(b) [**b**] $H = L^c_0$.

(c) [**c**] $[L^c_\alpha, L^c_\beta] \leq L^c_{\alpha+\beta}$ *for all* $\alpha, \beta \in \Lambda$.

Proof: (a) This follows from 2.10.3 applied with $V = L$,$L = H$ and $N = H$.

(b) By 1.7.11(e), $L_0(H)$ is the largest H-submodule of L, such that all compostion factors for H on $L_0(H)$ are trivial. Since H is nilpotent, all composition factors for H an H are trivial. Thus $H \leq L^c_0$. Suppose $H \neq L_0$ and let A/H be a simple submodule of L_0/H. By definition of L_0, A/H is a trivial module. Thus $[A, H] \leq H$, a contradiction since H is selfnormalizing..

(c) This follows from 2.3.6. □

Lemma 2.11.3 [**simple properties**] *Let* $\alpha, \beta \in \Lambda$ *and* $h \in H$.

(a) [**a**] f *is non-degenerate.*

(b) [**b**] $\mathrm{tr}_{L^c_\alpha}(h) = \alpha(h) \dim L^c_\alpha$.

(c) [**c**] *If* $\alpha(h) \neq 0$, *then* $[h, L^c_\alpha] = L^c_\alpha$.

(d) [**d**] *If* $\beta \neq -\alpha$, $L^c_\alpha \perp L^c_\beta$.

(e) [**e**] $f|_H$ *is non-degenerate.*

(f) [**f**] H *is abelian.*

(g) [**g**] $-\alpha \in \Lambda$.

(h) [**h**] $L_\alpha \neq 0$.

Proof: (a) : 2.7.11(a).

(b) Obvious.

(c) By 2.11.2(c), $[h, L^c_\alpha] \leq L^c_\alpha$. Since $\alpha(h) \neq 0$, $C_{L^c_\alpha}(h) = 0$ and so the action of h on L^c_α is invertible.

(d) By 2.3.5 $f(L^c_\alpha, L^c_\beta) \leq (\mathbb{K})^c_{\alpha+\beta}$. If $\alpha + \beta \neq 0$, $(\mathbb{K})^c_{\alpha+\beta} =$ and (d) holds.

(e) By (d), and 2.11.2(a), $L = H \bigoplus H^\perp$. So since f is non-degenerate, (e) holds.

(f) This follows from (e) and 2.7.4(c).

(g) Otherwise (d) would imply $L^c_\alpha \leq L^\perp = 0$.

(h) Follows from the definition of $\Lambda = \Lambda_L(H)$. □

Notation 2.11.4 [**ta for l**] *Since* $f|_H$ *is nondegenerate 2.6.1 yields a* $\mathbb{K}$*-isomorphism* $t : H^* \to H, \alpha \to t_\alpha$ *with* $\alpha(h) = f(t_\alpha, h)$ *for all* $\alpha \in H^*, h \in H$. *For* $\alpha, \beta \in H^*$ *and* $h \in H$ *define* $q(h) = \frac{1}{2} f(h, h)$, $f^*(\alpha, \beta) = f(t_\alpha, t_\beta)$ *and* $q^*(\alpha) = q(t_\alpha)$. *Recall the definition of* $\check{h}$ *and* ω_α *in 2.6.2. Note also that* $\Lambda \subset \Lambda(H) = H^*$.

Lemma 2.11.5 [**x,y**] *Let $\alpha \in \Phi$, $x \in L^c_\alpha$ and $y \in L_{-\alpha}$. Put $h = [x, y]$. Then*

(a) [**a**] $h \in H$.

(b) [**b**] *Let $\beta \in \Lambda$. Then there exists $q \in \mathbb{Q}$ with $\beta(h) = q\alpha(h)$.*

(c) [**c**] *$h = 0$ if and only if $\alpha(h) = 0$.*

(d) [**d**] $h = [x, y] = -f(x, y)t_\alpha$.

(e) [**e**] *$h = 0$ if and only if $x \perp y$.*

Proof: (a) follows from 2.11.2.

(b) Put $V := \sum_{n\in\mathbb{Z}} L^c_{\beta+n\alpha}$. By 2.11.2(c), V is invariant under x and y and so under h. We compute:

$$0 = \mathrm{tr}_V(h) = \sum_{n\in\mathbb{Z}} \mathrm{tr}_{L^c_{\beta+n\alpha}} h = \sum_{n\in\mathbb{Z}} (\beta(h) + n\alpha(h)) \dim L^c_{\beta+n\alpha}$$

and so

$$\beta(h) \sum_{n\in\mathbb{Z}} \dim L^c_{\beta+n\alpha} = -\alpha(h) \sum_{n\in\mathbb{Z}} n \dim L^c_{\beta+n\alpha}.$$

So (b) holds.

(c) Suppose $\alpha(h) = 0$, then by (b), $\beta(h) = 0$ for all $\beta \in \Lambda$. Hence h acts nilpotently on L. Since H is abelian we get $\mathbb{K}h \leq \mathrm{Nil}_H(L)$. Since $f|_H$ is non-degenerate, 2.7.4(b) implies $\mathrm{Nil}_H(L) = 0$. So $h = 0$.

(d) Let $a \in H$. Since $y \in L_{-\alpha}$, $[a, y] = -\alpha(a)y$. Since f is L-invariant we obtain:

$$f(h, a) = f([x, y], a)) = f(x, [y, a]) = -f(x, [a, y]) = -f(x, -\alpha(a)y) = \alpha(a)f(x, y)$$

On the otherhand,

$$f(f(x, y)t_\alpha, a) = f(x, y)f(t_\alpha, a) = f(x, y)\alpha(a) = \alpha(a)f(x, y)$$

Since $f\mid_H$ is non-degenerate, this implies $h = f(x, y)t_\alpha$.

(e) follows immediately from (d). □

Lemma 2.11.6 [**dim la**] *Let $a \in \Phi$.*

(a) [**a**] *$L_\alpha = L^c_\alpha$ is 1-dimensional.*

(b) [**c**] *Let $n \in \mathbb{Z}$. Then $n\alpha \in \Phi$ if and only if $n = \pm 1$.*

(c) [**b**] $[L_\alpha, L_{-\alpha}] = \mathbb{K}t_\alpha$.

(d) [**d**] $f(t_\alpha, t_a) \neq 0$.

Proof: Pick $0 \neq y \in L_{-\alpha}$. By 2.11.3(d), $L = y^{\perp} + L_{\alpha}^{c}$. Hence there exists $x \in L_{\alpha}^{c}$ with $x \not\perp y$. Put $h = [x, y]$. By 2.11.5(c) and (e), $h \neq 0$ and $\alpha(h) \neq 0$. Put

$$V := \mathbb{K}y \oplus H \oplus \bigoplus_{n \in \mathbb{Z}_+} L_{n\alpha}.$$

By 2.11.2(c), V is invariant under x. Since $y \in L_{-\alpha}$, $[y, H] \leq \mathbb{K}y$. Also $[y, y] = 0$ and so V is also invariant under y and h. Thus

$$0 = \mathrm{tr}_V(h) = -\alpha(h) + 0 + \sum_{n \in \mathbb{Z}_+} n\alpha(h) \dim L_{n\alpha}^{c}.$$

Since $\alpha(h) \neq 0$ we can divide by $\alpha(h)$ to obtain:

$$\sum_{n \in \mathbb{Z}_+} n \dim L_{n\alpha}^{c} = 1$$

Thus $\dim L_{n\alpha}^{c} = 0$ for $n > 1$ and $\dim L_{\alpha}^{c} = 1$. So (a) holds. Also (b) holds for positive n. Applying this result to $-\alpha$ we see that (b) holds. As L_α and $L_{-\alpha}$ are 1-dimensional, $[L_\alpha, L_{-\alpha}]$ is at most 1-dimensional. But $h = [x, y] \neq 0$ and so $[L_\alpha, L_{-\alpha}] = \mathbb{K}h$.

By 2.11.5(d)], $h = f(x, y)t_\alpha$ and hence (c) is proved.

Finally $0 \neq \alpha(h) = \alpha(f(x, y)t_\alpha) = f(x, y)\alpha(t_\alpha) = f(x, y)f(t_\alpha, t_\alpha)$ and so also (d) holds. □

Lemma 2.11.7 [**xa =slw**] *Let $\alpha \in \Phi$. Define $H_\alpha = \mathbb{K}t_\alpha$, $X_\alpha = L_\alpha + L_{-\alpha} + H_\alpha$ and $h_\alpha = \check{t}_\alpha = \frac{2}{f(t_\alpha, t_\alpha)}t_\alpha$. Then X_α is a subalgebra of L, $X_\alpha \cong \mathfrak{sl}(\mathbb{K}^2)$ and $\alpha(h_\alpha) = 2$. More precisely, if $x_\alpha \in L_\alpha$ and $x_{-a} \in L_{-a}$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$, then there exists an isomorphism form X_α to $\mathfrak{sl}(\mathbb{K}^2)$ with*

$$x \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } h_\alpha \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Proof: Note first that by 2.11.6(d), $f(t_\alpha, t_\alpha) \neq 0$, so h_α is well defined. By 2.11.6(c) we can choose $x_{\pm\alpha}$ as in the lemma. Now $\alpha(h_\alpha) = f(t_\alpha, h_\alpha) = f(t_\alpha, \frac{2}{f(t_\alpha, t_\alpha)}t_\alpha) = 2$ and so $[h_\alpha, x_\alpha] = \alpha(h_\alpha)x_\alpha = 2x_\alpha$ and $[x_{-\alpha}, h_\alpha] = -[h_\alpha, x_{-\alpha}] = -(-\alpha(h_\alpha))y = 2x_{-\alpha}$ and so the lemma holds. □

Notation 2.11.8 [**def: a string**] *Let $\alpha \in \Phi$. We define an equivalence relation $\sim_\alpha$ on Λ by $\beta \sim_\alpha \gamma$ if $\beta - \gamma \in \mathbb{Z}\alpha$. We denote the set of equivalence classes by $\Lambda/\mathbb{Z}\alpha$. The equivalence classes for of $\sim_\alpha$ are called α-strings. If β, γ are in the same α-string we say that $\beta \leq_\alpha \gamma$ if $\gamma - \beta \in \mathbb{N}\alpha$. For $\beta \in \Phi$ let $\beta - r_{\alpha\beta}\alpha$ be the minimal and $\beta + s_{\alpha\beta}\alpha$ be the maximal element (with respect to $\leq_\alpha$) in the α-string through β. For an α string Δ define $L_\Delta = \sum_{\delta \in \Delta} L_\delta$,*

Lemma 2.11.9 [**xa on l**] *Let $\alpha \in \Phi$ and Δ an α-string.*

(a) [**a**] *L_Δ is an X_α-submodule and*

$$L = \bigoplus_{\Delta \in \Lambda/\mathbb{Z}\alpha} L_\Delta.$$

(b) [**b**] *Let $\beta \in \Delta$ and $i \in \mathbb{Z}$ with $i\alpha + \beta \in \Delta$. Then $L_{\beta+i\alpha}$ is the eigenspaces for h_α on L_Δ corresponding to the eigenvector $\beta(h_\alpha) + 2i$.*

(c) [**c**] *Suppose $\alpha \in \Delta$, then*

- *(a)* [**a**] $\Delta = \{-\alpha, 0, \alpha\}$.
- *(b)* [**b**] $L_\Delta = X_\alpha \oplus \ker \alpha$.
- *(c)* [**c**] $\ker \alpha = C_H(X_\alpha) = H \cap H_\alpha^\perp$.

(d) [**d**] *Suppose that $\alpha \notin \Delta$ and let $\beta \in \Delta$.*

- *(a)* [**b**] $\Delta = \{\beta + i\alpha \mid -r_{\alpha\beta} \leq i \leq s_{\alpha\beta}\}$
- *(b)* [**c**] *L_Δ is a simple X_α-module of dimension $|\Delta| = r_{\alpha\beta} + s_{\alpha\beta} + 1$.*
- *(c)* [**d**] $C_{L_\Delta}(L_\alpha) = L_{\beta+s_{\alpha\beta}\alpha}$.
- *(d)* [**e**] $[L_\alpha, L_\beta] = L_{\alpha+\beta}$

Proof: (a) follows immediately from 2.11.2.

(b) Since $\alpha(h_\alpha) = 2$, $L_{\beta+i\alpha}$ is contained in the eigenspace for h_α corresponding to $\beta(h_\alpha) + 2i$. Since the $\beta(h_\alpha) + 2i$, $i \in \mathbb{Z}$ are pairwise distinct we conclude that $L_{\beta+i\alpha}$ is the eigenspace corresponding to $\beta(h_\alpha) + 2i$.

(c) By 2.11.6(b) , $\Delta = \{\alpha, 0, -\alpha\}$. Since $\alpha(h_\alpha) = 2 \neq 0$, $H = \ker \alpha \oplus H_\alpha$ and so $L_\Delta = X_\alpha \oplus \ker \alpha$.

$$C_H(X_\alpha) = C_H(L_\alpha) \cap C_H(L_{-\alpha}) \cap C_H(H) = \ker \alpha \cap \ker -\alpha \cap H = \ker \alpha.$$

By definition of t_α, $f(t_\alpha, h) = \alpha(h)$ and so $\ker \alpha = H \cap t_\alpha^\perp = H \cap H_\alpha^\perp$. Thus all parts of (c) are proved.

(d) For an X_α-module I and $k \in \mathbb{K}$ let $d_\kappa(I)$ be the number of composition factor (in a given composition series) for H_α on I on which h_α acts by multiplication by k. Then

$$d_k(L_\Delta) = \sum_{W \in \mathrm{Comp}_{L_\Delta}(X_\alpha)} d_k(W).$$

Let $W \in \mathrm{Comp}_{L_\Delta}(X_\alpha)$ and let $m_W = \dim W - 1$. Then by 2.5.4

$$d_W(k) = \begin{cases} 1 & \text{if } k \text{ is an integer between } -m_W \text{ and } m_W \text{ with } k \equiv m_W \pmod 2 \\ 0 & \text{otherwise} \end{cases}.$$

In particular, $d_W(0) + d_W(1) = 1$. Thus $d_{L_\Delta}(0) + d_{L_\delta}(1)$ is the number of composition factor for X_α on L_Δ. On the otherhand by (b) $d_{L_\Delta}(k) = 1$ if $k = \beta(h_\alpha) + 2i$ for some $i \in \mathbb{Z}$ such that $\beta + i\alpha$ is a root and $d_{L_\Delta}(k) = 0$ otherwise. Hence $d_{L_\Delta}(0) + d_{L_\delta}(1) \leq 1$. So there exists a unique composition factor for X_α on $\mathbb{L}_\Delta$. Hence L_Δ is simple and (d) follows from 2.5.4 □

Lemma 2.11.10 [**f(ta,hb)**] *Let $\alpha \in \Phi$, Δ an α-string in Λ and $\beta \in \Delta$.*

(a) [**a**] $\omega_\alpha(\beta) \in \Lambda$.

(b) [**b**] *Let $\Delta = \{\beta_0, \beta_1, \dots \beta_k\}$ with $\beta_0 <_\alpha \beta_1 <_\alpha \dots <_\alpha \beta_k$. Then $\omega_\alpha(\beta_i) = \beta_{k-i}$.*

(c) [**c**] $\beta(h_\alpha) = f^*(\beta, \check{\alpha}) = r_{\alpha\beta} - s_{\alpha\beta} \in \mathbb{Z}$.

Proof: If $\alpha \in \Delta$, this is readily verified. So suppose $\alpha \notin \Delta$. Let $i = \beta(h_\alpha)$. Then i is an eigenvalue for h_α on L_Δ and so by 2.5.4 also $-i$ is an eigenvalue. Since $\alpha(h_\alpha) = 2$ we have $(\beta - i\alpha)(h_\alpha) = -i$ and we conclude from 2.11.9(b) that $\beta - i\alpha \in \Delta$. Also

$$i = \beta(h_\alpha) = f(t_\beta, h_\alpha) = f^*(\beta, \check{\alpha})$$

and so

$$\beta - i\alpha = \beta - f^*(\beta, \check{\alpha})\alpha = \omega_\alpha(\beta)$$

Thus (a) holds.

(b) follows easily from the proof of (a).

(c) From (b), $\omega_\alpha(\beta + s_{\alpha\beta}\alpha) = \beta - r_{\alpha\beta}\alpha$. Hence

$$\beta + s_{\alpha\beta}\alpha - f^*(\beta + s_{\alpha\beta}\alpha, \check{\alpha})\alpha = \beta - r_{\alpha\beta}\alpha$$

$$s_{\alpha\beta} - \beta(h_\alpha) - 2s_{\alpha\beta} = -r_{\alpha\beta}$$

$$r_{\alpha\beta} - s_{\alpha\beta} = \beta(h_\alpha) = f^*(\beta, \check{a}).$$

□

Lemma 2.11.11 [**h=sum ha**] $H = \sum_{\alpha \in \Phi} H_\alpha$.

Proof: Let $h \in H$ with $h \perp H_\alpha = 0$ for all $\alpha \in \Phi$. Then $\alpha(h) = 0$ for all $\alpha \in \Lambda$. So $[h, L_\alpha] = 0$ and $h \in Z(L) = 0$. Thus $h = 0$. Since $f|_H$ is non-degenerate and H is finite dimensional the lemma now follows from 2.6.1(c). □

Lemma 2.11.12 [**q rational**]

(a) [**a**] $f^*(\alpha, \beta) \in \mathbb{Q}$ *for all* $\alpha, \beta \in \Phi$.

(b) [**b**] *The restriction of $f^*_{\mathbb{Q}}$ of f^* to $\mathbb{Q}\Phi$ is a positive definite symmetric $\mathbb{Q}$-bilinear form on $\mathbb{Q}\Phi$.*

(c) [**c**] *Any $\mathbb{Q}$-basis of $\mathbb{Q}\Phi$ is a $\mathbb{K}$-basis of H^*.*

Proof: Let $h \in H$. Then $f(h,h) = \mathrm{tr}_L(h^2)$. Since h acts trivially on H and acts as $\beta(h)$ on the 1-dimensional space L_β we have

(1) $$f(h,h) = \sum_{\beta\in\Phi} \beta(h)^2.$$

Since $t_\alpha \in H_\alpha = [L_a, L_a]$ we can apply 2.11.5(b) to $h = t_\alpha$. So there exists $q_\beta \in \mathbb{Q}$ such that

(2) $$f(t_\beta, t_\alpha) = \beta(t_\alpha) = q_\beta \alpha(t_\alpha) = q_\beta f(t_\alpha, t_\alpha)$$

Plucking (2) into (1) with $h = t_\alpha$ we obtain

$$f(t_\alpha, t_a) = \sum_{\beta\in\Phi} q_\beta^2 f(t_\alpha, t_\alpha)^2.$$

Since $(ft_\alpha, t_\alpha)) \neq 0$ we can divided by $f(t_\alpha, t_\alpha)$ to conclude $f(t_\alpha, t_\alpha) \in \mathbb{Q}$. From (2) we get $f^*(\beta, \alpha) = f(t_\beta, t_\alpha) \in \mathbb{Q}$. Put $H_{\mathbb{Q}} = \sum_{\alpha\in\Phi} \mathbb{Q}t_\alpha$ and note that $H_{\mathbb{Q}} = t(\mathbb{Q}\Phi)$. Then $f(h,h') \in \mathbb{Q}$ for all $h, h' \in H_{\mathbb{Q}}$. Thus also $\beta(h) = f(t_\beta, h) \in \mathbb{Q}$ for all $b \in \Phi$, $h \in H_{\mathbb{Q}}$. (1) now implies that $f(h,h) \geq 0$. Suppose $f(h,h) = 0$ then $\beta(h) = 0$ for all $\beta \in \Phi$. 2.11.11 shows that $h \in H \cap H^\perp = 0$. So $f\mid_{H_{\mathbb{Q}}}$ is positive definite and so (b) holds.

Let $\mathcal{B}$ be a $\mathbb{Q}$-basis for $H_{\mathbb{Q}}$. By 2.11.11, $\mathbb{K}\mathcal{B} = H$ and so $\mathcal{B}$ contains a $\mathbb{K}$-basis $\mathcal{D}$ for H. Let $h \in H_{\mathbb{Q}}$ with $h \perp \mathbb{Q}\mathcal{D}$. Then $H = \mathbb{K}\mathcal{D} \leq h^\perp$. Hence $h = 0$ and by (b) and 2.6.1(c), $\mathbb{Q}\mathcal{D} = H_{\mathbb{Q}}$. So $\mathcal{B} = \mathcal{D}$ and (c) is proved.

Chapter 3

Rootsystems

3.1 Definition and Rank 2 Rootsystems

Throughout this chapter $\mathbb{F}$ is a subfield of $\mathbb{R}$, E a finite dimensional vector space over $\mathbb{F}$ and $(\cdot,\cdot)$ a positive definite symmetric bilinear form on E. $E^\sharp = E \setminus \{0\}$.

Definition 3.1.1 [**def:root system**] *A subset Φ of $E^\sharp$ is called a root system in E provided that for all $\alpha, \beta \in \Phi$:*

(i) [**a**] $\omega_\beta(\alpha) \in \alpha + \mathbb{Z}\beta$ *(that is $(\alpha, \check{\beta}\,) \in \mathbb{Z}$)*

(ii) [**b**] $\omega_\alpha(\beta) \in \Phi$.

(iii) [**c**] $E = \mathbb{F}\Phi$

(iv) [**d**] $\mathbb{F}\alpha \cap \Phi = \{\alpha, -\alpha\}$.

If only (i) to (iii) hold, then Φ is called a weak root system. If only (i) holds then Φ is called a pre-root system.

Note that by 2.11.10, 2.11.11 and 2.11.12 the non-zero weights of a perfect semisimple standard Lie algebra are a root system in the dual of the Cartan subalgebra. The purpose of this chapter is determine all the roots system up to ismomophism. This will be used in later chapters to complete the classifications of the perfect semisimple standard Lie algebras.

Throughout this chapter Φ denotes a weak root system in E.

Definition 3.1.2 [**def:weyl groups**] *Let $\Delta \subseteq E^\sharp$ and $\alpha, \beta \in \mathbb{E}^\sharp$.*

(a) [**a**] $W(\Delta)$ *is the subgroups of the isometry group of $(\cdot,\cdot)$ generated by the $\omega_\alpha, \alpha \in \Delta$.*

(b) [**b**] $W = W(\Phi)$ *is called the Weyl group of Φ.*

(c) [**c**] $\langle\Delta\rangle = \bigcup \Delta^{W(\Delta)} = \{w(\delta) \mid w \in W(\Delta), \delta \in \Delta\}$.

(d) [**d**] *$\vartheta_{\alpha\beta}$ is the angle between α and β, that is the real number θ with $0 \le \theta \le 180$ and $\cos\theta = \frac{(\alpha,\beta)}{\sqrt{(\alpha,\alpha)}\sqrt{(\beta,\beta)}}$.*

(e) [**e**] *$m_{\alpha\beta} = (\alpha, \check{\beta})(\beta, \check{\alpha})$.*

Lemma 3.1.3 [**rank 2 root**] *Let $\alpha, \beta \in E^\sharp$. Then*

(a) [**a**] *$\cos^2 \vartheta_{\alpha\beta} = \frac{1}{4} m_{\alpha\beta}$.*

(b) [**b**] *$0 \le m_{\alpha\beta} \le 4$.*

(c) [**c**] *If $\alpha \not\perp \beta$ then $\frac{(\alpha,\alpha)}{(\beta,\beta)} = \frac{(\alpha,\check{\beta})}{(\beta,\check{\alpha})}$.*

(d) [**d**] *$\mathbb{F}a = \mathbb{F}b$ iff $m_{\alpha\beta} = 4$. In this case $\alpha = \frac{1}{2}(\alpha, \check{\beta})\beta$.*

Proof:

$$\cos^2 \vartheta_{\alpha\beta} = \frac{(\alpha,\beta)(\beta,\alpha)}{(\alpha,\alpha)(\beta,\beta)} = \frac{1}{4}(\alpha,\check{\beta})(\beta,\check{\alpha}) = \frac{1}{4} m_{\alpha\beta}$$

and so (a) holds. (b) follows immediately from (a). (c) follows easily from the definition of $\check{\alpha}$. For (d) note that $\mathbb{F}a = \mathbb{F}\beta$ iff $\vartheta_{\alpha\beta} \in \{0°, 180°\}$, that is iff $\cos^2 \vartheta_{\alpha\beta} = 1$. By (a) this holds iff $m_{\alpha\beta} = 4$. Suppose now that $\mathbb{F}\alpha = \mathbb{F}\beta$. Then $(\frac{1}{2}(\alpha,\check{\beta})\beta, \check{\alpha}) = \frac{1}{2}(\alpha,\check{\beta})(\beta,\check{\alpha}) = 2 = (\alpha,\check{\alpha})$ and so $\alpha = \frac{1}{2}(\alpha,\check{\beta})\beta$ □

Lemma 3.1.4 [**sab**] *Let $\{\alpha, \beta\} \in E^\sharp$ with $(\alpha,\check{\beta}) \in \mathbb{Z}$ and $(\beta,\check{\alpha}) \in \mathbb{Z}$ and $(\alpha,\alpha) \ge (\beta,\beta)$. Then one of the following holds:*

$(\alpha,\check{\beta})$	$(\beta,\check{\alpha})$	$\cos\vartheta_{\alpha\beta}$	$\vartheta_{\alpha\beta}$	$\frac{(\alpha,\alpha)}{(\beta,\beta)}$	
0	0	0	90°	?	
1	1	$\frac{1}{2}$	60°	1	
-1	-1	$-\frac{1}{2}$	120°	1	
2	1	$\frac{1}{\sqrt{2}}$	45°	2	
-2	-1	$-\frac{1}{\sqrt{2}}$	135°	2	
3	1	$\frac{\sqrt{3}}{2}$	30°	3	
-3	-1	$-\frac{\sqrt{3}}{2}$	150°	3	
2	2	1	0°	1	$\alpha = \beta$
-2	-2	-1	180°	1	$\alpha = -\beta$
4	1	1	0°	4	$\alpha = 2\beta$
-4	-1	-1	180°	4	$\alpha = -2\beta$

In particular, if $\mathbb{F}\alpha \ne \mathbb{F}\beta$, then $|(\beta,\check{\alpha})| = 1$.

Proof: Note that $|(\alpha,\check{\beta})| \ge |(\beta,\check{\alpha})|$. This follows easily from 3.1.3 □

Definition 3.1.5 [**def:discret**] *For $D \subseteq E$ define*

$$\text{min-d}(D) = \inf\{(e-d, e-d) \mid d \neq e \in D\}$$

and

$$\text{max-d}(D) = \sup\{(e-d, e-d) \mid d \neq e \in D\}.$$

We say that D is discret *if* $\text{min-d}(D) > 0$ *and that D is* bounded *if* $\text{max-d}(D) < \infty$

Lemma 3.1.6 [**discret**] *Let Δ be linear indepdendent subset of E. Then $\mathbb{Z}\Delta$ is discret.*

Proof: Since $\mathbb{Z}\Delta$ is closed under subtraction we need to show that $\inf_{t \in \mathbb{Z}\Delta^\sharp} (t, t) > 0$.

Let $d \in \Delta$ and put $\Sigma = \Delta \setminus d$. For $e \in E$ define $f_e \in \mathbb{F}$ and $\tilde{e} \in d^\perp$ by $e = f_e d + \tilde{e}$. Then Σ is a linear indepedent. By induction on Δ, $m := \inf_{s \in \mathbb{Z}\Sigma^\sharp} (s, s) > 0$. Let $0 \neq t \in \mathbb{Z}\Delta$. If $\tilde{t} \neq 0$, then $(t, t) \geq (\tilde{t}, \tilde{t}) \geq m$. If $\tilde{t} = 0$, then $t \in \mathbb{F}d \cap \mathbb{Z}\Delta^\sharp = \mathbb{Z}d$ and so $(t, t) \geq (d, d)$. $\square$

Lemma 3.1.7 [**discret and bounded**] *Let $D \subseteq E$ be discret and bounded. Then $|D|$ is finite and bounded by a function of $\frac{\text{max-d}(D)}{\text{min-d}(D)}$ and $\dim E$.*

Proof: Let $l = \text{min-d}(D)$, $u = \text{max-d}(D)$ and $n = \dim \mathbb{E}$. Let $\mathbb{E}_1$ be a 1-dimensional supspace of $\mathbb{E}$ and put $E_2 = E_1^\perp$. Let π_i the the projection of $\mathbb{E}$ onto $\mathbb{E}_i$. Let D_1 be a subset of $\pi_1(D)$ with $(d-e, d-e) \geq \frac{l}{4}$ for all $d, e \in D_1$. Since $(d-e, d-e) \leq u$ for all $d, e \in \pi_1(D)$, D_1 is a set in a 1-dim vector space within an interval of length u and the distance between each pair of point in it is greater than $\frac{l}{4}$, we have

$$(*) \quad |D_1| \leq \frac{4u}{l}$$

In particular, we can choose a maximal such D_1. For $e \in D_1$ let $D(e) = \{d \in D \mid (\pi_1(d) - e, \pi_1(d) - e) < \frac{l}{4}$. We claim that

$$(**) \quad D = \bigcup_{e \in D_1} D(e).$$

For if there is a $d \in D / \bigcup_{e \in D_1} D(e)$, then $(\pi_1(d) - e, \pi_1(d) - e) \geq \frac{l}{4} \forall e \in D_1$, by the maximality of D_1, $\pi_1(d)$ must be in D_1 and hence $(\pi_1(d) - e, \pi_1(d) - e) = 0$ for some $e \in D_1$ which is a contradiction. Now let $g, f \in D(e)$. Then $(\pi_1(g) - \pi_1(f), \pi_1(g) - \pi_1(f)) \leq \frac{l}{2}$ and so $(\pi_2(g) - \pi_2(f), \pi_2(g) - \pi_2(f)) \geq \frac{l}{2}$. In particular, $\pi_2 |_{D(e)}$ is $1-1$ and by induction on n, $|\pi_2(D(e))|$ is bounded by a function of $\frac{2u}{l}$ and $n-1$. (*) and (**) now imply the lemma. $\square$

Lemma 3.1.8 [**finite**] *Let Ψ be pre-root system in $\mathbb{E}$ and $E = \mathbb{F}\Psi$. Then $\mathbb{Z}\Psi$ is discret and Ψ is finite.*

Proof: Since E is finite dimensional and $E = \mathbb{F}\Psi$ there exists a finite subset Δ of Ψ with $\mathbb{F}\Delta = \mathbb{F}\Psi$. Denote $\check{\Delta} = \{\check{v} | v \in \Delta\}$ Let Σ be the basis of $\mathbb{E}$ dual to $\check{\Delta}$. Since $(\check{d}, \psi) \in \mathbb{Z}$ for all $\psi \in \Psi$ and $d \in \Delta$ we have $\Psi \subseteq \mathbb{Z}\Sigma$. Hence also $\mathbb{Z}\Psi \leq \mathbb{Z}\Sigma$. By 3.1.6, $\mathbb{Z}\Sigma$ is discret and so also Ψ and $\mathbb{Z}\Psi$ are discret.

Put $u = \max_{\delta \in \Delta} (\delta, \delta)$ and $\alpha \in \Psi$. Then there exists $\delta \in \Delta$ with $(\alpha, \delta) \neq 0$ and so by 3.1.4

$$(\alpha, \alpha) \leq 4(\delta, \delta) \leq 4u$$

thus Ψ is bounded. So by 3.1.7 Ψ is finite.

Lemma 3.1.9 [**basic <>**] *Let* $\Delta \subseteq E^\sharp$.

(a) [**a**] $W(\langle \Delta \rangle) = W(\Delta)$.

(b) [**b**] $W(\Delta) = W(\check{\Delta})$.

(c) [**c**] $\langle \Delta \rangle\check{} = \langle \check{\Delta} \rangle$.

Proof: Put $\Psi = \langle \Delta \rangle$. Clearly $W(\Delta) \subseteq W(\Psi)$. Let $\alpha \in \Psi$. Then $\alpha = w(\beta)$ for some $w \in W(\Delta)$ and $\beta \in \Delta$. Then by 2.6.3(e) $\omega_\alpha = w\omega_\beta w^{-1} \in W(\Delta)$ and so $W(\Psi) \subseteq W(\Delta)$. Thus (a) holds. (b) follows from $\omega_\alpha = \omega_{\check{\alpha}}$ and (c) follows from $w\check{(\alpha)} = w(\check{\alpha})$. □

From now on, for a weak-root or root system Φ, we denote $W(\Phi)$ by W.

Lemma 3.1.10 [**basic root**]

(a) [**a**] *$\check{\Phi}$ is a weak root system in E. If Φ is a root system, so is $\check{\Phi}$.*

(b) [**b**] *Φ is W-invariant, that is $w(\Phi) = \Phi$ for all $w \in W$.*

(c) [**c**] *W acts faithfully on Φ. In particular, W is finite.*

Proof: (a) follows immediately from 2.6.1 and the definition of a root system.

(b) Put $T = \{w \in GL(E) \mid w(\Phi) = \Phi)\}$. Let $\alpha \in \Phi$. Note that $\omega_\alpha(\Phi) \subseteq \Phi$ and since $\omega_\alpha^2 = 1$

$$\Phi = \omega_\alpha(\omega_\alpha(\Phi)) \subseteq \omega_\alpha(\Phi) \subseteq \Phi$$

Thus $\omega_\alpha \in T$. As T is a subgroup of $GL(E)$, we have conclude $W \leq T$.

(c) Let $w \in W$ with $w(\alpha) = \alpha$ for all $\alpha \in \Phi$. Since Φ spans E we get $w(e) = 1$ for all $e \in E$ and so $w = 1$. Hence W acts faithfully on Φ and so the homomorphism from W to $\mathrm{Sym}(\Phi)$ is one to one. By 3.1.8, Φ is finite. Therefore also $\mathrm{Sym}(\Phi)$ and W are finite. □

Definition 3.1.11 [**def:span**] *Let $\Psi \subseteq \Phi$ and R a subring of $\mathbb{F}$ (with $1 \in R$ and so $\mathbb{Z} \leq R$). Then* *(a)* [**a**] *Ψ is called a (weak) root subsystem of Φ if Ψ is a (weak) root system in $\mathbb{F}\Psi$.*

(b) [**b**] *Ψ is called R-closed if $\Psi = R\Psi \cap \Phi$.*

(c) [**d**] *$\langle \Psi \rangle_R$ denotes the smallest R-closed subset of Φ containing Ψ. $\langle \Psi \rangle_R$ is called the R-closure of Ψ.*

We often just say "subsystem" for "weak root subsystem". Note that if Φ is a root system and Ψ a weak roots subsystem then Ψ is already a root subsystem.

Lemma 3.1.12 [**closure**] *Let $\Psi \subseteq \Phi$ and R a subring of $\mathbb{F}$.*

(a) [**a**] *Ψ is a subsystem iff $\omega_\alpha(\beta) \in \Psi$ for all $\alpha, \beta \in \Psi$.*

(b) [**b**] *Ψ is a subsystem iff Ψ is invariant under $W(\Psi)$.*

(c) [**c**] *$\langle \Psi \rangle \subseteq \Phi$ and Ψ is the smallest subsystem of Φ containing Ψ.*

(d) [**d**] *Let T be an R-submodule in E. Then $T \cap \Phi$ is an R-closed subsystem of Φ.*

(e) [**e**] *$\langle \Psi \rangle_R$ is a subsystem of Φ and $\langle \Psi \rangle \subseteq \langle \Psi \rangle_R = \Phi \cap R\Psi \subseteq R\Psi$.*

Proof: (a) The forward direction is obvious. For the backward let $\alpha, \beta \in \Psi$. Then $-\alpha = \omega_\alpha(\alpha) \in \Psi$ and all the axiom of a root system are fulfilled.

(b) follows from (a).

(c) Let Σ be a any root subsystem of Φ containing Ψ. Since Σ is invariant under $W(\Sigma)$, $w(\psi) \in \Sigma$ for all $w \in W(\Psi) \leq W(\Sigma)$ and $\psi \in \Psi$. Thus $\langle \Psi \rangle \subseteq \Sigma$. In particular, $\langle \Psi \rangle \subseteq \Phi$.

By definition of $\langle \Psi \rangle$, $\langle \Psi \rangle$ is invariant under $W(\Psi)$. By 3.1.9 $W(\Psi) = W(\langle \Psi \rangle)$ and so $\langle \Psi \rangle$ is invariant under $W(\langle \Psi \rangle)$. Thus by (b), $\langle \Psi \rangle$ is a root subsystem.

(d) Let $\alpha, \beta \in T \cap \Phi$. Then $\omega_\beta(\alpha) = \alpha - (\alpha, \check{\beta})\beta \in \alpha + \mathbb{Z}\beta \in R\alpha + R\beta \leq T$ and so $\omega_\alpha(\beta) \in T \cap \Phi$. So by (a), $T \cap \Phi$ is a subsystem of Φ. Clearly $T \cap \Phi$ is R-closed and so (d) holds.

(e) Follows from (d) applied to $T = R\Psi$. $\square$

Lemma 3.1.13 [**creating root systems**] *Let $\Delta \subseteq E^\sharp$.*

(a) [**a**] *$\Sigma := \{\sigma \in \mathbb{Z}\Delta^\sharp \mid (\delta, \check{\sigma}) \in \mathbb{Z} \forall \delta \in \Delta\}$ is a weak roots system in $\mathbb{F}\Sigma$.*

(b) [**c**] *Suppose Δ is a pre-root system. Then $\langle \Delta \rangle$ is a weak root subsystem of Σ.*

(c) [**d**] *Suppose Δ is linearly independent pre-root system. Then $\Psi = \langle \Delta \rangle$ is a root system in $\mathbb{F}\Delta$*

Proof: (a) Let $\alpha, \beta \in \Sigma$.

1° [**1**] $(\alpha, (\,)\check{\beta}) \in \mathbb{Z}$.

Since $\alpha \in \Sigma \subseteq \mathbb{Z}\Delta$, $\alpha = \sum_{\delta\in\Delta} n_\delta \delta$ for some $n_\delta \in \mathbb{Z}$, almost all 0. Since $\beta \in \Sigma$, $(\delta, \check{\beta}\,) \in \mathbb{Z}$ for all $\delta \in \Delta$. So

$$(\alpha, \check{\beta}\,) = \sum_{\delta\in\Delta} n_\delta (\delta, \check{\beta}\,) \in \mathbb{Z}.$$

2° [**2**] $\omega_\beta(\alpha) \in \Sigma$.

By (1°), $(\alpha, \check{\beta}\,) \in \mathbb{Z}$ and so $\omega_\beta(\alpha) = \alpha - (\alpha, \check{\beta}\,)\beta \in \mathbb{Z}\Delta$. Let $\delta \in \Delta$. Then

$$(\delta, \omega_\beta(\alpha)\check{}\,) = (\delta, \omega_\beta(\check{\alpha})\,) = (\omega_\beta(\delta), \check{\alpha}\,) = (\delta - (\delta, \check{\beta}\,)\beta, \check{\alpha}\,) = (\delta, \check{\alpha}\,) - (\delta, \check{\beta}\,)(\beta, \check{\alpha}\,) \in \mathbb{Z}.$$

Note that (1°) and (2°) imply (a).

(b) Since Δ is a pre-root system $\Delta \subseteq \Sigma$. So by 3.1.12(b), Ψ is a weak root subsystem of Σ.

(c) By (b), Ψ a weak root system and $\Psi \subseteq \Sigma \subseteq \mathbb{Z}\Delta$. Let $n \in \mathbb{F}$ and $\alpha \in \Psi$ with $n\alpha \in \Psi$. We need to show that $n = \pm 1$. Since α is conjugate under $W(\Delta)$ to some element in Δ we may assume that $\alpha \in \Delta$. As $n\alpha \in \Sigma \subseteq \mathbb{Z}\Delta$, $n\alpha = \sum_{\delta\in\Delta} n_\delta \delta$ for some $n_\delta \in \mathbb{Z}$. Since Δ is linearly independent $n = n_\alpha \in \mathbb{Z}$.

By 3.1.9 $\check{\Psi} = \langle \check{\Delta} \rangle$.

Also $\check{n\alpha} = \frac{1}{n}\check{\alpha}$ and a symmetric result shows $\frac{1}{n} \in \mathbb{Z}$. Thus $n = \pm 1$. □

Definition 3.1.14 [**def:a string 2**] *Let $\alpha, \beta \in \Phi$. Then $\Delta = (\beta + \mathbb{F}\alpha) \cap \Phi$ is called the α-sring through β. Define a total ordering $\leq_\alpha$ on Δ by $\gamma \leq_\alpha \delta$ if $\delta - \gamma \in \mathbb{F}^{\geq 0}\alpha$. Let $\beta - r_{\alpha\beta}\alpha$ and $\beta + s_{\alpha\beta}\alpha$ be the minimal and maximal element in Δ with respect to $\leq_\alpha$.*

Lemma 3.1.15 [**a string**] *Suppose Φ is a root system, $\alpha, \beta \in \Phi$ and Δ is the α-string through β.*

(a) [**d**] *Suppose $\alpha \neq \pm\beta$. If $(\alpha, \beta\,) < 0$, then $\alpha + \beta \in \Phi$ and if $(\alpha, \beta\,) > 0$, $\alpha - \beta \in \Phi$.*

(b) [**a**] *ω_α leaves Δ invariant and reverses the $\prec_\alpha$ ordering. So if $\Delta = \{\beta_0, \beta_1, \ldots \beta_k\}$ with $\beta_0 <_\alpha \beta_1 <_\alpha \ldots <_\alpha \beta_k$, then $\omega_\alpha(\beta_i) = \beta_{k-i}$.*

(c) [**b**] *If $\beta = \pm\alpha$ then $\Delta = \{\pm\alpha\}$. Otherwise*

$$\Delta = \{\beta + i\alpha \mid -r_{\alpha\beta} \leq i \leq s_{\alpha\beta}, i \in \mathbb{Z}\}$$

In particular, $r_{\alpha\beta}$ and $s_{\alpha\beta}$ are integers.

(d) [**c**] *$(\beta, \check{\alpha}\,) = r_{\alpha\beta} - s_{\alpha\beta}$.*

Proof: (a) Suppose that $(\alpha, \beta) < 0$. Without loss $(\alpha, \alpha) \geq (\beta, \beta)$. Then by 3.1.4 $(\beta, \check{\alpha}) = -1$. Thus $\beta + \alpha = \omega_\alpha(\beta) \in \Phi$. The second statement follows from the first applied to α and $-\beta$.

(b) Let $\delta \in \Delta$. Then $\omega_\alpha(\delta) = \delta + (\delta, \check{\alpha})\alpha \in \beta + \mathbb{F}\alpha$ and so $\omega_\alpha(\delta) \in \Delta$. If $\gamma \in \Delta$ with $\gamma \leq \delta$, then $\delta = \gamma + f\alpha$ for a nonnegative $f \in \mathbb{F}$. Thus $\omega_\alpha(\delta) = \omega_\alpha(\gamma) - f\alpha$ and so $\omega_\alpha(\delta) \leq \omega_\alpha(\gamma)$.

(c) The case $\beta = \pm\alpha$ is obvious. So suppose $\alpha \notin \Delta$. Without loss $\beta = \beta_0$. Then $r_{\alpha\beta} = 0$. Let $f \in \mathbb{F}$ with $0 \leq f \leq s_{\alpha\beta}$. We need to show that $\delta := \beta + f\alpha \in \Phi$ iff $f \in \mathbb{Z}$. Since $\omega_\alpha(\beta) = b_k$ we have $b_k = \beta + s_{\alpha\beta}\alpha$ and so $\omega_\alpha(\delta) = \beta + (s_{\alpha\beta} - f)\alpha$. So replacing δ by $\omega_\alpha(\delta)$ if necessary we may assume that $(\delta, \alpha) \leq 0$.

Pick $i \in \mathbb{N}$ maximal with $i \leq f$ and $\gamma := \beta + i\alpha \in \Phi$. Put $k = f - i$. Then $\delta = \gamma + k\alpha$ and $k \geq 0$. If $k = 0$ then $f \in \mathbb{Z}$ and $\delta \in \Phi$. So we may assume that $k > 0$. Then $(\gamma, \alpha) < (\delta, \alpha) \leq 0$ and so by (a) $\gamma + \alpha \in \Phi$. The maximality of i shows $i + 1 > f$ and so $k < 1$. It remains to show that $\delta \notin \Phi$. Suppose for a contradiction that $\delta \in \Phi$. Then $(\delta, \check{\alpha}) = (\gamma, \check{\alpha}) + 2k$. As $0 < k < 1$ and both $(\delta, \check{\alpha})$ and $(\gamma, \check{\alpha})$ are integers this implies $k = \frac{1}{2}$. Hence $(\delta, \check{\gamma}) = 2 + \frac{1}{2}(\alpha, \check{\gamma})$. Thus $(\alpha, \check{\gamma})$ is even. Since $(\alpha, \check{\gamma}) < 0$ and we conclude from 3.1.4 that $(\alpha, \check{\gamma}) = -2$. Hence

$$\omega_\gamma(\alpha) = \alpha + 2\gamma = 2(\gamma + \frac{1}{2}\alpha) = 2\delta$$

and we obtained a contradiction to the definition of a roots system.

(d) Same proof as for 2.11.10(c). □

Definition 3.1.16 [**def rank**]

(a) [**a**] *The rank of Φ is the minimal size of subset Δ of Φ with $\Phi = \langle\Delta\rangle$.*

(b) [**b**] *Φ is called disconnected if it is the disjoint union of two proper perpendicular subsets. Otherwise, Φ is called connected.*

Let Φ be rank two roots system and choose α, β with $\Phi = \langle\alpha, \beta\rangle$. If $\alpha \perp \beta$ then $\Phi = \{\pm\alpha\} \cup \{\pm\beta\}$ and Φ is disconnected. Also $\beta \neq \pm\alpha$ since otherwise $\Phi = \langle\alpha\rangle$ as rank 1. Using 3.1.4 and 3.1.12(c) one now easily obtains a complete list of connected rank 2 root systems. See Figure 3.1.

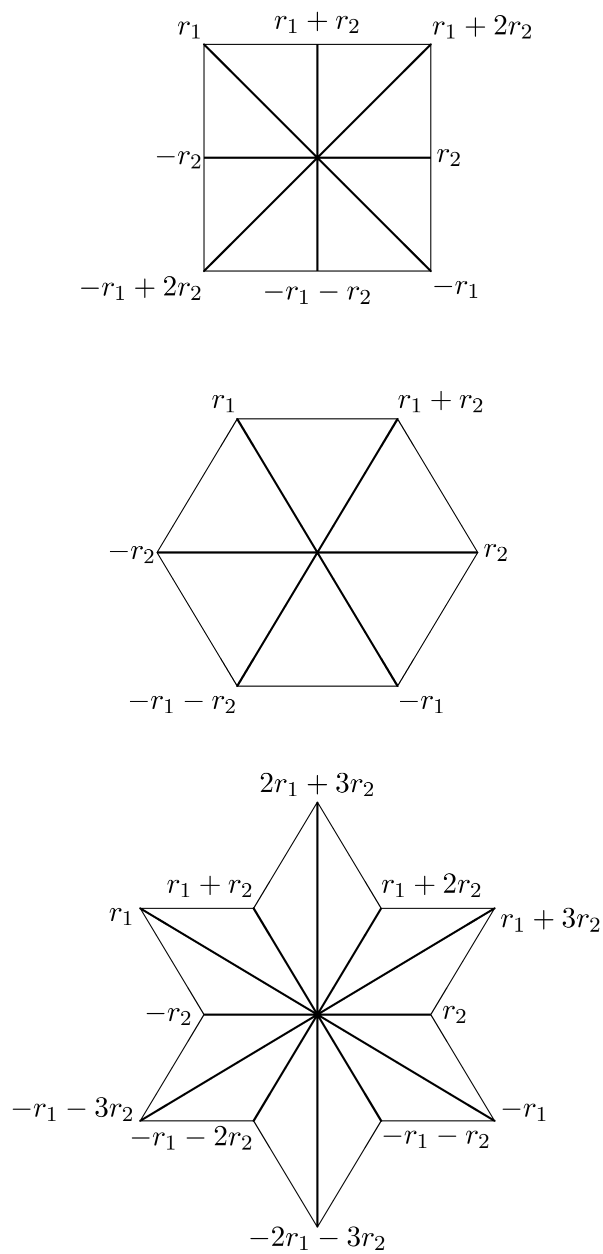
3.2 A base for root systems

Definition 3.2.1 [**def:base**]

(a) [**a**] *A subset Π of Φ is called base for Φ provided that Π is an $\mathbb{F}$-basis for E and $\Phi = \Phi^+ \cup \Phi^-$ where for $\Phi^+ = \mathbb{N}\Pi \cap \Phi$ and $\Phi^- = -\Phi^+$.*

[**rank2**]

Figure 3.1: The connected Rank 2 Root Systems



(b) [**b**] *Let* Π *be a base for* Φ*. The elements of* Π *are called simple roots and the element of* Φ^+ *are called positive roots. For* $e = \sum_{\alpha\in\Pi} f_\alpha \alpha$ *define* $\operatorname{ht} e = \sum_{\alpha\in\Pi} f_\alpha$*.* $\operatorname{ht} e$ *is called the height of* e *with respect to the base* Π*.*

In this section we show that Φ has a base and that any two base are conjugate under W.

Lemma 3.2.2 [**no finite cover**] *Let* V *be an finite dimensional vector sapce over an infinite field* $\mathbb{K}$ *and let* $\mathcal{H}$ *a finite set of proper subspace of* V*. Then* $V \neq \bigcup \mathcal{H}$*.*

By induction $\dim V$. Each $H \in \mathcal{H}$ lies in a hyperplane $\tilde{H}$ of V. Since K is infinite there exists infintely many hyperplane in V. So we can choose a hyperplane W of V with $W \neq \tilde{H}$ for all $H \in \mathcal{H}$. Then $W \neq W \cap H$ and so by induction there exists $w \in W$ with $w \notin W \cap H$ for all $H \in \mathcal{H}$. Thus $w \notin \bigcup \mathcal{H}$. □

Definition 3.2.3 [**def:regular**] $e \in E$ *is called regular if* $(\alpha, e) \neq 0$ *for all* $\alpha \in \Phi$*.*

Lemma 3.2.4 [**not perp**] *Let* S *be finite subset of* $E \setminus \{0\}$*. Then there exists* $e \in E$ *with* $(s, e) \neq 0$ *for all* $s \in S$*. In particular, there exist regular elements in* E*.*

Proof: By 3.2.2 $V \neq \bigcup_{s\in S} \alpha^\perp$. □

Lemma 3.2.5 [**s linear indep**] *Let* S *be a finite subset of* E *and* $e \in E$*. Suppose that*

$$(s, e) > 0 \quad \text{and} \quad (s, t) \leq 0$$

for all $s \neq t \in S$*. Then* S *is linearly independent.*

Proof: Let $f_s \in \mathbb{F}$ with $\sum_{s\in S} f_s s = 0$. Let $S_+ = \{s \in S \mid f_s > 0\}$ and $S_- = S \setminus S_+$. Then

$$u := \sum_{s_+ \in S^+} f_{s_+} s_+ = \sum_{s_- \in S_-} (-f_{s_-}) s_-$$

and

$$0 \leq (u, u) = \sum_{s_+ \in S_+} \sum_{s_- \in S_-} (-f_{s_+} f_{s_-})(s_+, s_-) \leq 0.$$

Therefore $u = 0$ and $0 = (u, e) = \sum_{s_+ \in S_+} f_{s_+}(s_+, e) \geq 0$. Hence $f_{s_+} = 0$ for all $s_+ \in S_+$. By symmetry, $f_{s_-} = 0$ for all $s_- \in S_-$ and so S is linearly independent. □

Proposition 3.2.6 (Existence of Bases) [**existence of bases**] *Let* $e \in E$ *be regular. with* $(\alpha, e) \neq 0$ *for all* $\alpha \in \Phi$*. Put* $\Phi_e^+ = \{\alpha \in \Phi \mid (\alpha, e) > 0\}$ *and* $\Pi_e = \Phi_e^+ \setminus (\Phi_e^+ + \Phi_e^+)$*. Then* Π_e *is a base for* Φ *and* $\Phi^+(\Pi_e) = \Phi_e^+$*.*

Proof: Let $\alpha, \beta \in \Pi_e$. Since $\alpha = (\alpha - \beta) + \beta$ we have that $\alpha - \beta \notin \Phi^+_e$. Also $\beta = (\beta - \alpha) + \alpha$ and so $\beta - \alpha \notin \Phi^+_e$. So $\alpha - \beta \notin \Phi$ and by 3.1.15(a), $(\alpha, \beta) \leq 0$. Thus by 3.2.5 Π_e is linearly independent.

Let $\alpha \in \Phi^+_e$. We will show by induction on (α, e) that $\alpha \in \mathbb{N}\Pi$. If $\alpha \in \Pi_e$, this is obvious. So suppose $\alpha = \beta + \gamma$ for some $\beta, \gamma \in \Phi^+_e$. Then $(\alpha, e) = (\beta, e) + (\gamma, e)$, $(\beta, e) < (\alpha, e)$ and $(\gamma, e) < (\alpha, e)$. So by induction $\beta \in \mathbb{N}\Pi$, $\gamma \in \mathbb{N}\Pi$ and so also $\alpha \in \mathbb{N}\Pi$.

Hence $\Phi^+_e = \mathbb{N}\Pi_e \cap \Phi = \Phi^+(\Pi_e)$. Thus $\Phi = \Phi^+_e \cup \Phi^-_e = \Phi^+(\Pi_e) \cup \Phi^-(\Pi_e)$. In particular as Φ spans E, so does Π_e and Π_e is a base for Φ. $\square$

3.3 Elementary Properties of Base

Lemma 3.3.1 [**ch and sum**] *Let Δ be a linearly independent subset of Φ and $e \in \langle \Delta \rangle$. Write $e = \sum_{\alpha \in \Delta} n_\alpha \alpha$. Then*

$$\check{e} = \sum_{\alpha \in \Delta} \frac{(\alpha, \alpha)}{(e, e)} n_\alpha \check{\alpha}.$$

and $\frac{(\alpha,\alpha)}{(e,e)} n_\alpha$ is an integer.

Proof: $\check{e} = \frac{2}{(e,e)} e = \sum_{\alpha \in \Delta} \frac{2(\alpha,\alpha)}{(e,e)(\alpha,\alpha)} n_\alpha \alpha = \sum_{\alpha \in \Delta} \frac{(\alpha,\alpha)}{(e,e)} n_\alpha \check{\alpha}$.

By 3.1.9(c), $\check{e} \in \langle \check{\Delta} \rangle$ and hence by 3.1.13, $\check{e} \in \mathbb{Z}\check{\Delta}$. The linear independence of Δ now shows that $\frac{(\alpha,\alpha)}{(e,e)} n_\alpha$ is an integer. $\square$

Lemma 3.3.2 [**basic base**] *Let Π be a base for Φ.*

(a) [**z**] *$\check{\Pi}$ is a base for $\check{\Phi}$ and $(\Phi^+)\check{} = (\check{\Phi})^+$.*

(b) [**a**] *Let $\alpha \neq \beta \in \Pi$. Then $\alpha - \beta \notin \Phi$ and $(\alpha, \beta) \leq 0$.*

(c) [**b**] *Let $\alpha \in \Phi$. Then $\operatorname{ht}\alpha$ is an integer, $\operatorname{ht}\alpha$ is positive if and only if α is positive and $\alpha \in \Pi$ if and only if $\operatorname{ht}\alpha = 1$.*

(d) [**c**] *Let $\alpha \in \Pi$. Then $\Phi^+ \setminus \{\alpha\}$ is ω_α invariant.*

(e) [**d**] *Let $\beta \in \Phi^+ \setminus \Pi$. Then there exists $\alpha \in \Pi$ with $(\alpha, \beta) > 0$. For any such α, both $\omega_\alpha(\beta)$ and $\beta - \alpha$ are in Φ^+ and $\operatorname{ht}(\omega_\alpha(\beta) \leq \operatorname{ht}(\beta - \alpha) = \operatorname{ht}\beta - 1$.*

(f) [**e**] *Let $\beta \in \Phi^+$. Then there exists $\alpha_1, \alpha_2, \ldots \alpha_k \in \Pi$ such that $\beta = \sum_{i=1}^k \alpha_i$ and for all $1 \leq j \leq k$, $\sum_{i=1}^j \alpha_i \in \Phi$.*

(g) [**f**] *Let $\beta \in \Phi^+$. Then there exists $\alpha_0, \alpha_1, \alpha_2, \ldots \alpha_k \in \Pi$ such that if we inductively define $\beta_0 = \alpha_0$ and $\beta_i = \omega_{\alpha_i}(\beta_{i-1})$ then $\beta = \beta_k$ and $\beta_i \in \Phi^+$ for all $0 \leq i \leq k$.*

(h) [**g**] $\Phi = \langle \Pi \rangle$, $W = W(\Pi)$ *and each root is conjugate under* W *to some root in* Π.

(i) [**h**] *Put* $\delta_\Pi = \frac{1}{2} \sum \Phi^+$. *Then for all* $\alpha \in \Pi$, $\omega_\alpha(\delta) = \delta - \alpha$.

Proof: (a) This follows from 3.3.1

(b) Note that neither $\alpha - \beta$ nor $\beta - \alpha$ is in $\mathbb{N}\Pi$. So the definition of a base implies $\alpha - \beta \notin \Phi$. The second statement now follows from 3.1.15.

(c) Obvious.

(d) Let $\beta = \sum_{\gamma \in \Pi} n_\gamma \gamma$ and $i = (\beta, \check{\alpha})$.

$$\omega_\alpha(\beta) = \beta - i\alpha = (n_\alpha - i)\alpha + \sum_{\alpha \neq \gamma \in \Pi} n_\gamma \gamma.$$

Suppose that $\omega_\alpha(\beta) \in \Phi^-$. Then $n_\gamma = 0$ for all $\alpha \neq \gamma \in \Phi$ and so $\beta \in \mathbb{N}\alpha \cap \Phi = \{\alpha\}$ and $\beta = \alpha$, contrary to the assumptions.

(e) $0 < (\beta, \beta) = \sum_{\alpha \in \Pi} n_\alpha (\beta, \alpha)$ and so $(\beta, \alpha) > 0$ for some $\alpha \in \Pi$. Suppose now that $\alpha \in \Pi$ with $i := (\beta, \check{\alpha}) > 0$. By (d), $\beta - i\alpha = \omega_\alpha(\beta) \in \Phi^+$. By 3.1.15(a), $\beta - \alpha \in \Phi$ and so $\beta - \alpha = \omega_\alpha(\beta) + (i-1)\alpha \in \Phi^+$.

(f) By induction on $\operatorname{ht} \beta$. If $\operatorname{ht} \beta = 1$, then $\beta \in \Pi$ and (f) holds with $k = 1$ and $\alpha_1 = \beta$. So suppose $\operatorname{ht} b > 1$ and thus $\beta \notin \Pi$. Choose α as in (e). By induction (f) holds for $\beta - \alpha$ and so also for β.

(g) By induction on $\operatorname{ht} \beta$. If $\operatorname{ht} \beta = 1$, then $\beta \in \Pi$ and (g) holds with $k = 0$ and $\alpha_o = \beta$. So suppose $\operatorname{ht} b > 1$ and thus $\beta \notin \Pi$. Choose α as in (e). By induction (f) holds for $\omega_\alpha(\beta)$ and so also for β.

(h) This follows from (g) and 3.1.12(c).

(i) By (d), ω_α fixes $\frac{1}{2} \sum_{\alpha \neq \beta \in \Phi^+} \beta$. Also $\omega_\alpha(\frac{1}{2}\alpha) = \frac{1}{2}\alpha - \alpha$ and so (i) holds. □

Lemma 3.3.3 [**bases equals chambers**] *Any bases is of the form* Π_e *for some regular element.*

Proof: Let Π be a base. Note that there exists a regular element e with $\Pi \subseteq \Phi_e^+$. (Indeed choose $\alpha^* \in E, \alpha \in \Pi$ with $(\alpha^*, \beta) = \delta_{\alpha\beta}$ and put $e = \sum_{\alpha \in \pi} \alpha^*$.) Then $\Phi^+(\Pi) \subseteq \Phi_e^+$ and $\Phi^-(\Pi) \subseteq \Phi_e^-$. Hence $\Phi^+(\Pi) = \Phi_e^+$ and by 3.3.2(e), $\Phi_e^+ \setminus (\Phi_e^+ + \Phi_e^+) = \Pi$. □

3.4 Weyl Chambers

Define two regular elements e and d to be equivalent if $\Phi_e^+ = \Phi_d^+$. The equivalence classes of this relation are called Weyl chambers. Note that there is natural 1-1 correspondence between Weyl chambers and bases for Φ. Also the equivalence relation is invariant under W and so W acts on the set Weyl of chambers. For a regular element e let $\mathfrak{C}(e)$ be the Weyl chamber containing e. For $\alpha \in \Phi$ let $P_\alpha(e) = \{d \in E \mid (\alpha, e)(\alpha, d) > 0\}$. Then

$$\mathfrak{C}(e) = \bigcap_{\alpha\in\Phi} P_\alpha(e)$$

Define $\overline{P}_\alpha(e) = \{d \in E \mid (\alpha, e)(\alpha, d) \geq 0\}$ and $\overline{\mathfrak{C}}(e) = \bigcap_{\alpha\in\Phi} \overline{P}_\alpha(e)$. $\overline{\mathfrak{C}}(e)$ is called a closed Weyl chamber. Topologically, $P_\alpha(e)$ and so also $\mathfrak{C}(e)$ are open convex subsets of E. $\overline{P}_\alpha(e)$ and $\overline{\mathfrak{C}}(e)$ are their closures.

Definition 3.4.1 [**def:dominant**] *Given a base Π of Φ. Let $e, d \in E$. We say that is positive if $0 < e \in \mathbb{F}^{\geq 0}\Pi$. Define the relation $\prec$ on E by $d \prec e$ if $e - d$ is positive. e is called dominant if $(e, \check{\alpha}) \geq 0$ for all $\alpha \in \Pi$. e is strictly dominat if $(e, \check{\alpha}) > 0$ for all $\alpha \in \Pi$. Let $\overline{\mathfrak{C}}$ and $\mathfrak{C}$ be the set of dominant and strictly dominant elements in E.*

Let $w \in W$. If $w = \omega_\alpha$ for a simple root α, then w is called a simple reflection.

$n = l(w)$ is the minimal integer such that there are simple reflections $\omega_1, \dots, \omega_n$ with $w = \omega_n\omega_{n-1}\dots\omega_1$. $n(w) = |\Phi^- \cap w(\Phi^+)|$.

Observe that $\overline{\mathfrak{C}} = \overline{\mathfrak{C}}(\delta_\Pi)$ and $\mathfrak{C} = \mathfrak{C}(\delta_\Pi)$.

Also $l(1) = 0$ and $l(\omega) = 1$ if and only if ω is a simple reflection; and $n(w)$ is the number of positive roots whose image under w is negative.

Proposition 3.4.2 [**existence of dominant**] *Let $e \in E$ and d be an element of maximal height in $W(e)$. Then d is dominant. In particular, there exists $w \in W$ with $w(e) \in \overline{\mathfrak{C}}$. If e is regular, then d is regular and $w(e) \in \mathfrak{C}$.*

Let $\alpha \in \Pi$. Then $\omega_\alpha(d) \in W(e)$ and $\omega_\alpha(d) = d - (d, \check{\alpha})\alpha$ has height $\mathrm{ht} d - (d, \check{\alpha})$. The maximal choice of $\mathrm{ht}\, d$ implies that $(d, \check{\alpha}) > 0$. Thus $d \in \overline{P}(\alpha)$ and $d \in \overline{\mathfrak{C}}$. □

Lemma 3.4.3 [**reducing**] *Let $w \in W$ and $(\omega_1, \omega_2, \dots \omega_t)$ be a tuple of simple reflections with $w = \omega_t\omega_{t-1}\dots\omega_1$. Suppose that α a positive root with $w(\alpha)$ negative. Then there there exists $1 \leq s \leq t$ with*

$$w\omega_\alpha = \omega_t\omega_{t-1}\dots\omega_{s+1}\omega_{s-1}\omega_{s-2}\dots\omega_1.$$

Proof: Put $\rho_i = \omega_i\omega_{i-1}\dots\omega_1$ and $\beta_i = \rho_i(\alpha_0)$. Choose s minimal such that β_s is negative. Then β_{s-1} is positive and $\beta_s = \omega_s(\beta_{s-1})$ is negative. Since ω_s is a simple refections there exists $\delta \in \Pi$ with $\omega_s = \omega_\delta$ with $\delta \in \pi$. Since $\omega_\delta(\beta_{s-1})$ is negative, 3.3.2(d) implies that $\beta_{s-1} = \delta$. Thus

$$\omega_s = \omega_\delta = \omega_{\beta_{s-1}} = \omega_{\rho_{s-1}(\alpha)} = \rho_{s-1}\omega_\alpha\rho_{s-1}^{-1},$$

$\rho_s = \omega_s\rho_{s-1} = \rho_{s-1}\omega_\alpha$ and $\rho_s\omega_\alpha = \rho_{s-1}$. Multiplying the last equation with $\omega_t\omega_{t-1}\dots\omega_{s+1}$ from the left gives the lemma. □

Lemma 3.4.4 [**n(w)=l(w)**] *Let $w \in W$ and $\alpha \in \Pi$.*

(a) [**b**] *If $w(\alpha)$ is negative, then $l(w\omega_\alpha) = l(w) - 1$ and $n(w\omega_\alpha) = n(w) - 1$*

(b) [**c**] *If $w(\alpha)$ is positive, then $l(w\omega_\alpha) = l(w) + 1$ and $n(w\omega_\alpha) = n(w) + 1$.*

(c) [**a**] *$l(w) = n(w)$.*

Proof: (a) Let $t = l(w)$ and choose simple roots $\omega_1, \ldots, \omega_t$ with $w = \omega_t\omega_{t-1}\ldots\omega_1$. Then by 3.4.3 $l(w\omega_\alpha) \leq l - 1$. Since $w\omega_\alpha\omega_\alpha = w$, $l(w\omega_\alpha) \geq l(w) - 1$ and so the first statement in (a) hold.

Let $\Sigma = \Phi^+ \setminus \{\alpha\}$. By 3.3.2(d) $\omega_\alpha(\Sigma) = \Sigma$. Hence also $w(\Sigma) \cap \Phi^- = (w\omega_\alpha)(\Sigma) \cap \Phi^-$. Now $w(\alpha) \in \Phi^-$ while $(w\omega_\alpha)(\alpha) \notin \Phi^-$. So also the second statement in (c) holds,

(b) We have $w\omega_\alpha(\alpha) = w(-\alpha)$ is negative. So (b) follows from (a) applied to $w\omega_\alpha$.

(c) Since $l(1) = n(1)$ this follows from (a) and (b) and induction on $l(w)$ □

Theorem 3.4.5 [**transitivity on bases**]

(a) [**a**] *Let $w \in W$ and $e \in \overline{\mathfrak{C}}$ with $w(e) \in \overline{\mathfrak{C}}$. Then $w(e) = e$ and $w \in W(\Pi \cap e^\perp)$. If, in addition, $e \in \mathfrak{C}$, then $w = 1$.*

(b) [**b**] *Let $\mathfrak{D}$ and $\mathfrak{D}'$ be Weyl chambers. Then there exists a unique $w \in W$ with $w(\mathfrak{D}) = \mathfrak{D}'$.*

(c) [**c**] *Let Π and Π' be bases for Φ. Then there exists a unqiue $w \in W$ with $w(\Pi) = \Pi'$.*

(d) [**d**] *$|W|$ is the number of Weyl chambers.*

(e) [**e**] *There exists a unique element $w_0 \in W$ with $n(w_0)$ maximal. Moreover $n(w_0) = l(w_0) = |\Phi^+|$, $w_0(\Pi) = -\Pi$, $w_0(\Phi^+) = \Phi^-$ and $w_0^2 = 1$.*

Proof: (a) If $e \in \mathfrak{C}$, then $\Pi \cap e^\perp = 0$. So it suffices to proof first statement. If $l(w) = 0$, $w = 1$ and (a) holds. So suppose $l(w) > 0$ and pick a simple root α with $l(w\omega_\alpha) < l(w)$. Then by 3.4.4, $w(\alpha)$ is negative. As both e and $w(e)$ are in $\overline{\mathfrak{C}}$ we have

$$0 \leq (e, \alpha) = (w(e), w(\alpha)) \leq 0$$

Thus $\alpha \in e^\perp$, $w\omega_\alpha(e) = w(e)$ and the results follows by induction on $l(w)$.

(b) Without loss of generality $\mathfrak{D}' = \mathfrak{C}$. Pick $d \in \mathfrak{D}$. Then by 3.4.2 there exists $w \in W$ with $w(e) \in \mathfrak{C}$. Then $w(\mathfrak{D}) = \mathfrak{C}$. Let w' be any element of W with $w'(\mathfrak{D}) = \mathfrak{C}$. Then $(w'w^{-1})(w(e)) = w'(e) \in \mathfrak{C}$ and so by (a) appled to $w'w^{-1}$ and $w(e)$ we have $w'w^{-1} = 1$ and so $w' = w$.

(c) and (d) follow immediately from (b).

(e). Note that $n(w) \leq |\Phi^-| = |\Phi^+|$ for all $w \in W$. Also $n(w) = |\Phi^+|$ if and only if $w(\Phi^+) = \Phi^-$ and so if and only if $w(\Pi) = -\Pi$. By (c) such an w exists and is unique. Also $w^2(\Pi) = \Pi$ and so $w^2 = 1$. Thus (e) holds. □

Definition 3.4.6 [**def:obtuse**] *A subset S of $E \setminus \{0\}$ is called acute (obtuse) if $(s, t) \geq 0$ ($(s, t) \leq 0$) for all $s \neq t \in E$.*

Lemma 3.4.7 [**easy base**] *Let Δ be a linear independent obtuse preroot system in E. Then Δ is base for the root system $\langle \Delta \rangle$ in $\mathbb{F}\Delta$.*

Proof: Put $\Phi = \langle \Delta \rangle$. Then by 3.1.13 Φ is a root sytem.

Let $\alpha \in \Phi$ and write $\alpha = \sum_{\beta \in \Delta} n_\beta \beta$ with $n_\beta \in \mathbb{Z}$. We need to show that the non-zero n_β all have the same sign. Suppose not and choose such an α with $\sum_{\beta \in \Delta} |n_\beta|$ minimal. Since (α, α) is positive there exists $\delta \in \Delta$ with $n_\delta(\alpha, \delta) \geq 0$. Replacing α by $-\alpha$ if necessary we may assume that $n_\delta > 0$. Then also (α, δ) is positive. Note that $\alpha \notin \mathbb{F}\delta$ and so by 3.1.15(a), $\alpha - \delta$ is a root. Now

$$\alpha - \delta = (n_\delta - 1)\delta + \sum_{\delta \neq \beta \in \Delta} n_\beta \beta$$

By the minimal choice of α, the non zero coefficents of $\alpha - \delta$ are either all positive or all negative. Let $\delta \neq \beta \in \Delta$. If $n_\beta > 0$ then $n_\gamma > 0$ for all $\gamma \in \Delta$, contrary to our assumptions. Hence $n_\beta \leq 0$. Thus also $n_\delta - 1 \leq 0$. But $n_\delta - 1 \geq 0$ and so $n_\delta - 1 = 0$ and $n_\delta = 1$. Since $(\beta, \delta) \leq 0$ this implies that

$$(\alpha - \delta, \delta) = \sum_{\delta \neq \beta \in \Delta} n_\beta (\beta, \delta) \geq 0.$$

So $(\alpha, \check{\delta}) = (\alpha - \delta, \check{\delta}) + (\delta, \check{\delta}) \geq (\delta, \check{\delta}) = 2$. Note also that $\alpha \neq \delta$ and so 3.1.4 implies that $(\delta, \delta) < (\alpha, \alpha)$. On the other hand by 3.3.1 $\frac{(\delta,\delta)}{(\alpha,\alpha)} n_\delta$ is an integer. Since $n_\delta = 1$, this implies $(\delta, \delta) \geq (\alpha, \alpha)$, a contradiction. □

3.5 Orbits and Connected Components

Definition 3.5.1 [**def:coxeter graph**] *Let $\Sigma \subseteq \Phi$.*

$$\Gamma(\Sigma) = \{(\alpha, \beta, i) \mid \alpha, \beta \in \Sigma, (\alpha, \check{\beta}) \neq 0, i \in \mathbb{Z}_+, i \leq |(\alpha, \check{\beta})|\}$$

We view $\Gamma(\Sigma)$ as a multiple edged directed graph on Σ, namely each $(\alpha, \beta, i) \in \Gamma(\Sigma)$ is an edge from α to β. So if $(\alpha, \check{\beta})) = 0$, then there exists no edge from α to β, and if $(\alpha, \check{\beta}) \neq 0$, then there exists $|(\alpha, \check{\beta})|$ edges from α to β. $\Gamma(\Phi)$ is called the Coxeter graph of Φ. $\Gamma(\Pi)$ is called the Dynkin diagram of Φ. For $S \subseteq E$ let $\Gamma^0(S)$ be the undirect graph without multiply edges, where s, t are adjacent if and only if $(s, t) \neq 0$.

Note that for $\Sigma \subseteq \Phi$, $\Gamma(\Sigma)$ and $\Gamma^0(\Sigma)$ have the same connected component.

Lemma 3.5.2 [**connected components**] *Let Φ be a root system.*

(a) [**a**] *Let $\alpha, \beta \in \Phi$ with $\alpha \not\perp \beta$. Then α, β and $\omega_\alpha(\beta)$ are in the same connected component (with respect to the coxeter graph $\Gamma(\Phi)$).*

(b) [**b**] *Let $\mathcal{D}$ be the set of connected components of Φ . Then $E = \bigoplus_{\Lambda\in\mathcal{D}} \mathbb{F}\Lambda$ and $W = \bigtimes_{\Lambda\in\mathcal{D}} W(\Lambda)$.*

(c) [**c**] *Let $\tilde{\Delta}$ be a connected component of Φ. Then $\tilde{\Delta}$ is invariant under W, $\tilde{\Delta}$ is a subsystem of Φ, $\tilde{\Delta}\cap\Pi$ is a base for $\tilde{\Delta}$, $\tilde{\Delta} = \langle\tilde{\Delta}\cap\Pi\rangle$ and $\tilde{\Delta}\cap\Pi$ is connected.*

(d) [**d**] *The map $\Delta \to \langle\Delta\rangle$ is one 1-1 correspondence between the connected components of Π and the connected components of Φ.*

(e) [**e**] *Φ is connected iff Π is connected.*

Proof: Since $\alpha \not\perp \beta$ we also have $\alpha \not\perp \omega_\alpha(\beta)$ and so (a) holds.

Let Δ be a connected component of Π. Also let $\tilde{\Delta}$ be the connected component of Φ containing Δ . We claim that $\tilde{\Delta}$ is W-invariant. For this let $\alpha \in \Phi$ and $\beta \in \tilde{\Delta}$. If $\alpha \perp \beta$ then $\omega_\alpha(\beta) = \beta \in \tilde{\Delta}$. If $\alpha \not\perp \beta$, then by (a), $\omega_\alpha(\beta)$ and β are in the same connected component of $\Gamma(\Phi)$ and again $\omega_\alpha(\beta) \in \tilde{\Delta}$. So $\tilde{\Delta}$ is invarinat under all $\omega_\alpha, \alpha \in \Phi$ and so also under W. Thus $\langle\Delta\rangle = \bigcup \Delta^{W(\Delta)} \subseteq \tilde{\Delta}$.

Put $\Sigma = \Pi \setminus \Delta$. Then $\Sigma \perp \Delta$. Hence $W(\Delta)$ centralizes $W(\Sigma)$, $W = W(\Delta)W(\Sigma)$. In particular, $\langle\Delta\rangle \perp \langle\Sigma\rangle$ and $\langle\Delta\rangle$ and $\langle\Sigma\rangle$ are $W(\Delta)W(\Sigma) = W$ invariant

Thus
$$\Phi = \bigcup \Pi^W = \bigcup \Delta^W \cup \bigcup \Sigma^W = \langle\Delta\rangle \cup \langle\Sigma\rangle$$

Since $\tilde{\Delta}$ is connected, this implies $\tilde{\Delta} \subseteq \langle\Delta\rangle$ and $\tilde{\Delta}\cap\langle\Sigma\rangle = \emptyset$. Hence $\tilde{\Delta} = \langle\Delta\rangle$ and so by 3.4.7 Δ is a base for $\tilde{\Delta}$. Moreover $\Delta\cap\Pi = \Delta \cup (\tilde{\Delta}\cap\Sigma) = \Delta$ and so (c) holds.

Note that $W(\Delta)\cap W(\Sigma)$ centralizes $\mathbb{F}\Delta + \mathbb{F}\Sigma = E$ and so $W(\Delta)\cap W(\Sigma) = 1$ and $W = W(\Delta)\times W(\Sigma)$. An easy induction proof now shows that (b) holds.

(d) and (e) follow easily from from (c). □

Lemma 3.5.3 [**z closed**] *Let Φ be a root system and $\Psi \subseteq \Phi$. Then Ψ is $\mathbb{Z}$-closed iff $-\Psi \subseteq \Psi$ and $\alpha+\beta \in \Psi$ for all $\alpha, \beta \in \Psi$ with $\alpha+\beta \in \Phi$.*

Proof: One direction id obvious. For the other suppose that $-\alpha \in \Psi$ for all $\alpha \in \Psi$ and $\alpha+\beta \in \Psi$ for all $\alpha, \beta \in \Psi$ with $\alpha+\beta \in \Phi$. Let $\alpha \in \langle\Psi\rangle_\mathbb{Z}$. Then $\alpha = \sum_{\beta\in\Psi} n_\beta\beta$ with $n_\beta \in \mathbb{Z}$. Since $n_\beta\beta = (-n_\beta)(-\beta)$ we may assume that $n_\beta \geq 0$ for all $\beta \in \Psi$.

Since $\sum_{\beta\in\Psi} n_\beta(\alpha, \beta) = (\alpha, \alpha) > 0$ there exists $\delta \in \Psi$ with $n_\delta(\alpha, \delta) > 0$. Thus $n_\delta \geq 1$ and $(\alpha, \delta) > 0$. If $\alpha = \pm\delta$, then $\alpha \in \Psi$. If $\alpha \neq \pm\delta$ then by 3.1.15(a), $\alpha - \delta \in \Phi$. Thus $\alpha - \delta \in \langle\Psi\rangle_\mathbb{Z}$ and by induction on $\sum_{\beta\in\Psi} n_\beta$ we conclude that $\alpha - \delta \in \Psi$. Thus $\alpha = (\alpha-\delta)+\delta \in \Psi$. □

Lemma 3.5.4 [**root lengths**] *Let Φ be a connected root system, $\mathcal{L}(\Phi) = \{(\alpha, \alpha) \mid \alpha \in \Phi\}$. Let $r \in \mathcal{L}(\Phi)$ and put $\Phi_r = \{\alpha \in \Phi \mid (\alpha, \alpha) = r\}$.*

(a) [**a**] $E = \mathbb{F}\Phi_r$ *and* $\Phi = \langle \Phi_r \rangle_{\mathbb{Q}}$

(b) [**b**] *If* r *is minimal in* $\mathcal{L}(\Phi)$, *then* $\Phi = \langle \Phi_r \rangle_{\mathbb{Z}} \leq \mathbb{Z}\Phi_r$.

(c) [**c**] W *acts transitively on* Φ_r.

(d) [**e**] *If* r *is maximal in* $\mathcal{L}(\Phi)$, *then* $\Phi_r = \langle \Phi_r \rangle_{\mathbb{Z}}$ *is* $\mathbb{Z}$*-closed.*

(e) [**d**] $|\mathcal{L}(\Phi)| \leq 2$.

Proof: Let Σ be an orbit for W on Φ_r and let $\alpha \in \Phi$. Suppose that $\alpha \not\perp \sigma$ for some $\sigma \in \Sigma$. Then

$$(*) \qquad \alpha = \frac{1}{(\sigma, \check{\alpha}\,)}(\sigma - \omega_\alpha(\sigma)) \in \mathbb{Q}\Sigma$$

Thus $\Phi = (\Phi \cap (\mathbb{Q}\Sigma)^{\perp}) \cup \langle \Sigma \rangle_{\mathbb{Q}}$ and since Φ is connected, $\Phi = \langle \Sigma \rangle_{\mathbb{Q}} \subseteq \langle \Phi_r \rangle_{\mathbb{Q}} \subseteq \Phi$ and so (a) holds. In particular, $\Sigma^{\perp} \cap \Phi = \emptyset$. If r is minimal in $\mathcal{L}(\Phi)$, then $(\alpha, \alpha\,) \geq (\sigma, \sigma\,)$. So by 3.1.4 either $\alpha = \pm\sigma$ or $(\sigma, \check{\alpha}\,) = \pm 1$. From (*) we get in any case that $\alpha \in \mathbb{Z}\Sigma$ and so (b) holds.

Suppose now that $\alpha \in \Phi_r$. Then either $\alpha = \pm\sigma$ or $\langle \alpha, \sigma \rangle$ is a root system of type A_2. In either case α and σ are conjugate in $W(\langle \alpha, \sigma \rangle)$. Thus (c) holds.

Suppose that r is maximal. Let $\alpha, \beta \in \Phi_r$ with $\alpha + \beta \in \Phi$. Then $\alpha \neq \pm\beta$ and since $(\alpha, \alpha\,) = (\beta, \beta\,)$ 3.1.4 implies $(\alpha, \check{\beta}\,) \geq -1$. Thus

$$\begin{aligned}(\alpha+\beta, \alpha+\beta\,) = {} & (\alpha, \alpha\,) + 2(\alpha, \beta\,) + (\beta, \beta\,) & = (\alpha, \alpha\,) + (\alpha, \check{\beta}\,)(\beta, \beta\,) + (\beta, \beta\,) \\ = {} & (2 + s\alpha\beta)(\beta, \beta\,) & \geq \quad (\beta, \beta\,)\end{aligned}$$

So $\alpha + \beta \in \Phi_r$. (d) now follows from 3.5.3.

Let $s, l \in \mathcal{L}(\Phi)$ with $s < l$. Then by (a) we can choose $\beta \in \Phi_s$ and $\alpha \in \Phi_l$ with $\beta \not\perp \alpha$. Then by 3.1.4, $\frac{l}{s} = \frac{(\alpha,\alpha\,)}{(\beta,\beta\,)} \in \{2, 3\}$. If $|\mathcal{L}(\Phi)| > 2$, we can choose $s < l < t \in \mathcal{L}(\Phi)$. But then $\frac{t}{s} = \frac{l}{s}\frac{t}{l}$ is not a prime, a contradiction and (e) holds. $\square$

Lemma 3.5.5 [**dominant**] *Let* Φ *be a root system in* E *and* $\tilde{E}$ *an Euclidean* $\mathbb{F}$ *space with* $E \leq \tilde{E}$. *Let* Δ *be an orbit for* W *on* $\tilde{E}$ *and* $e \in \overline{\mathfrak{C}}$. *Then*

(a) [**a**] Δ *contains a unique dominant member* d.

(b) [**b**] $b \prec d$ *and for all* $b \in \Delta$.

(c) [**c**] $(e, b\,) \leq (e, d\,)$ *for all* $b \in \Delta$.

(d) [**d**] *Let* $\Pi \cap e^{\perp}$ *is a basis for* $\Phi \cap e^{\perp}$.

(e) [**e**] *Let $b \in \Delta$ with $(e, b) = (e, d)$. Then there exists $w \in W(\Pi \cap e^\perp)$ with $w(d) = b$.*

Proof:

(a) Let e and d be dominant in Δ. Then $d = w(e)$ for some $w \in W$ and 3.4.5(a) implies that $d = e$.

(b) Choose $a \in \Delta$ such that $b \prec a$ and a is maximal in Δ with respect to $\prec$. We claim that a is dominant. Otherwise there exists $\beta \in \Pi$ with $(a, \check{\beta}) < 0$. Then $a \prec a - (a, \check{\beta})\beta = \omega_\beta(a) \in \Delta$, a contradiction to the maximality of a. Thus a is dominant and so by (a) $a = d$ and (b) holds.

(c) By (b) $d - b \in \mathbb{N}\Pi$ and since e is dominant, $(e, d - b) \geq 0$.

(d) Let $\beta \in \Phi^+ \cap e^\perp$ we need to show that $\beta \in \mathbb{N}(\Pi \cap e^\perp)$. If $\mathrm{ht}\,\beta = 1$, then $\beta \in \Pi \cap e^\perp$. So suppose $\mathrm{ht}\,\beta > 1$, that is $\beta \notin \Pi$. Then by 3.3.2(e) there exists $\alpha \in \Pi$ with $\delta = \beta - \alpha \in \Phi^+$. Since $e \in \overline{\mathfrak{C}}$ both (e, α) and (e, δ) are non-negative. Since $0 = (e, \beta) = (e, \alpha) + (e, \delta)$ we conclude that both α and δ are in $e^\perp$. $\mathrm{ht}\,\delta < \mathrm{ht}\,\beta$ and so by induction on $\mathrm{ht}\,\beta$, $\delta \in \mathbb{N}(\Pi \cap e^\perp)$. Hence $\beta \in \mathbb{N}(\Pi \cap e^\perp)$

(e) If b is dominant, then by (a) $b = d$ and we are done. So suppose that b is not dominant and and choose exists $\alpha \in \Pi$ with $(\beta, \check{\alpha}) < 0$. Then $c := \omega_\alpha(b) \in \Delta$, $(e, c) = (e, b) - (\beta, \check{\alpha})(e, \alpha) \geq (e, b)$. On the other hand, by (c) $(e, c) \leq (e, d)$. This implies $(e, c) = (e, d)$ and $(e, \alpha) = 0$. So $\alpha \in \Pi \cap e^\perp$ and by induction on $\mathrm{ht}\,b$, $c = w(d)$ for some $w \in W(\Pi \cap e^\perp)$. Hence $b = \omega_\alpha(c) = (\omega_\alpha w)(d)$ and (e) holds. □

3.6 Cramer's Rule and Dual Bases

Lemma 3.6.1 (Cramer's Rule) [**cramer rule**] *Let I a finite set, R a commutative ring with 1, $A : I \times I \to R$ be $I \times I$-matrix over R. Define $(i, j) \in I \times I$ to be an edge if $a_{ij} \neq 0$. Let $S(i, j)$ be the set of all direct paths $s = (i_0, i_1, \ldots, i_n)$ from i to j, where $i_0 = i, i_n = j, (i_{k-1}, i_k)$ is an edge $\forall k = 1, \ldots, n$ and the i_k's are pairwise distinct. Put $|s| = n$, $m(s) = \prod_{k=1}^n a_{i_{k-1} i_k}$ and $I - s = I \setminus \{i_0, i_1, \ldots i_n\}$. For $J \subseteq I$ let A_J be the restriction of A to $J \times J$. Define*

$$b_{ij} = \sum_{s \in S(i,j)} (-1)^{|s|} m(s) \det A_{I-s}.$$

and $B = (b_{ij})$. Then $AB = \det(A)\,\mathrm{Id}_I$, where Id_I is the $I \times I$ identity matrix.

Proof: Let $i, j \in I$ and define the matrix $D = D^{ij}$ by $d_{kl} = a_{kl}$ if $k \neq j$ and $d_{jl} = \delta_{il}$. We will show that $b_{ij} = \det D$. For $K \subseteq I$ and $\sigma \in \mathrm{Sym}(K)$ define

$$a(\sigma) = \mathrm{sgn}(\sigma) \prod_{k \in K} a_{k\sigma(k)}.$$

Similarly define $d(\sigma)$. Then

$$\det D = \sum_{\pi \in \mathrm{Sym}(I)} d(\pi)$$

We investigate $d(\pi)$ for $\pi \in \mathrm{Sym}(I)$. If $\pi(j) \neq i$, then $d_{j\pi(j)} = \delta_{i\pi(j)} = 0$ and so also $d(\pi) = 0$.

Suppose $\pi(j) = i$. Let $n \in \mathbb{N}$ be minimal with $\pi^{n+1}(i) = i$. For $0 \leq k \leq n$, put $i_k = \pi^k(i)$ and $s = (i_0, i_1, \ldots, i_n)$. The i_k are pairwise distinct, $i_0 = i$ and $i_n = j$. If (i_{k-1}, i_k) not an edge for some $1 \leq k \leq n$, then $d_{i_{k-1}\pi(i_{k-1})} = a_{i_{k-1}i_k} = 0$ and so also $d(\pi) = 0$.

Suppose s is a string and view s as a cycle in $\mathrm{Sym}(\{i_0, \ldots, i_n\})$. Then $\pi = s\sigma$ for a unique $\sigma \in \mathrm{Sym}(I - s)$. Now $d(\pi) = d(s)d(\sigma)$, $\mathrm{sgn}(s) = (-1)^n = (-1)^{|s}|$ and $d_{ji} = 1$. Thus $d(s) = (-1)^{|s|}m(s)$, $d(\sigma) = a(\sigma)$ and so

$$d(\pi) = (-1)^{|s|}m(s)a(\sigma).$$

It follows that

$$\begin{aligned}
\det D &= \sum_{\pi\in\mathrm{Sym}(I)} d(\pi) \\
&= \sum_{s\in S(i,j)} \sum_{\sigma\in\mathrm{Sym}(I-s)} (-1)^{|s|}m(s)a(\sigma) \\
&= \sum_{s\in S(i,j)} (-1)^{|s|}m(s)\det A_{I-s}
\end{aligned}$$

Thus indeed $b_{ij} = \det D^{ij}$.

Note that $\sum_{j\in J} a_{ij}b_{jk} = \sum_{j\in J} a_{ij} \det D^{jk}$ is the determinant of the matrix E^{ik} obtained from A by replacing row k of A by row i. Now $\det E^{ik} = \delta_{ik}\det(A)$ and so $AB = \det A\, \mathrm{Id}_I$.
□

Lemma 3.6.2 [**dual basis**] *Let $\mathcal{B}$ be a basis for E. For $b \in \mathcal{B}$ define $b^* \in E$ by $(b^*, a) = \delta_{ba}$ $\forall a \in \mathcal{B}$. Put $\mathcal{B}^* = \{b^* \mid b \in \mathcal{B}\}$ and let $A(\mathcal{B})$ be the $I \times I$ matrix $((a, b))$, $a, b \in \mathcal{B}$.*

(a) [**a**] *Then $d = \sum_{b\in\mathcal{B}} (d, b)b^* = \sum_{b\in\mathcal{B}} (d, b^*)b$.*

(b) [**b**] $A(\mathcal{B}^*) = A(\mathcal{B})^{-1}$.

(c) [**c**] $\det A(\mathcal{B}) > 0$.

(d) [**d**] *Suppose that $\mathcal{B}$ is obtuse. Then $\mathcal{B}^*$ is acute and, for $a, b \in \mathcal{B}$, $(a^*, b^*) > 0$ if and only if a and b lie in the same connected component of the $\perp$-graph on $\mathcal{B}$.*

Proof: (a) Let $d = \sum_{b\in\mathcal{B}} f_b b$ and let $a \in \mathcal{B}$. Then $(d, a^*) = f_a$. Also $\mathcal{B}^{**} = \mathcal{B}$ and so (a) holds.

(b) Let $a \in \mathcal{B}$. Then by (a),

$$a = \sum_{b\in\mathcal{B}} (a, b)b^* = \sum_{b\in\mathcal{B}} \sum_{d\in\mathcal{B}} (a, b)(b^*, d^*)d$$

and so (b) holds.

(c) Let $\mathcal{E}$ be an orthogonal basis for E and let D be the $\mathcal{B} \times \mathcal{E}$ matrix defined by $b = \sum d_{be} e$. Then $A(\mathcal{B}) = DA(\mathcal{E})D^{\mathrm{T}}$ and so $\det A(\mathcal{B}) = (\det D)^2 \det A(\mathcal{E})$. Since $A(\mathcal{E})$ is a diagonal matrix with positive diagonal elements, $\det A(\mathcal{E})$ is positive and so (c) holds.

(d) Let $A = A(\mathcal{B})$. From (b) and 3.6.1 we have

$$(a^*, b^*) = \frac{1}{\det}(A) \sum_{s \in S(a,b)} (-1)^{|s|} m(s) \det A_{\mathcal{B}-s}.$$

Since $\mathcal{B}$ is obtuse $m(s)$ is the product of $|s|$ negative elements. Hence $(-1)^{|s|}m(s)$ is positive. By (c) also $\det A_{I-s}$ and $\det(A(\mathcal{B}))$ are positive. Hence (a^*, b^*) is non-negative and $(a^*, b^*) = 0$ if and only if $S(a,b) = \emptyset$. So (d) holds. □

3.7 Minimal Weights

Throughout this section Φ is root system. We call a root α long (short) if $(\alpha, \alpha) \geq (\beta, \beta)$ ($(\alpha, \alpha) \leq (\beta, \beta)$) for all $\beta \in \Phi$, which are on the same connected component of Φ as α. Note that if Φ has only one root length then all roots are long and short. Φ_{l} and Φ_{s} denotes the sets of long and short roots in Φ, respectively.

Definition 3.7.1 [**def:weights for phi**] *Let $\lambda \in E$. We say that λ is an integral weight of Φ if $(\lambda, \alpha) \in \mathbb{Z}, \alpha \in \Phi$. For $\alpha \in \Pi$ define $\alpha^* \in E$ by $(\alpha^*, \beta) = \delta_{\alpha\beta}$ for all $\beta \in \Pi$. For $e = \sum_{\alpha \in \Pi} f_\alpha \alpha$ put $e^* = \sum_{\alpha \in \Pi} f_\alpha \alpha^*$.*

$\check{\Lambda} = \check{\Lambda}(\Phi)$ is the set of integral weights and $\Pi^* = \{\alpha^* \mid \alpha \in \Pi\}$. $\check{\Lambda}^+$ is the set of dominant integral weights.

Lemma 3.7.2 [**z basis**]

(a) [**a**] $\check{\Phi} \subset \check{\Lambda}$.

(b) [**b**] *Let $e \in E$. Then $e = \sum_{\alpha \in \Pi} (e, \alpha)\alpha^*$. In particular, $\overline{\mathfrak{C}} = \mathbb{F}^{\geq 0}\, \Pi^*$.*

(c) [**c**] Π^* *is a $\mathbb{Z}$-basis for $\check{\Lambda}$*

(d) [**d**] $\overline{\mathfrak{C}}^\sharp$ *is acute and, if Φ is connected, strictly acute.*

(e) [**e**] *Let $e \in \overline{\mathfrak{C}}^\sharp$ then $e = \sum_{\alpha \in \Pi} (e, \alpha^*)\alpha$, $(e, a^*) \geq 0$ and if Φ is connected, $(e, \alpha^*) > 0$.*

Proof: (a) follows directly from the definition of a root system.

(b) Follows from 3.6.2(a).

(c) Since Π is a base for Φ, every $\beta \in \Phi$ is a integral linear combination of Π. This implies that each α^* for α in Π is an integral weight. (c) now follows from (b).

(d) and (e) follows easily from 3.6.2 □

Lemma 3.7.3 [**along min**] *Let Φ be a connected root system.*

(a) [**a**] *Φ_{l} has a unique dominant root α_{l} and Φ_{s} has a unique dominant root α_{s}.*

(b) [**b**] *If $\alpha \in \Phi$ with $\alpha \neq \alpha_{\mathrm{l}}$ then there exists $\beta \in \Phi^+$ with $\alpha + \beta \in \Phi$.*

(c) [**c**] *Let $e \in \overline{\mathfrak{C}}^\sharp$ and $\alpha \in \Phi$. Then $-\alpha_{\mathrm{l}} \prec \alpha \prec \alpha_{\mathrm{l}}$ and $-(e, \alpha_{\mathrm{l}}) \leq (e, \alpha) \leq (e, \alpha_{\mathrm{l}})$.*

Proof: By 3.5.4(c) W is transitive on Φ_l. So (a) follows from 3.5.5.

For (b) suppose first that α is not dominant. Then there exists $\beta \in \Pi$ with $(\alpha, \beta) < 0$ and so by 3.1.15(b), $\alpha + \beta \in \Phi$. Suppose next that α is dominant. Since $\alpha \neq \alpha_{\mathrm{l}}$ we conclude that Φ has two root lengths and $\alpha = \alpha_{\mathrm{s}}$. By 3.7.2(d) $(\alpha_{\mathrm{l}}, \alpha) > 0$ and so by 3.1.4 $(\alpha, \check{\alpha}_{\mathrm{l}}) = 1$. Thus $\beta := \alpha_{\mathrm{l}} - \alpha$ is a roots and $(\beta, \check{\alpha}_{\mathrm{l}}) = 2 - 1 = 1 > 0$. Since α_{l} is dominant this implies $\beta \in \Phi^+$ and so (b) holds.

(c) Note that by (b), α_{l} is the unique element of maximal height in Φ. From (b) and induction on $\operatorname{ht}\alpha_{\mathrm{l}} - \operatorname{ht}\alpha$, $\alpha \prec \alpha_{\mathrm{l}}$ and so $\alpha_{\mathrm{l}} = \alpha + \phi$ for some $\phi \in \mathbb{N}\Pi$. Since $(e, \phi) > 0$, $(\epsilon, \alpha) \leq (e, \alpha_{\mathrm{l}})$. Note that this results also holds for the base $-\Pi$ and so (c) is proved. □

Definition 3.7.4 [**def:minimal**] *$\lambda \in \check{\Lambda}$ is called minimial if $(\lambda, \alpha) \in \{-1, 0, 1\}$ for all $\alpha \in \Phi$.*

Proposition 3.7.5 [**minimal 1**] *Let $0 \neq \lambda \in \check{\Lambda}^+$ Then the following are eqiuvalent*

(a) [**a**] *λ is minimal.*

(b) [**b**] *$(\lambda, \alpha_{\mathrm{l}}) = 1$.*

(c) [**c**] *$\lambda = \beta^*$ for some $\beta \in \Pi$ where $n_\beta = 1$ is defined by $\alpha_{\mathrm{l}} = \sum_{\delta \in \Pi} n_\delta \delta$.*

Proof: (a)$\Longrightarrow$ (b): Since λ is dominant and minimal, $(\lambda, \alpha_{\mathrm{l}}) \in \{0, 1\}$. If $(\lambda, \alpha_{\mathrm{l}}) = 0$, then 3.7.3 implies $\lambda = 0$.

(b)$\Longrightarrow$ (a): Suppose $(\lambda, \alpha_{\mathrm{l}}) = 1$. Then by 3.7.3(c) shows that λ is minimal.

(b)$\Longleftrightarrow$ (c): Note that $(\lambda, \alpha_{\mathrm{l}}) = \sum_{\delta \in \Pi} n_\delta (\lambda, \delta)$.

By 3.7.2(e) each n_δ is a positive integer. So we see that $(\lambda, \alpha_{\mathrm{l}}) = 1$ iff the following holds:

There exists a unique $\beta \in \Pi$ with $(\lambda, \beta) \neq 0$; and for this β, $(\lambda, \beta) = 1 = n_\beta$.

Note that this is equivalent to (c). □

Definition 3.7.6 [**def:affine**]

(a) [**a**] *$\Pi^\circ = \Pi \cup \{-\alpha_{\mathrm{l}}\}$. $\Gamma(\Pi^\circ)$ is called the affine diagram of Φ.*

(b) [**b**] *w_Π is the unique element in W with $w_\Pi(\Pi) = -\Pi$ (and so $(-w_\Pi)(\Pi) = \Pi$).*

For an example let $I = \{0, 1, \ldots n\}$ and let E_0 be the euclidean $\mathbb{F}$-space with orthonormal basis $(e_i \mid i \in I)$. For $0 \neq i \in I$ put $\alpha_i = e_{i-1} - e_i$. Put $\Pi = \{\alpha_i \mid 1 \leq i \leq n$ and $\Phi = \langle \Pi \rangle$. Note that $(\alpha_i, \alpha_j\,) = 2$ if $i = j$, -1 if $|i - j| = 1$ and 0 if $|i - j| > 1$. In particular, $\check{\alpha}_i = \alpha_i$ and Π is a linear independent pre-root system. Thus Φ is a root system. Note that $\omega_{\alpha_i}(e_i) = e_{i-1}$, $\omega_{\alpha_i}(e_{i-1}) = e_i$ and $\omega_{\alpha_i}(e_j) = e_j$ if $j \neq i, i-1$. Hence if we view $\mathrm{Sym}(I)$ as a subgroup of $GL(E_0)$, then ω_{α_i} is the cycle $(i-1, i)$ in $\mathrm{Sym}(I)$ and $W(\Pi) = \mathrm{Sym}(I)$. Thus the definition of $\langle \Pi \rangle$ implies that $\Phi = \{e_i - e_j \mid i \neq j \in I\}$. Let $e = -\sum_{i \in I} ie_i$. Then $(e, \alpha_i\,) = 1$ for all $i \in I$ and so e is a regular and dominant. Hence $\Phi^+ = \{\alpha \in \Phi \mid (e, \alpha\,) > 0\} = \{e_i - e_j \mid i < j \in I\}$. Let $\alpha = e_0 - e_n$. Suppose that $n > 1$. Then $(\alpha, \alpha_1\,) = (\alpha, \alpha_n\,) = 1$ and $(\alpha, \alpha_i\,) = 0$ for $1 < i < n$. If $n = 1$ then $(\alpha, \alpha_1\,) = 2$. In any case α is dominant, $\alpha = \alpha_\mathrm{l}$ and so $\Pi^\circ = \Pi \cup \{-\alpha\}$. Note that $\Gamma(\Pi)$ is a string of lenght n with only single bonds. If $n > 1$ then $\Gamma(\Pi^\circ)$ is circle of length $n + 1$ with only single bonds and if $n = 1$, then $\Gamma(\Pi^\circ)$ consist of two vertices with a double bond.

Let $w \in \mathrm{Sym}(I)$ be defined by $w(i) = n - i$. Then $w(\alpha_i) = e_{n-(i-1)} - e_{n-i} = -\alpha_{n+1-i}$. Thus $w(\Pi) = -\Pi$ and so $w_\Pi = w$. Note that $-w$ induces the unique non-trivial graph automorphism on $\Gamma(\Pi)$.

Lemma 3.7.7 [**phi-sigma invariant**] *Let $\Sigma \subseteq \Phi$. Then $\Phi^+ \setminus \langle \Sigma \rangle$ is invarinant under $W(\Sigma)$.*

Proof: By definition, $\langle \Sigma \rangle$ is invarinat under $W(\Sigma)$. Hence also $\Phi \setminus \langle \Sigma \rangle$ is $W(\Sigma)$-invariant. Let $\alpha \in \Phi^+ \setminus \langle \Sigma \rangle$ and $\sigma \in \Sigma$. Then $\alpha \neq \sigma$ and so by 3.3.2(d), $\omega_\sigma(\alpha) \in \Phi^+$. Thus $\Phi^+ \setminus \langle \Sigma \rangle$ is ω_σ-invariant and so also $W(\Sigma)$ invariant. $\square$

Proposition 3.7.8 [**minimal 2**] *Let $\beta \in \Pi$ and $\Sigma = \Pi - \beta$. Then the following are equivalent.*

(a) [**a**] *β^* is minimal.*

(b) [**b**] *β is long and $W(\Sigma)$ act transitively on $\Phi_\mathrm{l}^+ \setminus \langle \Sigma \rangle$.*

(c) [**c**] *$w_\Sigma(\beta) = \alpha_\mathrm{l}$.*

(d) [**d**] *Π° is invariant under $-w_\Sigma$*

(e) [**e**] *There exists a graph automorphism σ of $\Gamma(\Pi^\circ)$ with $\sigma(\beta) = -\alpha_\mathrm{l}$.*

(f) [**f**] *$\Gamma(\Pi^\circ - \beta)$ and $\Gamma(\Pi)$ are isomorphic graphs.*

(g) [**g**] *$\Phi = \langle \Pi^\circ - \beta \rangle$.*

Proof: (a)$\Longrightarrow$ (b): By 3.7.5 $n_\beta = 1$. By 3.3.1 $\frac{(\beta,\beta)}{(\alpha_l,\alpha_l)} n_\beta$ is an integer and so $(\beta,\beta) = (\alpha_l,\alpha_l)$ and β is long.

Let $\delta \in \Phi_l^+ \setminus \langle\Sigma\rangle$. By 3.5.5(d), $\langle\Sigma\rangle = \Phi \cap \beta^{*\perp}$ and so $(\beta^*,\delta) \neq 0$. Since β^* is minimal, $(\beta^*,\delta) = 1 = (\beta^*,\alpha_l)$. (b) now follows from 3.5.5(e).

(b)$\Longrightarrow$ (c): Since Π is obtuse and $w_\Sigma(\Sigma) = -\Sigma$, $w_\Sigma(\beta)$ is dominant on Σ. Since also α_l is dominant on Σ and since $w_\Sigma(\beta)$ and α_l are conjugate under $W(\Sigma)$, 3.5.5(a) (applied to $\langle\Sigma\rangle$ in place of Φ) implies $w_\Sigma(\beta) = \alpha_l$.

(c)$\Longrightarrow$ (d): We have $-w_\Sigma(\Sigma) = \Sigma$ and $-w_\Sigma(\beta) = -\alpha_l$. Since $-w_\Sigma$ has order 2, $-w_\Sigma(-\alpha_l) = \beta$ and so (d) holds.

(d)$\Longrightarrow$ (e): By 3.7.7 $w_\Sigma(\beta) \in \Phi^+$ and so $-w_\Sigma(\beta) \neq \beta$. Since $-w_\Sigma$ leaves Π° and Σ invariant we conclude $-w_\Sigma(\beta) = -\alpha_l$. Also $-w_\Sigma$ is an isometry and so $-w_\Sigma$ induces a graph automorphism on $\Gamma(\Pi^\circ)$. So (e) holds with $\sigma = -w_\Sigma \mid_{\Pi^\circ}$.

(e)$\Longrightarrow$ (f): Obvious.

(f)$\Longrightarrow$ (g): By (f) $\langle\Pi^\circ - \beta\rangle$ is a subroot system of Φ isomorphic to $\langle\Pi\rangle = \Phi$. As Φ is finite, (g) holds.

(g)$\Longrightarrow$ (a): We have $\beta \in \langle\Pi^\circ - \beta\rangle$ and so $\beta = n\alpha_l + \sigma$ for some $n \in \mathbb{Z}$ and $\sigma \in \mathbb{Z}\Sigma \subseteq b^{*\perp}$. Thus $1 = (\beta^*,\beta) = n(\beta^*,\alpha_l)$ and so $(\beta^*,\alpha_l) = 1$. Hence by 3.7.5, β^* is minimal. $\square$

For $\Phi = A_n$ we have that Π° is circle of lenght $n+1$. Hence for all $\alpha \in \Pi$, $\Pi^\circ - \alpha$ is a string of length n and so isomorphic to Π. Thus each α^* for $\alpha \in \Pi$ is a minimal weight. Also 0 is a minimal weight and hence A_n has $n+1$ minimal weights.

Definition 3.7.9 [**def:cartan matrix**] *C and E are the $\Pi \times \Pi$ matrix defined by $c_{\alpha\beta} = (\check{\alpha},\beta)$ and $e_{\alpha\beta} = \frac{(\alpha^*,\beta^*)(\beta,\beta)}{2}$. C is called the Cartan matrix of Φ. Put $e_\alpha := e_{\alpha\alpha}$.*

Lemma 3.7.10 [**basic cartan matrix**]

(a) [**a**] $\check{\alpha} = \sum_{\beta\in\Pi} c_{\alpha\beta}\beta^*$.

(b) [**b**] $\alpha^* = \sum_{\beta\in\Pi} e_{\alpha\beta}\check{\beta}$.

(c) [**c**] $E = C^{-1}$.

Proof: (a) follows from 3.6.2(a) applied to $\mathcal{B} = \Pi$.

(b) By 3.6.2(a), $\alpha^* = \sum_{\beta\in\Pi} (\alpha^*,\beta^*)\beta = \sum_{\beta\in\Pi} (\alpha^*,\beta^*)\frac{(\beta,\beta)}{2}\check{\beta} = \sum_{\beta\in\Pi} e_{\alpha\beta}\check{\beta}$.

(c) Follows easily from (a) and (b). $\square$

Proposition 3.7.11 [**decomposing pi**] *Suppose Φ is connected. Let $\alpha \in \Pi$ be long and let Δ be the set of neighbors of α in $\Gamma(\Pi)$. Put $\Sigma = \Pi - \alpha$ and let $\{\tilde{\sigma} \mid \sigma \in \Sigma\}$ the basis of $\mathbb{F}\Sigma$ dual to Σ. For $\delta \in \Delta$ define $\tilde{e}_\delta = \frac{(\delta,\delta)(\tilde{\delta},\tilde{\delta})}{2}$ and $r_\delta = \frac{(\alpha,\alpha)}{(\delta,\delta)}$. Then* *(a)* [**a**] *Each connected component of* Σ *contains exactly one element of* Δ*.*

(b) [**b**] $\tilde{\delta}$ *is a minimal dominant weight for* $\langle \Sigma \rangle$*.*

(c) [**c**] $\check{\alpha} = \frac{\alpha^*}{e_\alpha} - \sum_{\delta \in \Delta} \tilde{\delta}$*.*

(d) [**d**] $\frac{1}{e_\alpha} + \sum_{\delta \in \Delta} r_\delta \tilde{e}_\delta = 2$.

Proof: Let $\mathcal{D}$ be the set of connected components of Σ. Note that

$$E = \mathbb{F}\alpha^* \oplus \mathbb{F}\Sigma = \mathbb{F}\alpha^* \oplus \bigoplus_{D \in \mathcal{D}} \mathbb{F}D.$$

and so

$$(*) \qquad \check{a} = m\alpha^* - \sum_{D \in \mathcal{D}} \lambda_D$$

for some $m \in \mathbb{F}$ and $\lambda_D \in \mathbb{F}D$. Let $\beta \in \langle D \rangle$. The $(\lambda_D, \beta) = -(\beta, \check{\alpha})$. Since Π is linearly independent, $\beta \notin \mathbb{F}\alpha$ and since α is long we conclude that $(\beta, \check{\alpha}) \in \{-1, 0, 1\}$. Thus λ_D is a minimal weight for $\langle D \rangle$. Since Π is obtuse, λ_D is dominant for D. From 3.7.5 we conclude that $\lambda_D = \tilde{\delta}$ for some $\delta \in D$. Then clearly δ is the unique element of Δ contained in D and so (a) and (b) hold.

Note that $1 = (\alpha^*, \alpha) = (\alpha^*, \check{\alpha})\frac{(\alpha,\alpha)}{2}$ and so by (*), $1 = m(\alpha^*, \alpha^*)\frac{(\alpha,\alpha)}{2} = me_\alpha$. Thus $m = \frac{1}{e_\alpha}$ and (c) follows from (*).

Note that

$$(\check{\alpha}, \check{\alpha}) = \frac{4}{(\alpha, \alpha)},$$

$$\left(\frac{\alpha^*}{e_\alpha}, \frac{\alpha^*}{e_\alpha}\right) = \frac{1}{e_\alpha}\frac{(\alpha^*, \alpha^*)}{e_\alpha} = \frac{1}{e_\alpha}\frac{2(\alpha^*, \alpha^*)}{(\alpha^*, \alpha^*)(\alpha, \alpha)} = \frac{2}{(\alpha, \alpha)}\frac{1}{e_\alpha}$$

and

$$(\tilde{\delta}, \tilde{\delta}) = \frac{2\tilde{e}_\delta}{(\delta, \delta)} = \frac{2}{(\alpha, \alpha)} r_\delta \tilde{e}_\delta$$

Computing the squared lengths of both sides in (c) we now obtain

$$\frac{4}{(\alpha, \alpha)} = \frac{2}{(\alpha, \alpha)}\frac{1}{e_\alpha} + \sum_{\delta \in \Delta} \frac{2}{(\alpha, \alpha)} r_\delta \tilde{e}_\delta$$

Multiplying with $\frac{(\alpha,\alpha)}{2}$ we get

$$2 = \frac{1}{e_\alpha} + \sum_{\delta \in \Delta} r_\delta \tilde{e}_\delta.$$

Thus (d) holds. $\square$

Proposition 3.7.12 [**composing pi**] *Let I be a finite set. For $i \in i$ let E_i an euclidean $\mathbb{F}$-space, Φ_i a connected root system in E_i with base Π_i and $\delta_i \in \Pi_i$. Let $\{\tilde{\delta} \mid \delta \in \Pi_i\}$ be the basis dual to Π_i in E_i and put $\tilde{e}_i = \frac{(\delta_i,\delta_i)(\tilde{\delta}_i,\tilde{\delta}_i)}{2}$. Also let l in $\mathbb{F}$ be positive. Suppose that for all $i \in I$*

(i) [**a**] *$\tilde{\delta}_i$ is a minimal dominant weight for Φ_i.*

(ii) [**b**] *$r_i := \frac{l}{(\delta_i,\delta_i)}$ is an integer.*

(iii) [**c**] *$\sum_{i\in I} r_i\tilde{e}_i < 2$.*

Define $e \in \mathbb{F}$ by $\frac{1}{e} + \sum_{i\in I} r_i\tilde{e}_i = 2$. Choose a one dimensional euclidean $\mathbb{F}$ space X and $x \in X$ with $(x,x) = \frac{2e}{l}$. Put $E = X \oplus \bigoplus_{i\in I} E_i$. Put $\alpha = \frac{l}{2}(\frac{x}{e} - \sum_{i\in I}\tilde{\delta}_i)$, $\Pi = \{\alpha\} \cup \bigcup_{i\in I} \Pi_i$ and $\Phi = \langle \Pi \rangle$. Then Φ is a root system with base Π, α is a long root with $(\alpha,\alpha) = l$, $e_\alpha = e$, $\alpha^* = x$, $\{\Pi_i \mid i \in I\}$ is the set of connected components of $\Pi - \alpha$ and, for $i \in I$, $(\delta_i, \check{\alpha}) = -1$, and δ_i is the unique neighbor of α in Π_i.

Proof: A straight forward calculation shows that $(\alpha,\alpha) = l$ and so

$$\check{\alpha} = \frac{x}{e} - \sum_{i\in I}\tilde{\delta}_i.$$

Hence $(\delta_i,\check{\alpha}) = -1$ and $(\delta,\check{\alpha}) = 0$ for all other $d \in \Pi - \alpha$. Also $(\alpha,\check{d}_i) = r_i(\delta_i,\check{\alpha}) = -r_i$ is a negative integer. Hence Π is a linearly independent, obtuse pre-root system and so by 3.1.13 Φ is a root system.$x \perp \Pi - \alpha$ and $(x,\alpha) = \frac{l}{2e}(x,x) = 1$. So $x = \alpha^*$ and $e_\alpha = \frac{(\alpha,\alpha)(\alpha^*,\alpha^*)}{2} = \frac{l\frac{2e}{l}}{2} = e$. □

Lemma 3.7.13 [**echa**] *Let $((\check{\alpha})^* \mid \alpha \in \Pi)$ be the basis for E dual to $\check{\Pi}$. Then $(\check{\alpha})^* = \frac{(\alpha,\alpha)}{2}\alpha^*$ and $e_{\check{\alpha}} = e_\alpha$.*

Proof: Let $r := \frac{(\alpha,\alpha)}{2}$. Then $r\check{\alpha} = \alpha$. Clearly $\check{\beta} \perp r\alpha^*$ for all $\alpha \neq \beta \in \Pi$. Also

$$(r\alpha^*,\check{\alpha}) = (\alpha^*, r\check{\alpha}) = (\alpha^*,\check{\alpha}) = 1$$

and so $(\check{\alpha})^* = r\alpha^*$.

$$\begin{aligned} 2e_{\check{a}} &= (\check{a},\check{a})\cdot((\check{a})^*,(\check{\alpha})^*) &&= (\check{a},\check{a})\cdot(r\alpha^*, r\alpha^*) \\ &= (r\check{a}, r\check{a})\cdot(\alpha^*,\alpha^*) &&= (\alpha,\alpha)\cdot(\alpha^*,\alpha^*) &&= 2e_\alpha. \end{aligned}$$

So $e_{\check{a}} = e_\alpha$ and the lemma is proved. □

Lemma 3.7.14 [**pi a tree**] *Let Φ be a connected root system.*

(a) [**a**] $\Gamma^0(\Pi)$ *is a tree.*

(b) [**z**] *Let* $\alpha \in \Pi_l$. *Then* $e_\alpha \geq \frac{1}{2}$ *with equality iff* $\Pi = \{\alpha\}$.

(c) [**b**] *Suppose* $\alpha \in \Pi_l$ *with* $e_\alpha < 1$. *Then* $\Phi \cong A_n$, α *is an end-node of* Π *and* $e_\alpha = \frac{n}{n+1}$.

(d) [**c**] *Exactly one of the following holds:*

1. [**a**] $\check{\alpha}_l = \beta^*$ *for a long root* $\beta \in \Pi$.
2. [**b**] $\Phi \cong A_n$ *and* $\check{\alpha}_l = \beta_1^* + \beta_n^*$ *where* β_1 *and* β_n *are the end nodes of* Π *(with* $\check{\alpha}_l = 2\beta_1^*$ *if* $|\Pi| = 1$*).*
3. [**c**] $\Gamma(\Pi^\circ) = \bullet\!\Rightarrow\!\circ - \circ - \quad \cdots \quad - \circ - \circ \!\Leftarrow\! \circ$ *and* $\check{\alpha}_l = \beta^*$ *for the short end-node* β *of* Π.

(e) [**y**] *If* $\Phi \ncong A_n$ *then* α_l *is an end-node of* Π° *and* $\Gamma(\Pi^\circ)$ *is a tree.*

(f) [**d**] *Suppose* $\Phi \ncong A_n$ *and* $\beta \in \Pi$ *such that* β^* *is a minimal weight. Then* β *is an end-node of* Π° *and* Π.

Proof: (a) Let $\alpha \in \Pi$ be long. By induction each connected component of $\Pi - \alpha$ is a tree. Also by 3.7.11, α is joint to exactly one vertex from each connected component of $\Pi - \alpha$. Thus also Π is tree.

By 3.7.13 $e_\alpha = e_{\check{\alpha}}$. So (b) and (c) are true for (α, Φ) iff they are true for $(\check{\alpha}, \check{\Phi})$. So for (b) and (c) we assume without loss that α is long.

(b) By 3.7.11(d) $\frac{1}{e_\alpha} + \sum_{\delta \in \Delta} r_\delta \tilde{e}_\delta = 2$. So $e_\alpha \geq \frac{1}{2}$ with equality iff $\Delta = \emptyset$. Since Π is connected (b) holds.

(c) If $\Delta = \emptyset$, (c) holds with $n = 1$. Suppose that $|\Delta| > 0$. Then $e_\alpha < 1$ implies $\sum_{\delta \in \Delta} r_\delta \tilde{e}_\delta < 1$. Thus $\Delta = \{\delta\}$, $r_\delta = 1$ and $\tilde{e}_\delta < 1$. So by induction on Π, $\langle \Pi - \alpha \rangle \cong A_m$, δ is an end-node of $\Pi - \alpha$ and $e_\delta = \frac{m}{m+1}$. Thus $\Phi \cong A_{m+1}$ and $\frac{1}{e_\alpha} = 2 - \frac{m}{m+1} = \frac{m+2}{m+1}$ and (c) is proved.

(d) Since $\check{\alpha}_l$ is a dominant intergral weight $\check{\alpha}_l = \sum_{i=1}^k \beta_i^*$ for some $\beta_i \in \Pi$. Since $2 = (\alpha_l, \check{\alpha}_l) = 2 = \sum_{i=1}^l (\beta_i^*, \alpha_l)$ and $((\beta_i^*, \alpha_l)$ is a positive integer we get, $k \leq 2$.

If $k = 2$, then $(\beta_i^*, \alpha_l) = 1$, β_i^* is a minimal weight and so by 3.7.8(b), β_i^* is long. Also by 3.7.2(d), $(\beta_1^*, \beta_2^*) > 0$ and so $(\check{\alpha}_l, \check{\alpha}_l) > (\beta_1, \beta_1) + (\beta_2, \beta_2)$. Since β_i is long, $(\alpha_l, \alpha_l) = (\beta_i, \beta_i)$ and multiplication with $\frac{(\alpha_l, \alpha_l)}{2}$ gives $2 > e_{\beta_1} + e_{\beta_2}$. So $e_{\beta_i} < 1$ for at least one i. By (c), $\Phi \cong A_n$. For $\Phi = A_n$ we have $\alpha_l = e_0 - e_n$ and (d:2) holds in this case.

So suppose $k = 1$ and put $\beta = \beta_1$. If β is long, (d:1) holds. So suppose that β is not long. Put $r = \frac{(\alpha_l, \alpha_l)}{(\beta, \beta)}$. By 3.7.13,

$$(\check{\beta})^* = \frac{(\beta, \beta)}{2} \beta^* = \frac{(\beta, \beta)}{2} \check{\alpha}_l = \frac{1}{r} \alpha_\lambda.$$

Hence $\alpha_l = r(\check{\beta})^*$. Since $(\check{\beta})^*$ is an integral weight on $\check{\Phi}$ we conclude that r divides $(\alpha_l, \check{\alpha})$ for all $\alpha \in \Phi$. Choosing $\alpha = \alpha_l$ we see that $r = 2$. If $\alpha \in \Phi_l$ with $\alpha \neq \pm\alpha_l$ we get

$\alpha \perp \alpha_{\rm l}$. Let δ be a long root of minimal distance from β in $\Gamma^0(\Pi)$. Let Σ be the set of vertices of the path from β to δ. By 3.5.4 a we have $\Sigma \subseteq \mathbb{F}(\langle\Sigma\rangle_l)$ and so there exists a long root $\epsilon \in \Sigma^+$ with $\alpha_l \not\perp \epsilon$. Then $\alpha_{\rm l} = \epsilon \in \mathbb{F}\Sigma$. Suppose $\rho \in \Pi \setminus \Sigma$. Then $\alpha_{\rm l} \in \mathbb{F}\Sigma$ implies $(\rho^*, \alpha_{\rm l}) = 0$, a contradiction to 3.7.2(d). Thus $\Sigma = \Pi$ and (d) holds.

(e) By (d), $\check{\alpha}_{\rm l} = \beta^*$ for some $\beta \in \Pi$. Thus β is the unique neighbor of $-\alpha_{\rm l}$ in $\Gamma^0(\Pi^\circ 0$. By (a), $\Gamma^0(\Pi)$ is a tree and so (e) holds.

(f) By (e), $-\alpha_{\rm l}$ is an end-node of Π°. Hence by 3.7.8(e), also β is an end-node of Π°.

□

Lemma 3.7.15 [**w pi**] *Let Φ be a connected root system with $|\Pi| > 1$. Put $\Sigma = \Pi \cap \alpha_{\rm l}^\perp$ and let $\alpha \in \Pi \setminus \Sigma$.*

(a) [**a**] $w_\Pi = \omega_{\alpha_{\rm l}} w_\Sigma = \omega_{\alpha_{\rm l}} w_\Sigma$

(b) [**b**] $\alpha_{\rm l} = (-w_\Pi)(\alpha) + w_\Sigma(\alpha)$.

(c) [**c**] $\Pi \setminus \Sigma = \{\alpha, (-w_\Pi)(\alpha)\}$.

(d) [**d**] $(-w_\Pi)\mid_\Sigma = (-w_\Sigma)\mid_\Sigma$.

(e) [**e**] *Each connected component of $\Gamma(\Sigma)$ is invariantt under $-w_\Pi$.*

Proof: (a) Let $\beta \in \Phi^+$ and put $\delta = w_\Sigma(\beta)$. We claim that $(\omega_{\alpha_{\rm l}} w_\Sigma)(\beta) = \omega_\alpha(\delta) \in \Phi^-$. Since $\Sigma \perp \alpha_{\rm l}$ we have $w_\Sigma(\alpha_{\rm l}) = \alpha_{\rm l}$. Since w_Σ is an isometry,

$$(*) \quad (\beta, \alpha_{\rm l}) = (w_\Sigma \delta, w_\Sigma(\alpha_{\rm l})) = (\delta, \alpha_{\rm l})$$

Suppose first that $\beta \perp \alpha_{\rm l}$. By 3.5.5(d), $\Phi \cap \alpha_{\rm l}^\perp = < \Pi \cap \alpha_{\rm l}^\perp\rangle = \langle\Sigma\rangle$ and so $\beta \in \langle\Sigma\rangle$. Thus by definition of w_Σ, $\delta = w_\Sigma(\beta) \in \Phi^-$. By (*), $\delta \perp \alpha_{\rm l}$ and so $\omega_{\alpha_{\rm l}}(\delta) = \delta \in \Phi^-$.

Suppose next that $(\beta\alpha_{\rm l}, >)0$. Then since ω_{α_l} is an isomoetry and has order two

$$(\omega_{\alpha_{\rm l}}(\delta), \alpha_{\rm l}) = (\delta, \omega_{\alpha_{\rm l}}(\alpha_{\rm l}) = -(\delta, (\,)\alpha_{\rm l}) = -(\beta, \alpha_{\rm l}) < 0$$

and again $\omega_{\alpha_{\rm l}}(\delta) \in \Phi^-$.

This proves the claim and so $w_\Pi = \omega_{\alpha_{\rm l}} w_\Sigma$. Taking the inverse on both sides of this equation gives $w_\Pi = w_\Sigma \omega_{\alpha_{\rm l}}$.

(b) Since $\Pi \neq \{\alpha\}$, 3.7.2(e) implies $\alpha_{\rm l} \neq \alpha$ and so $(\alpha, \check{\alpha}_{\rm l}) = 1$. Thus $w_{\alpha_{\rm l}}(\alpha) = \alpha - \alpha_{\rm l}$. Also $w_\Sigma(\alpha_{\rm l}) = -\alpha_{\rm l}$ and so by (a) $w_\Pi(\alpha) = w_\Sigma(w_{\alpha_{\rm l}}(\alpha)) = w_\Sigma(\alpha) - \alpha_{\rm l}$.

(c) Let $\Pi' = \Sigma \cup \{\alpha, (-w_\Pi)(\alpha)\}$. Note that $w_\Sigma(\alpha) \leq \langle\alpha, \Sigma\rangle \leq \mathbb{F}\Pi'$. So by (a) also $\alpha_{\rm l} \leq \mathbb{F}\Pi'$. Suppose that $\beta \in \Pi \setminus \Pi'$, then $(\alpha_{\rm l}, \beta^*) = 0$, a contradiction to 3.7.2(e).

(d) Since $\omega_{\alpha_{\rm l}}$ acts trivially on Σ, this follows from (a).

(e) Let $\mathcal{D}$ be the set of connected componenent of Σ. Then $-w_\Sigma = -\prod_{\Delta\in\mathcal{D}} w_\Delta$ fixes each $\Delta \in \mathcal{D}$. So (e) follows from (d). □

Proposition 3.7.16 [**decomposing affine**] *Suppose that $\check{\alpha}_l = \alpha^*$ for a long root α. Retain the notation from 3.7.11 and for $\delta \in \Delta$ let Π_δ be the connected component of Σ containing δ.*

(a) [**a**] $e_\alpha = 2$.

(b) [**b**] $-w_\Sigma(\delta) = \delta$ *for all* $\delta \in \Delta$.

(c) [**c**] $\sum_{\delta\in\Delta} r_\delta \tilde{e}_\delta = \frac{3}{2}$.

(d) [**d**] *One of the following holds.*

1. [**a**] $|\Delta| = 3$, δ *is long and* $\Pi_\delta = \{\delta\}$ *for all* $\delta \in \Delta$.
2. [**b**] $\Delta = \{\delta, \epsilon\}$, δ *and* ϵ *are long,* $\Pi_\delta = \{\delta\}$ *and* $\tilde{e}_\epsilon = 1$.
3. [**c**] $\Delta = \{\delta, \epsilon\}$, $\Pi_\delta = \{\delta\}$, $\Pi_\epsilon = \{\epsilon\}$, δ *is long and* $r_\epsilon = 2$.
4. [**d**] $\Delta = \{\delta\}$, δ *is long and* $\tilde{e}_\delta = \frac{3}{2}$.
5. [**e**] $\Delta = \{\delta\}$, $\Pi_\delta = \{\delta\}$ *and* $r_\delta = 3$.

Proof: (a) $\epsilon_\alpha = \frac{(\alpha,\alpha)(\alpha^*,\alpha^*)}{2} = \frac{(\alpha_l,\alpha_l)(\check{\alpha}_l,\check{\alpha}_l)}{2} = 2$.

(b) Note that $(-w_\Pi)$ fixes α_l and Π and so also α and Δ. By 3.7.15(e), w_Π also fixes Π_δ and so $\Pi\delta \cap \Delta = \{\delta\}$. Thus $(-w_\Pi)(\delta) = \delta$ and (b) follows from 3.7.15(d).

(c) Follows (a) and 3.7.11(d).

(d) Let $\delta \in \Delta$. If $\tilde{e}_\delta < 1$, then by 3.7.14(c) $\langle \Pi_d \rangle \cong A_n$ and δ is an end-node in Π_δ. Thus (b) implies that $n = 1$ and so $\langle \Pi_\delta \rangle = \{d\}$. (d) now follows easily from (d).

Proposition 3.7.17 [**composing affine**] *Retain the assumptions and notations of 3.7.12. Suppose in addition that for all $i \in I$,*

(iii') [**a**] $\sum_{i\in I} r_i \tilde{e}_i = \frac{3}{2}$

(iv) [**b**] $-w_i\Pi_i(\delta_i) = \delta_i$.

Then $\alpha_l = \alpha + w(\alpha)$ *and* $\check{a}_l = \alpha^*$.

Proof: Put $\lambda = \sum_{i\in I} \tilde{\delta}_i$ and $w = \prod_{i\in I} w_{\Pi_i}$. Since $-w_{\Pi_i}$ normalizes Π_i and by (iv) fixes δ_i we have $-w_{\Pi_i}(\tilde{\delta}_i) = \tilde{\delta}_i$. Thus $w(\lambda) = -\lambda$. From (iii') we have $e = 2$ and $\check{a} = \frac{1}{2}x - \lambda$. Hence $\check{\alpha} + w(\check{\alpha}) = x = \alpha^*$. Since $(x, x) = \frac{4}{l} = (\check{\alpha}, \check{\alpha})$ we see that $\check{x} = \alpha + w(\alpha)$. By 3.7.12, α is long and so also and x is a long root. Since $x = \alpha^*$ is dominant and α_l is the unique dominant long root, $\check{x} = \alpha_l$. □

Lemma 3.7.18 [**l-m**] *Let λ and μ dominant minimal integral weights on Φ. Then also $\lambda - \mu$ is minimal.*

Proof: Let $\alpha \in \Phi^+$. Then $(\lambda, \alpha) \in \{0,1\}$ and $(\mu, \alpha) \in \{0,1\}$ and so $(\lambda - \mu, \alpha) \in \{-1, 0, 1\}$. $\square$

Lemma 3.7.19 [**basic min**]

(a) [**a**] *Let $a, b \in E$ with $a \check{\preceq} b$. Then $a + \mathbb{Z}\check{\Phi} = b + \mathbb{Z}\check{\Phi}$.*

(b) [**b**] *If W acts trivially on $\breve{\Lambda}/\mathbb{Z}\check{\Phi}$.*

(c) [**c**] *Let $e \in \overline{\mathfrak{C}}$. Then $\{b \in \overline{\mathfrak{C}} \mid b \check{\preceq} e\}$ is finite.*

(d) [**d**] *Every coset of $\mathbb{Z}\check{\Phi}$ in $\breve{\Lambda}$ contains a dominant integral weight which is minimal in $\breve{\Lambda}^+$ wit respect to $\check{\preceq}$.*

Proof: (a) By definition of $\check{\preceq}$, $b - a \in \mathbb{N}\check{\Pi} \leq \mathbb{Z}\check{\Phi}$.

(b). Let $\lambda \in \breve{\Lambda}$ and $\alpha \in \Phi$. Then $\omega_\alpha(\lambda) = \alpha - (\lambda, \alpha)\check{\alpha} \in \lambda + \mathbb{Z}\check{\Phi}$.

(c) Let $b \in \overline{\mathfrak{C}}$ with $b \check{\preceq} e$. Then $e - b \in \mathbb{N}\check{\Pi}$ and $e + b \in \overline{\mathfrak{C}}$. Thus $(e + b, e - b) \geq 0$ and $(b, b) \leq (e, e)$. By 3.1.6, $\mathbb{Z}\check{\Pi}$ is discret Hence also $b + \mathbb{Z}\Pi$ is discret. Therefore $\{b \in \overline{\mathfrak{C}} \mid b \check{\preceq} e\}$ is discret and bounded and so by 3.1.7 finite.

(d). Let $\lambda \in \breve{\Lambda}$. Then $w(\lambda)$ is dominant for some $w \in W$. By (c) there exists $b \prec w(\lambda)$ such that b is $\check{\preceq}$-minimal in $\breve{\Lambda}^+$. Then by (a) and (b), $b, w(\lambda)$ and λ all lie in the same coset of $\mathbb{Z}\check{\Phi}$. So (d) holds.

Lemma 3.7.20 [**min equal min**]

(a) [**b**] *Let $\lambda \in \mathbb{Z}\check{\Phi}$ be minimal. Then $\lambda = 0$.*

(b) [**a**] *Let $\lambda \in \breve{\Lambda}^+$. Then λ is $\check{\preceq}$- minimal if and only if λ is minimal.*

(c) [**c**] *Every coset of $\mathbb{Z}\check{\Phi}$ in $\breve{\Lambda}$ contains a unique dominant minimal weight.*

Proof: Without loss Φ is connected.

(a) By 3.4.2 there exists $w \in W$ such that $w(\lambda$ dominant. Then also $w(\lambda)$ is minimal and we may assume that λ is dominant. Let $\lambda = \sum_{a \in \Pi} n_\alpha \check{\alpha}$ with $n_\alpha \in \mathbb{Z}$. Suppose that $\lambda \neq 0$. Let $a \in \Pi$. By 3.7.2(e), $(\lambda, \alpha^*) > 0$. So also $n_\alpha = \frac{2}{\alpha\alpha}(\lambda, \alpha^*) > 0$. Also $(\lambda, \alpha_l) = 1$ and since $(\check{\alpha}, \alpha_l) \in \mathbb{N}$ we conclude that there existts a unique $\alpha \in \Pi$ with $(\check{\alpha}, \alpha_l) \neq 0$. Moreover, $n_\alpha = 1 = (\check{\alpha}, \alpha_{\mathrm{l}})$. and α is long. As α is long $-1 \leq (\beta, \check{\alpha}) \leq 1$ for all $\pm\alpha \neq p \in \Phi$. Also Π is obtuse and so $-\check{\alpha}$ is a dominant minimal weight on $\Sigma := \Pi - \alpha$. Hence also $-\omega_\Sigma(-\check{\alpha}) = \omega_\Sigma(\check{\alpha})$ is a dominant minimal weight on Σ. Since $n_\alpha = 1$ we have $\lambda - \check{\alpha} \in \mathbb{Z}\check{\Sigma}$. By 3.7.19(b), α and and $\omega_\Sigma(\check{\alpha})$ lie in the same coset of $\mathbb{Z}\check{\Sigma}$. Thus $\lambda - \omega_\sigma(\check{\alpha}) \in \mathbb{Z}\check{\Sigma}$. By 3.7.18 $\lambda - \omega_\Sigma(\check{\alpha})$ is a minimal weight on Σ. Thus by induction $\lambda - \omega_\Sigma(\check{\alpha}) = 0$. So $\lambda = \omega_\Sigma(\check{\alpha})$. Thus $\check{\alpha}$ is a mimimal weight a contradiction to $(\alpha, \check{\alpha}) = 2$.

(b) and (c): We frist show that

(**) If $\lambda \in \breve{\Lambda}^+$ is $\check{\preceq}$- minimal then λ is minimal.

For this it suffices to show that $(\lambda, \alpha_l) \leq 1$. Choose a long root δ of minimal height with $(\lambda, \delta) = (\lambda, \alpha)_\lambda$. Since λ is $\check{\prec}$-minimal, $\lambda - \check{\delta}$ is not dominant and so there exists $\beta \in \Pi$ with $(\lambda - \check{d}, \beta) < 0$. So

$$(*). \qquad (\lambda, \beta) < (\beta, \check{\delta}).$$

Suppose that $\delta \neq \beta$. Then since δ is long, $(\beta, \check{\delta}) = 1$ and so $(\lambda, \beta) = 0$. Hence $(\lambda, \omega_\beta(\delta)) = (\lambda, \delta) = (\lambda, \alpha)$ and $\omega_\beta(\delta)$ is a positive long root of smaller height than δ, a contradiction to the choice of δ. Hence $\delta = \beta$. So by (*) $(\lambda, \alpha_l) = (\lambda, \delta) < (\delta, \check{\delta}) = 2$. Hence λ is minimal.

Next we show that

(***) Every coset of $\mathbb{Z}\check{\Phi}$ in $\check{\Lambda}$ contains at most one minimal dominant weight.

For this let λ and μ be minimal dominant weights in the same coset of $\mathbb{Z}\check{\Phi}$. Then $\lambda - \mu \in \mathbb{Z}\check{\Phi}$ and by 3.7.18, $\lambda - \mu$ is minimal. So by (a), $\lambda - \mu = 0$ and $\lambda = \mu$.

Now let λ be any dominant minimal weight in $\check{\Lambda}$. By 3.7.19(d), $\lambda + \mathbb{Z}\check{\Phi}$ contains a $\check{\prec}$-minimal element μ. By (**) μ is minimal and by (***) $\lambda = \mu$. Thus (b) holds.

(c) follows from (b), 3.7.19(d) and (***). $\square$

Definition 3.7.21 [**o ab**]

(a) [**a**] *For a path* $p = (\alpha_0, \alpha_1, \ldots \alpha_n)$ *in* $\Gamma^0(\Pi)$ *define* $s(p) = \prod_{i=1}^n |(\check{\alpha}_{i-1}, \alpha_i)|$.

(b) [**b**] *If* $\alpha, \beta \in \Pi$ *be in the same connected component of* $\Gamma^(\Pi)$*, then* $\overline{\alpha\beta}$ *denotes the unique path in* $\Gamma^0(\Pi)$ *from* α *to* β.

(c) [**c**] $\det \Pi$ *is the number of minimal dominant weights for* Φ.

Lemma 3.7.22 [**basic det pi**]

(a) [**a**] $\det \Pi = |\check{\Lambda}/\mathbb{Z}\check{\Phi}| = \det C$.

(b) [**b**] *Let* $\alpha, \beta \in \Pi$ *If* α *and* β *are in the same connected componenent of* $\Gamma^0(\Pi)$*, then* $e_{\alpha\beta} = s(\overline{\alpha\beta}) \frac{\det(\Pi - \overline{\alpha\beta})}{\det \Pi}$. *Otherwise* $e_{\alpha\beta} = 0$.

Proof: (a) By 3.7.20(c), $\det \Pi = |\check{\Lambda}/\mathbb{Z}\check{\Phi}|$.

Define $T \in \mathrm{End}_{\mathbb{Z}}(\check{\Lambda})$ by $T(\alpha^*) = \check{\alpha} = \sum_{\beta \in \Pi} c_{\alpha\beta}\beta^*$. Then $T(\check{\Lambda}) = \mathbb{Z}(\check{\Phi})$ and so

$$|\check{\Lambda}/\mathbb{Z}\check{\Phi}| = |\det T| = \det C$$

Thus (a) holds.

By 3.7.10(c) $E = C^{-1}$. Let $\alpha, \beta \in \Pi$. Then there either exists no path or exactly one path from α to β in $\Gamma^0(\Pi)$. In the first case 3.6.1 implies $e_{\alpha\beta} = 0$. In the second let $\overline{\alpha\beta} = (\alpha_0, \alpha_1 \ldots, \alpha_n)$. Then since Π is obtuse,

$$(-1)^n \prod_{i=1}^{n} c_{\alpha_{i-1}\alpha_i} = \prod_{i=1} |(\check{a}_{i-1}, \check{\alpha}_i)| = s(\overline{\alpha\beta}).$$

Thus by 3.6.1

$$e_{\alpha\beta} = s(\overline{\alpha\beta}) \frac{\det C(\Pi - \overline{\alpha\beta})}{\det C(\Pi)}.$$

(b) now follows from (a). □

3.8 The classification of root system

In the section we determine all the connected roots systems up to isomorphism. We also determine the affine diagrams, the action of $-w_\Pi$ on Π and the minimal weights. we combine all thus information in what we call the labeled affine diagram:

Recall that the non-zero minimal weights are all of the from α^* for some root $\alpha \in \Pi$. We will label such an α with $\det(\Pi - \alpha)$. We also label $-\alpha_{\mathrm{l}}$ with $\det \Pi$. We use a filled node to distinguish $-\alpha_{\mathrm{l}}$ from the remaining vertices for Π°. We also draw a dotted line betweeen any two distinct elements of Π which are interchanged by $-w_\Pi$.

Theorem 3.8.1 [**labeled affine**] *The labeled affine diagrams of the connected root systems are exactly as listed in Figure 3.8.*

Proof: By induction we assume that labeled affine diagrams of rank smaller than n are exactly as in Figure 3.8.

Suppose we know the affine diagrams for the rank connected roots sytems. Then 3.7.8[f] gives us $\det \Pi$ and all $\alpha \in \Pi$ such that α^* is minimal. From 3.7.15 and induction we obtain the action of $-w_\Pi$ on Π. Also by induction we can compute $\det(\Pi - \alpha)$.

So it remains to determine the affine diagrams.

In case 3.7.14(d:2),d:2 we see that the $\Pi^\circ = A_n^\circ$ or $\Pi^\circ = C_n^\circ$.

So suppose that $\alpha_{\mathrm{l}} = \alpha^*$ for a long root α.

We now consider the different case of 3.7.16(d).

In case d:1 $\Pi^\circ \cong D_4^\circ$.

In case d:2 Π_ϵ is a connected rank$n-2$ root system, $\tilde{e}_\epsilon = 1$ and $w_{\Pi_\epsilon}(\epsilon) = \epsilon$. Note that by 3.7.22(b), $\tilde{e}_\epsilon = \frac{\det(\Pi_\epsilon - \epsilon)}{\det \Pi_\epsilon}$ and so e_ϵ can be computed from the labeled affine diagram of Π_ϵ.

Suppose that $\Pi_\epsilon = A_{n-2}$. Then since w_{Π_ϵ} fixes ϵ, we get $n-2 = 2k+1$ and

$$1 = \tilde{e}_\epsilon = \frac{(\det \Pi(A_k))^2}{\det \Pi(A_{2k+1})} = \frac{(k+1)^2}{2k+2} = \frac{k+1}{2}.$$

Thus $k = 1$, $n = 5$ and $\Pi^\circ = D_5^\circ$.

If $\Pi_\epsilon = B_{n-2}$, then ϵ is the long end-node and $\Pi^\circ = B_n^\circ$ for $n \geq 5$.
If $\Pi_\epsilon = C_{n-2}$, then again ϵ is the long end-node, $\frac{n-2}{2} = 1$ and so $n = 4$ and $\Pi^\circ = B_4^\circ$.
If $\Pi_\epsilon = D_{n-2}$ then either ϵ is the left end-node or $n - 2 = 4$. In any case $\Pi^\circ = D_n^\circ$.
According to Figure 3.8 no other possibilities occur in the current case.

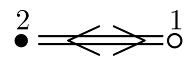
In case d:3 $\Pi^\circ = B_3^\circ$.

In case d:4 Π_δ is a connected rank$n-1$ root system, $\tilde{e}_\delta = \frac{3}{2}$ and $w_{\Pi_\delta}(\delta) = \delta$.
If $\Pi_\delta = A_{n-1}$. $n - 1 = 2k + 1$ and $\frac{3}{2} = \tilde{e}_\delta = \frac{(k+1)^2}{2k+2} = \frac{k+1}{2}$. Thus $k = 2$, $n = 6$ and $\Pi^\circ = E_6^\circ$.
If $\Pi_\delta = C_{n-1}$, then ϵ is the long end-node, $\frac{n-1}{2} = \frac{3}{2}$ and so $n = 4$ and $\Pi^\circ = F_4^\circ$.
If $\Pi_\delta = D_{n-1}$ then ϵ is one of the right end-nodes end-node and $\frac{n-1}{4} = \frac{3}{2}$ so $n = 7$ and $\Pi^\circ = E_7^\circ$.
If $\Pi_\delta = E_7$, then δ is the right end-node and $\Pi^\circ = E_8^\circ$.
According to Figure 3.8 no other possibilities occurs in the current case.

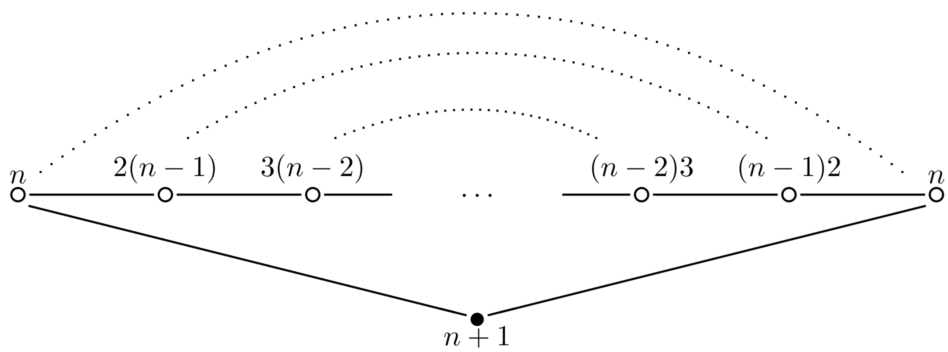
In case d:5 $\Pi^\circ = G_2^\circ$.
Finally we remark that 3.7.17 ensures that all the root systems encounter actually do exit. $\square$

Figure 3.2: The labeled affine diagrams

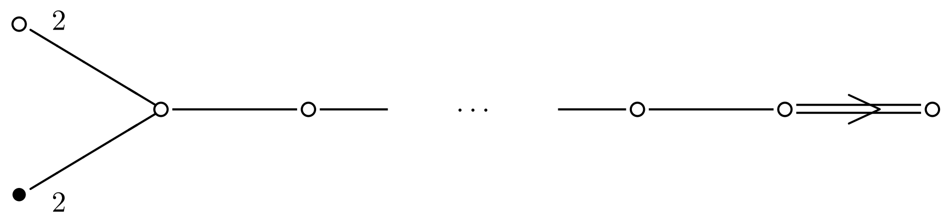
A_1



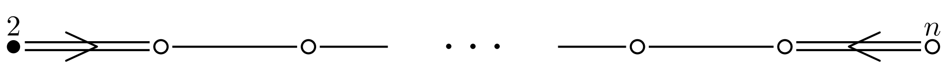
$A_n, n \geq 2$:



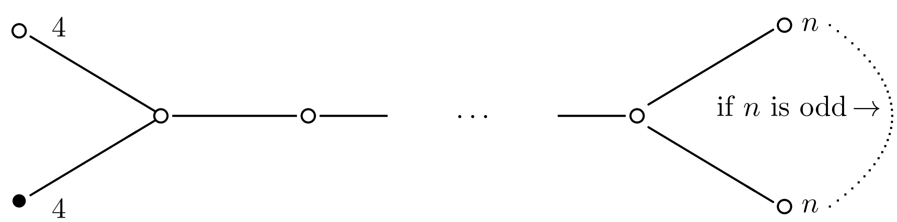
$B_n, n \geq 3$:



$C_n, n \geq 2$:



$D_n, n \geq 4$:



E_6:

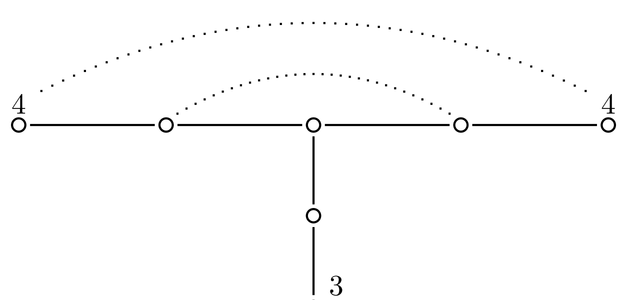
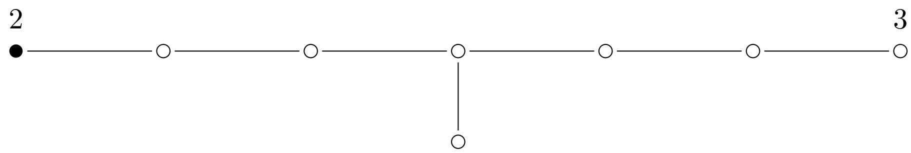
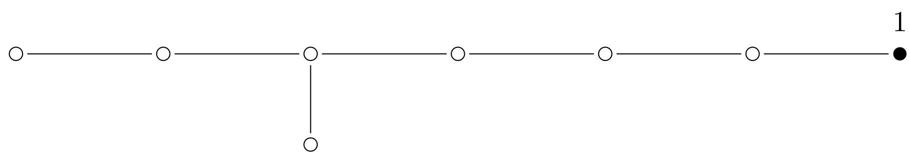


Figure 3.2: The labeled affine diagrams

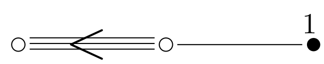
E_7:



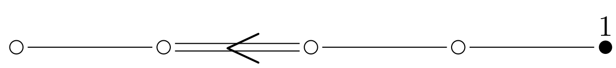
E_8:



G_2:



F_4:



Chapter 4

Uniqueness and Existence

4.1 Simplicity of semisimple Lie algebras

We say that a subset Ψ of a root system is *closed* if $\alpha + \beta \in \Psi$ whenever $\alpha, \beta \in \Psi$ with $\alpha + \beta \in \Phi$.

Proposition 4.1.1 [**h invariant in l**] *Let L be a standard Liealgebra with $\mathrm{Sol}(L) = 0$. Let H be a Cartan Subalgebra and Φ the set of roots for H on L. Let $\Psi \subseteq \Phi$ be closed. Put $L(\Psi) = \langle L_\alpha \mid \alpha \in \Psi \rangle_{Lie}$ and $H(\Psi) = \langle H_\alpha \mid \alpha \in \Psi \cap -\Psi \rangle_{Lie}$.*

(a) [**a**] *$L(\Psi) = \bigoplus_{\alpha \in \Psi} L_\alpha \oplus H(\Psi)$. In partcular, if $\beta \in \Phi$, then $L_\beta \in L(\Psi)$ if and only if $\beta \in \Psi$.*

(b) [**b**] *Let $A \leq L$ with $[A, H] \leq A$ and put $\Delta = \{\alpha \in \Phi \mid L_\alpha \leq A\}$. Then Δ is closed and $A = L(\Delta) + (A \cap H)$. If, in addition, A is perfect, then $A = L(\Delta)$.*

(c) [**c**] *Let $A \leq L$. Then A is an ideal iff $A = A(\Psi)$ for some closed $\Psi \subseteq \Phi$ with $\Phi = \Psi \cup (\Phi \cap \Psi^\perp)$.*

(d) [**d**] *Let $\mathcal{D}$ be the set of connected components of Φ. Then $L(\Delta), \Delta \in \mathcal{D}$ are the simple ideals in L and $L = \bigoplus_{\Delta \in \mathcal{D}} L(\Delta)$.*

Proof: (a): Let $A := L(\Psi) \bigoplus_{\alpha \in \Psi} L_\alpha \oplus H(\Psi)$. Let $\alpha \in \Psi \cap -\Psi$. Then $H_\alpha = [L_\alpha, L_a] \in L(\Psi)$ and so $A \leq L(\Psi)$. So it suffices to show that A is a subalgebra of L. Clearly $[H(Psi), A] \leq A$. Now let $\alpha, \beta \in \Psi$. If $\alpha + \beta \in \Phi$, then since Φ is closed, $\alpha + \beta \in \Psi$ and so $[L_\alpha, L_\beta] = L_{\alpha+b} \leq A$. If $\alpha + b = 0$, then $\alpha = b \in \Psi \cap -\Psi$, $[L_\alpha, L_\beta] = H_\alpha \leq H(\Psi) \leq A$. If $0 \neq \alpha + \beta \notin \Phi$, then $[L_\alpha, L_\beta] = 0 \leq A$. Thus A is a subalgebra and (a) is proved.

(b). Let $\alpha, \beta \in \Delta$ with $\alpha + \beta \in \Phi$. Then $L_{\alpha+\beta} = [L_\alpha, L_\beta] \leq A$ and so Δ is closed. Since A is invariant under H, $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$. For $\alpha \in \Phi$, L_α is 1-dimensional so either $L_\alpha = A_\alpha$ or $A_\alpha = 0$. Also $A_0 = A \cap H$ and so $A = \bigoplus_{\alpha \in \Delta} L_\alpha \oplus (H \cap A) = L(\Delta) + (H \cap A)$.

Since $[L(\Psi), H] \leq L(\Psi)$ and H is abelian we conclude that $[A, A] \leq L(\Psi)$. So if A is perfect, $A = L(\Psi)$.

(c) Let $\Psi = \{\alpha \in \Phi \mid L_\alpha \leq A$ and $\Sigma = \Phi \cap \Psi^\perp$.

Suppose that $A = L(\Psi)$ and $\Phi = \Psi \cap \Sigma$. Note that $[L(\Psi), H] \leq L(\Psi)$. Let $\beta \in \Sigma$ and $a \in \Psi$. Then $\alpha + \beta$ is neither perpendicular to Σ nor to Ψ and so $\alpha + \beta \notin \Phi$. Also $a \neq -\beta$ and so $[L_\alpha, L_\beta] = 0$. Hence $L(\Psi), L_\beta] = 0$ and $L(\Psi)$ is an ideal.

Suppose next that A is an ideal. By 2.7.11 A is perfect and so by (b), $A = L(\Psi)$. Let $\alpha \in \Phi$ and suppose that $\beta \not\perp \alpha$ for some $\beta \in \Psi$. We claim that $\alpha \in \Psi$. If $-\alpha \in L_\Psi$, then $L_\alpha = [[L_\alpha, L_{-\alpha}], L_{-\alpha}] \leq A$ and so $\alpha \in \Psi$. So we may assume that $(\alpha, \beta\,) < 0$ and so by 3.1.15(b), $\alpha + \beta \in \Phi$. So $L_{\alpha+b} = [L_\alpha, L_\beta] \leq L(\Psi)$. Also $-\beta \in \Psi$ and so $\alpha = (\alpha + \beta) + (-\beta) \in \Psi$. Hence $\Phi = \Psi \cup \Sigma$.

(d) By 2.7.11, $L = \bigoplus_{i=1}^n L_i$, where $L_1, \ldots, L_n$ are the simple ideals in L. By (b), $L_i = L(\Psi_i)$, where Ψ_i is a closed subset of Φ. Moreover, $\Psi_i \perp \Psi_j$ for all $i \neq j$ and $\Phi = \bigcup_{i=1}^n \Psi_i$. Let Δ be a connected component of Ψ. Then clearly $\Delta \subseteq \Psi_i$ for some i. By (c) $L(\Delta)$ is an ideal and so since L_i is simple, $L(\Delta) = L_i = L(\Psi_i)$. (a) implies $\Delta = \Psi_i$. Therefore the Ψ_i are exactly the connected components of Φ. □

4.2 Generators and relations

Proposition 4.2.1 [**relations for l**] *Let L be a standard Lie algebra with* $\mathrm{Sol}(L) = 0$. *Let H be a Cartan subalgebra, Φ be the set of roots for H on L and Π an base for Φ. For $\alpha \in \pm\Pi$ pick $0 \neq x_\alpha \in L_\alpha$ with $[x_\alpha, x_{-\alpha}] = h_\alpha$. Then for all $\alpha, \beta \in \Pi$:*

(Rel:a) [**a**] $[h_\alpha, h_\beta] = 0$.

(Rel:b) [**b**] $[h_\alpha, x_\beta] = c_{\alpha\beta} x_\beta$ *and* $[h_\alpha, x_{-\beta}] = -c_{\alpha\beta} x_{-\beta}$.

(Rel:c) [**c**] $[x_\alpha, x_{-\beta}] = \delta_{\alpha\beta} h_\alpha$.

(Rel:d) [**d**] *If $\alpha \neq \beta$ then* $x_\alpha^{-c_{\alpha\beta}+1} * x_\beta = 0$ *and* $x_{-\alpha}^{-c_{\alpha\beta}+1} * x_{-\beta} = 0$.

Proof: (a) holds since H is abelian. (b) follows since $x_{\pm\beta} \in L_{\pm b}$.

Note that for $\beta \neq \alpha$, $0 \neq \alpha - \beta \notin \Phi$ and so $[x_\alpha, x_{-\beta}] \in L_{\alpha-\beta} = 0$. By choice of the $x_{\pm\alpha}$, $[x_\alpha, x_{-\alpha}] = h_\alpha$ and so (c) holds.

By induction on i we have $x_\alpha^i * x_b \in L_{\beta+i\alpha}$. Since $\alpha - \beta$ is not a root, $r_{\alpha\beta} = 0$. So by 2.11.10(c), $s_{\alpha\beta} = -c_{\alpha\beta}$ and hence $\beta + (-c_{\alpha\beta} + 1)\alpha \notin \Phi$. Thus the first statement in (d) holds. By symmetr y also the second holds. □

Definition 4.2.2 [**def:locally nilpotent**] *Let V be a vector space and $\phi \in \mathrm{End}(V)$. Then ϕ is called locally nilpotent if for all $v \in V$ there exists $n \in \mathbb{N}$ with $\phi^n(v) = 0$.*

Lemma 4.2.3 [**e phi**] *Let V be a vectorspace over the field $\mathbb{K}$ and ϕ, ψ locally nilpotent endomorphism. Suppose that* $\mathrm{char}\,\mathbb{K} = 0$.

(a) [**a**] *There exists a unique* $\rho \in \mathrm{End}(V)$ *with* $\rho(v) = \sum_{i=0}^{n} \frac{1}{i!}\phi^i(v)$ *whenever* $v \in V$ *and* $n \in \mathbb{N}$ *with* $\phi^{n+1}(v) = 0$. *We denote* ρ *by* e^ϕ *and by* $\sum_{i=0}^{\infty} \frac{\phi^i}{i!}$.

(b) [**b**] *If* $[\phi, \psi] = 0$ *then* $e^{\phi+\psi} = e^\phi e^\psi$.

(c) [**c**] e^ϕ *is invertible with inverse* $e^{-\psi}$.

Proof: Readily verified.

Lemma 4.2.4 [**inner aut**] *Let* A *be a* $\mathbb{K}$ *algebra,* δ *a derivation and suppose that* $\mathrm{char}\,\mathbb{K} = 0$. *Then*

(a) [**a**] $\displaystyle \frac{\delta^n}{n!}(ab) = \sum_{i+j=n} \frac{\delta^i}{i!}(a)\frac{\delta^j}{j!}(b)$

(b) [**b**] *If* δ *is locally nilpotent then* e^δ *is an automorphism of* A.

Proof: (a) This is the usually product rule for differentiation. Indeed for $n = 1$, it is just the defintion of a derivation and the general formula can be established by induction on n.

(b)

$$\begin{aligned} e^\delta(a)e^\delta(b) = \quad & \sum_{i=0}^{\infty} \frac{\delta^i}{i!}(a) \cdot \sum_{j=0}^{\infty} \frac{\delta^j}{j!}(b) & = \sum_{n=0}^{\infty} \sum_{i+j=n} \frac{\delta^i}{i!}(a)\frac{\delta^j}{j!}(b) \\ = \quad & \sum_{n=0}^{\infty} \frac{\delta^n}{n!}(ab) & = e^\delta(ab) \end{aligned}$$

□

Lemma 4.2.5 [**locally nilpotent and semisimple**] *Suppose that* $\mathbb{K}$ *is standard and* $L \cong \mathfrak{sl}(\mathbb{K}^2)$ *with Chevalley basis* (x, y, h). *Let* V *be an* L*-module. Then the following are equivalent:*

(a) [**a**] V *is the sum of the finite dimensional* L*-submodule.*

(b) [**b**] V *is the direct sum of finite dimensional simple* L*-submodules.*

(c) [**c**] x *and* y *act locally nilpotent on* V *and* h *is diagonliazible on* V.

By 2.9.3, (a) implies (b). Also by 2.5.4, (b) implies (c).

Suppose now that (c) holds. Let W be the sum of the finite dimensional L-submodule. Suppose $V \neq W$. Since x is locally nilpotent on V, there exists $v \in V \setminus W$ with $xv \in W$. Then there exists a finite dimensional L-submodule T of W with $xv \in T$. Hence $C_{V/T}(x) \not\leq V/T$ and since h is diagonalizible on V, there exists eigenvector $u+T \in C_{V/T}(x)$ with $u \notin W$. Let U/T be the smallest L-submodule of V/T containing $u+T$. Since y is locally nilpotent, there exists $m \in \mathbb{N}$ with $y^m u = 0$. 2.5.4 now implies that U/T is finite dimensional. But then U is is finite dimensional and so $u \in U \leq W$, a contradiction.

Thus $V = W$ and (c) implies (a). □

Lemma 4.2.6 [**e and w**] *Let $\mathbb{K}$ be standard, S and H subalgebra of V, $\alpha \in \Lambda(H)$ and V an L-module. Suppose that*

(i) [**i**] $S \cong \mathfrak{sl}(\mathbb{K}^2)$ *with Chevalley basis* (x, y, h).

(ii) [**ii**] *H is abelian, $h \in H$, $x \in L_\alpha$ and $y \in L_{-\alpha}$*

(iii) [**iii**] *V is the direct sum of the weight spaces for H.*

(iv) [**iv**] *x and y are locally nilpotent.*

Then for each $\lambda \in H^*$ $\dim V_\lambda = \dim V_{\lambda-\lambda(h)\alpha}$.

Proof: Without loss $L = S + H$. If the lemma is true for the composition factors for L on V its also true for V. So we may assume that V is a simple L-module. Since x is locally nilpotent, $C_V(x) \neq 0$ and so by (iii) there exists $\mu \in H^*$ and $0 \neq v \in V_\mu$ with $xv = 0$. Since y is locally nilpotent, there exists $m \in \mathbb{N}$ minimal with $y^{m+1}v = 0$. Note that $v_i := y^i v$ is a weightvector for H of weight $\mu - i\alpha$. By 2.5.4(b) $W = \mathbb{K}v_i \mid 0 \leq i \leq m\rangle$ is S-invarinat and $\mu(h) = m$. Since W is also H invariant, W is an L-submodule and so $W = V$. Let $\lambda \in \Lambda_V(H)$. Then $\lambda = \mu - i\alpha$ for some $0 \leq i \leq m$ and $V_\lambda = \mathbb{K}v_i$ is 1-dimensional. From $x \in L_\alpha$ and $[h, x] = 2x$ we have $\alpha(h) = 2$ and so $\lambda(h) = m - 2i$. Thus $\lambda - \lambda(h)\alpha = \mu - i\alpha - (m - 2i)\alpha = \mu - (m - i)\alpha$ and so $\lambda - \lambda(h)\alpha \in \Lambda_V(H)$. □

Definition 4.2.7 [**def:iso of root systems**] *Let Φ_1 and Φ_2 be root systems over the same field $\mathbb{F}$. An* isomomorphism of root systems *is a bijection $\rho : \Phi_1 \to \Phi_2$ which extends to a $\mathbb{F}$-linear map from $\mathbb{F}\Phi_1$ to $\mathbb{F}\Phi_2$.*

Theorem 4.2.8 (Serre) [**serre**] *Let Φ be a root system with basis Π and L the Lie-algebra over $\mathbb{K}$ generated by elements $x_\alpha, x_{-\alpha}, h_\alpha, \alpha \in \Pi$ subject to the relations (a)-(d) in 4.2.1. Suppose that* $\operatorname{char} \mathbb{K} = 0$. *Then*

(a) [**a**] $\dim L = |\Phi| + |\Pi|$.

(b) [**b**] $\mathrm{Sol}(L) = 0$.

(c) [**c**] *$H := \langle h_\alpha \mid \alpha \in \Pi\rangle_{Lie}$ is a Cartan subalgebra.*

(d) [**d**] *There exists a $\mathbb{Q}$-linear map $\rho : \Phi \to H^*$ with $\rho(\alpha)(h_\beta) = (\alpha, \check{\beta})$ for all $\alpha \in \Phi, \beta \in \Pi$.*

(e) [**e**] *$\rho(\Phi)$ is the set of roots $\Phi_L(H)$ for H on L and Φ.*

(f) [**f**] *$\rho : \Phi \to \Phi_L(H)$ is an isomorphism of roots systems over $\mathbb{Q}$.*

Proof: Let $\hat{L}$ be the Lie-algebra over $\mathbb{K}$ generated by elements $\hat{x}_\alpha, \hat{x}_{-\alpha}, \hat{h}_\alpha, \alpha \in \Pi$ subject to the relations (a)-(c) in 4.2.1. Note that L isomorphic to a quotient of $\hat{L}$. As a first step in determining the structure of L we derive some information about $\hat{L}$. Let $\hat{H} = \langle \hat{h}_\alpha \mid \alpha \in \Pi \rangle_{Lie}$. Note the relation (a) ensures that $\hat{H}$ is abelian and spanned by the $\hat{h}_\alpha$ as a $\mathbb{K}$-space.

Let V be a $\mathbb{K}$-space with basis $v(\alpha), \alpha \in \Pi$. Let T be the tensor algebra over V, so T has basis $\mathcal{B} = \{\otimes_{i=1}^n v(\alpha_i) \mid n \in \mathbb{N}, \alpha_i \in \Pi\}$. For $b = \otimes_{i=1}^n v(\alpha_i) \in \mathcal{B}$ and $\alpha \in \Pi$ define $c_{\alpha b} = \prod_{i=1}^n c_{\alpha\alpha_i}$. In particular (for the case $n = 0$) $c_{\alpha 1} = 0$. Define endomorphism $\tilde{x}_\alpha, \tilde{h}_\alpha, \tilde{x}_{-\alpha}, \alpha \in \Pi$ of T via:

(i) [**0:a**] $\tilde{h}_\alpha(b) = -c_{\alpha b} b$.

(ii) [**0:b**] $\tilde{x}_{-\alpha}(b) = v(\alpha) \otimes b$.

(iii) [**0:c**] $\tilde{x}_\alpha(1) = 0$ and $\tilde{x}_\alpha(v(\beta) \otimes b) = v(\beta) \otimes \tilde{x}_\alpha(b) - \delta_{\alpha\beta} c_{\alpha b} b$.

To establish $\hat{L} \neq 0$ we show:

1° [**1**] *There exists a Lie-homomorphism $\hat{L} \to \mathfrak{gl}(T)$ with $\hat{r}_\alpha \to \tilde{r}_\alpha$ for all $r \in \{x, h\}$ and $\alpha \in \pm\Pi$.*

We merely need to verify that the $\tilde{r}_\alpha$ fulfill the relations 4.2.1(a) to 4.2.1(c). By (i) the h_α act diagonally on T with respect to $\mathcal{B}$ and so commute. Thus 4.2.1(a) holds.

$$\begin{aligned} (\tilde{x}_\alpha(\tilde{x}_{-\beta}(b)) - \tilde{x}_{-\beta}(\tilde{x}_\beta(b)) &= \tilde{x}_\alpha(v(\beta) \otimes b) - v(\beta) \otimes \tilde{x}_\alpha(b) \\ &= v(\beta) \otimes \tilde{x}_\alpha(b) - \delta_{\alpha\beta} c_{\alpha b} b - v(\beta) \otimes \tilde{x}_\alpha(b) \\ &= \delta_{\alpha\beta} \tilde{h}_\alpha(b) \end{aligned}$$

and so $[\tilde{x}_\alpha, \tilde{x}_{-\beta}] = \delta_{\alpha\beta} h_\alpha$. That is 4.2.1(c) holds.

Note that $c_{\alpha v(\beta)\otimes b} = c_{\alpha\beta} + c_{\alpha b}$ and so

$$\begin{aligned} \tilde{h}_\alpha(\tilde{x}_{-\beta}(b)) - \tilde{x}_{-\beta}(\tilde{h}_\alpha)(b) &= \tilde{h}_\alpha(v(\beta) \otimes b) - \tilde{x}_{-\beta}(c_{\alpha b} b \\ &= -(c_{\alpha\beta} + c_{\alpha b}) v(\beta) \otimes b) - (-c_{\alpha b} v(\beta) \otimes b) \\ &= -c_{\alpha\beta} \tilde{x}_{-\beta}(b). \end{aligned}$$

Hence $[\tilde{h}_\alpha, \tilde{x}_{-\beta}] = -c_{\alpha b} x_{-\beta}$ and the second part of 4.2.1(b) holds.

From the definiton of $\tilde{x}_\alpha$ and induction we see that

$$\tilde{x}_\beta(b) = -(c_{\beta b} - c_{\beta\beta}) \sum_{k \text{ with } \alpha_k = \beta} v(\alpha_1) \otimes \ldots \otimes v(\alpha_{k-1}) \otimes v(\alpha_{k+1} \otimes \ldots \otimes v(\alpha_n)$$

Each of the summands are eigenvectors with eigenvalue $-(c_{\alpha b} - c_{\alpha\beta}$ for $\tilde{h}_\alpha$ and so the same is true for $x_\beta(b)$. Thus

$$\tilde{h}_\alpha(\tilde{x}_\beta(b)) - \tilde{x}_\beta(\tilde{h}_\alpha(b)) = -(c_{\alpha b} - c_{\alpha\beta})x_b(b) - (-c_{\alpha b}\tilde{x}_\beta(b)) = c_{\alpha\beta}\tilde{x}_\beta(b)$$

and so $[\tilde{h}_\alpha, \tilde{x}_\beta] = c_{\alpha\beta}\tilde{x}_\beta$. Therefore the $\tilde{r}_\alpha$ fulfill all the required relations and (1°) is proved.

2° [**2**] *Let $k_\alpha \in K, \alpha \in \Pi$ such that $[\sum_{\alpha\in\Pi} k_\alpha \hat{h}_\alpha, \hat{x}_\beta] = 0$. Then $k_\alpha = 0$ for all $\alpha \in \Pi$. In particular, $\hat{h}_\alpha \mid \alpha \in \Pi)$ is linearly independent.*

Let k be the $\{1\} \times \Pi$-matrix $(k_\alpha)_{\alpha\in\Pi}$. We have

$$0 = [\sum_{\alpha\in\Pi} k_\alpha \hat{h}_\alpha, \hat{x}_\beta] = \sum_{\alpha\in\Pi} k_\alpha c_{\alpha\beta} = kC$$

Since C is invertible, this implies $k = 0$ and (2°) is proved.

3° [**3**] *There exist $\mathbb{Q}$-linear monomorphism $\rho : \mathbb{Q}\Phi \to \hat{H}^*$ and $\check{\rho} : \mathbb{Q}\check{\Pi} \to \hat{H}$ such that $\rho(d)(\check{\rho}(e)) = (d, e\,)$ for all $d \in \mathbb{Q}\Phi$ and $e \in \mathbb{Q}\check{\Phi}$.*

By (2°), $\hat{h}_\beta, \beta \in \Pi$ is a $\mathbb{K}$-basis for $\hat{H}$. Hence for each $\alpha \in \Pi$ there exists a unique $\rho(\alpha) \in H^*$ with $\rho(\alpha)(h_\beta) = (\alpha, \check{\beta}\,)$ for all $\beta \in \Pi$. Since Π is a $\mathbb{Q}$ basis for $\mathbb{Q}\Phi$ we can extend ρ uniquely to a linear map from $\mathbb{Q}\Phi$ to $\hat{H}^*$. Define $\check{\rho}(\beta) = h_\beta$ and extend to a linear map $\check{\rho} : \mathbb{Q}\check{\Phi} \to \hat{H}$. Then for all $\alpha, \beta \in \Pi$

$$\rho(\alpha)(\check{\rho}(\check{\beta})) = \rho(\alpha)(h_\beta) = (\alpha, \check{\beta}\,).$$

Since both $\rho(d)(\check{\rho}(e))$ and $(d, e\,)$ are $\mathbb{Q}$ linear in d and e we conclude that $\rho(d)(\check{\rho}(e)) = (d, e\,)$ for all $d \in \mathbb{Q}\Phi, e \in \mathbb{Q}\check{\Phi}$. Since $(\cdot, \cdot\,)$ is non-degenerate, ρ and $\check{\rho}$ must be 1-1. So (3°) holds.

Let $n \in \mathbb{Z}^+$ and $a = (\alpha_1, a_2, \ldots a_n) \in \pm\Pi^n$ we define $\hat{x}_a = [\hat{x}_{\alpha_1}, [\hat{x}_{\alpha_2}, \ldots, \hat{x}_{\alpha_n}] \ldots]$ and $a^\circ = \alpha_1 + \alpha_2 + \ldots \alpha_n \in \pm\mathbb{N}\Pi$. Note that $n = \operatorname{ht} a^\circ$. Also put $\hat{X}_0 = \hat{H}$ and $\hat{X}_{\pm n} = \mathbb{K} < \hat{x}_{\pm a} \mid a \in \Pi^n\rangle$. Let $\mu \in \pm\mathbb{N}\Pi$. If $\mu = 0$ put $\hat{X}(\mu) = \hat{X}$ and if $\mu \neq 0$ put $\hat{X}(\mu) = \mathbb{K}\langle \hat{x}_a \mid a \in \pm\Pi^{\operatorname{ht}\mu}, a^\circ = \mu\rangle$.

4° [**4**]

(a) [**4:a**] $\hat{x}_a \in \hat{L}_{\rho(a^\circ)}$.

(b) [**4:b**] $\hat{X}(\mu) = \hat{L}_\rho(\mu)$.

(c) [**4:c**] *Let $m \in \mathbb{Z}$. Then $[\hat{X}_m, \hat{X}_0] \leq \hat{X}_m$, $[\hat{X}_m, \hat{X}_1] \leq \hat{X}_{m+1}$ and $[\hat{X}_m, \hat{X}_{-1}] \leq \hat{X}_{m-1}$.*

(d) [**4:d**] $\hat{L} = \bigoplus_{m\in\mathbb{Z}} \hat{X}_m = \bigoplus_{\mu\in\pm\mathbb{N}\Pi^n} \hat{X}(\mu)$

(e) [**4:e**] *Let λ be a weight for $\hat{H}$ on $\hat{L}$. Then $\lambda = \rho(\mu)$ for some $\mu \in \pm\mathbb{N}\Pi$.*

(f) [**4:f**] *Let $\alpha \in \Pi$. Then $\hat{L}_{\rho(\alpha)} = \mathbb{K}\hat{x}_\alpha$ is 1-dimensional, while $\hat{L}_{\rho(k\alpha)} = 0$ for all $k \in \mathbb{Z}$ with $k > 1$.*

By Rel 4.2.1(b), $[\hat{h}_\beta, \hat{x}_\alpha] = (\alpha, \check{\beta}\,)x_\alpha = \rho(\alpha)(h_\beta)x_\alpha$ for all $\alpha, \beta \in \Pi$ and so $\hat{x}_\alpha \in \hat{L}_{\rho(\alpha)}$. Let $a \in \Pi^n$. Thus by induction on n

$$\hat{x}_{(\alpha,a)} = [\hat{x}_\alpha, x_a] \in [\hat{L}_{\rho(\alpha)}, \hat{L}_{\rho(a^\circ)}] \leq \hat{L}_{\rho(\alpha)+\rho(\alpha^\circ)} = \hat{L}_{\rho((\alpha,a)^\circ)}.$$

So (a) holds. In particular, $\hat{X}(\mu) \leq \hat{L}(\rho(\mu))$. Thus $[\hat{X}(\mu), \hat{H}] \leq \hat{X}(\mu)$ and since $\hat{X}_m = \sum_{\operatorname{ht}\mu=m} X(\mu)$, $[X_m, \hat{H}] \leq \hat{X}_m$.

If $n \in \mathbb{Z}^+$ the definition of $\hat{X}_{n+1}$ implies $\hat{X}_{n+1} = [\hat{X}_a, \hat{X}_n]$. Also $[\hat{X}_1, \hat{X}_0] = [\hat{X}_1, \hat{H}] \leq \hat{X}_1$. From Rel 4.2.1(b) we have

$$[\hat{X}_1, \hat{X}_{-1}] = \mathbb{K}\langle [\hat{x}_\alpha, x_{-\beta}] \mid \alpha, \beta \in \Pi\rangle = \mathbb{K}\langle \hat{h}_\alpha \mid \alpha \in \Pi\rangle = \hat{X}_0.$$

By induction on n and the Jacobi identity

$$\begin{array}{rccc} [\hat{X}_1, \hat{X}_{-n-1}] = & [\hat{X}_1, [\hat{X}_{-1}, \hat{X}_{-n}] & \leq [\hat{X}_{-1}, [\hat{X}_{-n}, \hat{X}_1]] + [\hat{X}_{-n}, [\hat{X}_1, \hat{X}_{-1}]] \\ \leq & [\hat{X}_{-1}, \hat{X}_{-n+1}] + [\hat{X}_{-n}, \hat{X}_0] & \leq \hat{X}_n \end{array}$$

We proved that $[\hat{X}_m, \hat{X}_1] \leq \hat{X}_{m+1}$ for all $m \in \mathbb{Z}$. By symmetry also $[\hat{X}_m, \hat{X}_{-1}] \leq \hat{X}_{m-1}$ and so (c) holds.

Put $\hat{X} = \sum_{m\in\mathbb{Z}} \hat{X}_m$. Then by (c) $[\hat{X}, r_\alpha] \leq \hat{X}$ for all $r \in x, y, h$ and $\alpha \in \Pi$. Since $\hat{L}$ is generate by the $\hat{r}_\alpha$ we conclude that $\hat{X}$ is an ideal in $\hat{L}$ and in particular, a subalgebra. Since $\hat{X}$ contains all the $\hat{r}_\alpha$ conclude that $\hat{X} = \hat{L}$. Thus

$$\hat{L} = \hat{X} = \sum_{\mu\in\pm\mathbb{N}\Pi} \hat{X}(\mu) \leq \bigoplus_{\lambda\in\pm\rho(\mathbb{N}\Pi)} \hat{L}_\lambda.$$

This easily implies (b), (d) and (e).

Let $\alpha \in \Pi$ and $k \in \mathbb{Z}$. Note that $k = \operatorname{ht} k\alpha$ and $a_k := \underbrace{(\alpha, \alpha, \ldots, \alpha)}_{k-\text{times}}$ is the unique element of $\pm\Pi^k$ with $a_k^\circ = k\alpha$. If $k > 1$, then $\hat{x}_{a_k} = 0$ and so $\hat{L}_{\rho(\alpha)} = \hat{X}(k\alpha) = 0$. Now $\hat{x}_{a_1} = x_\alpha$. If $\hat{x}_\alpha = 0$ then also $\hat{h}_\alpha = [\hat{x}_\alpha, \hat{x}_{-\alpha}] = 0$, a contradiction to (2°). Thus $\hat{L}_{\rho(\alpha)} = \mathbb{K}\hat{x}_\alpha$ is one dimensional. So (f) holds and (4°) is proved.

5° **[5]** *Let $\alpha, \beta, \gamma \in \Pi$ with $\alpha \neq \beta$ and put $\hat{v}_{\alpha\beta} = \hat{x}_\alpha^{-c_{\alpha\beta}+1} * \hat{x}_\beta$. Then $[\hat{x}_{-\gamma}, \hat{v}_{\alpha\beta}] = 0$.*

Case 1: $\gamma \neq \alpha$. Then $[\hat{x}_{-\gamma}, \hat{x}_\alpha] = 0$ and so $\operatorname{ad} x_\alpha$ and $\operatorname{ad} x_{-\gamma}$ commutes. Hence

$$[\hat{x}_{-\gamma}, \hat{v}_{\alpha\beta}] = \hat{x}_\alpha^{-c_{\alpha\beta}+1} * [\hat{x}_{-\gamma}, \hat{x}_\beta] = -\delta_{\gamma\beta}\hat{x}_\alpha^{-c_{\alpha\beta}+1} * \hat{h}_\beta = -\delta_{\gamma\beta}c_{\beta\alpha}x_\alpha^{-c_{\alpha\beta}} * x_\alpha$$

If $c_{\beta\alpha} = 0$, this is zero as desired. If $c_{\beta\alpha} \neq 0$, then $c_{\alpha\beta} < 0$. Thus $\hat{x}_\alpha^{-c\alpha\beta} * \hat{x}_\alpha = \hat{x}_\alpha^{-c\alpha\beta-1} * [\hat{x}_\alpha, \hat{x}_\alpha] = 0$ and again we are done.

Case 2: $\gamma = \alpha$. Note that $\mathbb{K}x_\alpha + \mathbb{K}h_\alpha\mathbb{K}x_{-\alpha}$ is isomorphic to $\mathfrak{sl}(\mathbb{K}^2)$ with Chevalley basis $(\hat{x}_{-\alpha}, \hat{x}_\alpha, -\hat{h}_\alpha)$. Put $k = -c_{\alpha\beta}$. Since $[\hat{x}_{-\alpha}, \hat{x}_\beta] = 0$ and $[-\hat{h}_\alpha, \hat{x}_\beta] = kx_\beta$. So (5°) now folles from 2.5.4(b:d) applied with $i = k + 1$.

6° [**6**] *For $\epsilon = \pm$ let $X^\epsilon = \sum_{i=1}^\infty \hat{X}_{\epsilon i} = \langle \hat{x}_{\epsilon\alpha} \mid \alpha \in \Pi \rangle_{Lie} \leq \hat{L}$. Let I^+ be be the ideal in X^+ generate by the $v_{\alpha\beta}$ and similarly define X^-. Then I^ϵ is an ideal in $\hat{L}$.*

Let $\hat{I}_1^+ = \mathbb{K}\langle \hat{v}_{\alpha\beta} \mid \alpha \neq \beta \in \Pi \rangle$. Since $v_{\alpha\beta}$ are eigenvectors (of weight $\rho(\beta + (-c_{\alpha\beta}+1)\alpha)$ for $\hat{H}$ we conclude I_1^+ is invariant under $\hat{H}$. Put $\hat{I}_{k+1}^+ = [\hat{I}_k, \hat{X}_1]$. Since also $\hat{X}_1$ is invariant under $\hat{H}$ we get by induction on k that each $\hat{I}_k^+$ is invariant under $\hat{H}$. Put $\hat{I}_0^+ = 0$. We claim that $[\hat{I}_k^+, \hat{X}_{-1}] \leq \hat{I}_{k-1}^+$. For $k = 1$ this follows from (5°). Using induction

$$\begin{array}{rll} [I_{k+1}, \hat{X}_{-1}] = & [[\hat{I}_k^+, \hat{X}_1], \hat{X}_{-1}] & \leq [[\hat{X}_1, \hat{X}_{-1}], \hat{I}_k^+] + [[\hat{X}_{-1}, \hat{I}_k^+], \hat{X}_1] \\ \leq & [\hat{H}, \hat{I}_k^+] + [\hat{I}_{k-1}^+, \hat{X}_1] & \leq \hat{I}_k^+ \end{array}$$

It follows that $\sum_{i=0}^\infty \hat{I}_k^+$ is invarinat under $\hat{X}_1$ and $\hat{X}_1$ and so also under $\hat{L}$. Thus $\hat{I}^+ = \sum_{i=0}^\infty$ is an ideal in $\hat{L}$.

7° [**7**] $L = \hat{L}/(I^+ + I^-)$

By (6°), $\hat{I}^+ + \hat{I}^-$ is the ideal in I generated by the $x_\alpha^{-c_{\alpha}\beta+1} * x_\beta$ and $y_\alpha^{-c_{\alpha\beta}+1} * x_{-\beta}$ and so (7°) follows from the definition of L and $\hat{L}$.

If $\hat{T}$ is is any element or subset of $\hat{L}$ we denote the image of $\hat{T}$ in L by T. From (4°)(d) we have $\hat{H} \cap (\hat{I}^+ + \hat{I}^-) = 0$ and so $\hat{H}$ and H are isomorphic. It follows that (2°), (3°) and (4°) remain true for $\hat{L}$ replaced by L.

8° [**8**] *For each $a \in \Pi$, $\operatorname{ad} x_\alpha$ and $\operatorname{ad} x_{-\alpha}$ are locally nilpotent endomorphism of$L$.*

Let M be the set of all elements in L annihilated by some power of $\operatorname{ad} x_\alpha$. For $i = 1, 2$ let $m_i \in \mathbb{N}$, $l_i \in L$ with $x_\alpha^{m_i} * l_i = 0$. Then by 4.2.4(a) $x_\alpha^{m_1+m+2} * [l_1, l_2] = 0$. Thus M is a subalgebra of L. Let $\alpha \neq \beta \in \Pi$. Then $[x_\alpha, x_{-\beta}] = 0$ and so $x_{-\beta} \in M$. $[x_\alpha, x_{-\alpha}] = h_\alpha$, $[x_\alpha, h_\alpha] = -2x_\alpha$ and $[x_\alpha, x_\alpha] = 0$. So $x_\alpha^3 * x_{-\alpha} = 0$ and $x_\alpha, x_{-\alpha} \in M$. $x_\alpha^{-c_{\alpha\beta}+1} * x_\beta = 0$, $x_\beta \in M$ and so M contains all the generators of L. Thus $L = M$ and (8°) hold.

9° [**11**] *Let A be an ideal in L, $w \in W(\Phi)$, and $d \in \mathbb{Q}\Phi$. Then* $\dim A_{\rho(d)} = \dim A_{\rho(w(d))}$.

Suppose first that $w = \omega_\alpha, \alpha \in \Pi$. Then

$$\rho(\omega_\alpha(d)) = \rho(\delta - (d, \check{\alpha})\alpha) = \rho(d) - \rho(d)(\check{h}_\alpha)\rho(\alpha)$$

Thus 4.2.6 implies that (9°) holds in this case. As $W(\Phi)$ is generated by $\omega_\alpha, \alpha \in \Pi$, (1°)1 holds for all w.

10° [**12**] $\rho(\Phi) = \Phi_L(H)$ *and* $\dim L_{\rho(\alpha)} = 1$ *for all* $\alpha \in \Phi$.

Let $\alpha in \Pi$. Then there exists $w \in W(\Phi)$ with $w(\alpha) \in \Pi$. So (f) and (9°) imply that $\dim L_{\rho(\alpha)} = 1$ and $L_{\rho(\alpha)} = 0$ for all $k > 1$. In particular, $\rho(\Phi) \subseteq \Phi_L(H)$. Conversely, let $\lambda \in \Phi_L(H)$. Then by (4°)(e), $\lambda = \rho(\mu)$ for some $\mu \in \mathbb{Z}\Pi$. Let $w \in W(\Phi)$. By (9°) also $\rho(w(\mu))$ is a weight for H on L and so (4°)(e) implies $w(\mu) \in \pm\mathbb{N}\Pi$. Thus by Exercise II.1.3, we have $\mu = k\alpha$ for some $k \in \mathbb{N}$ and $\alpha \in \Pi$. As just seen this implies $k = 1$ and $\lambda = \rho(\alpha)$.

11° [**13**] $\dim L = |\Phi| + |Pi|$ *is finite.*

Thus follows from (10°) and (4°)(c).

12° [**14**] $\mathrm{Sol}(L) = 0$.

Let A be an abelian ideal in L. We need to show that $A = 0$. Since A is invariant under H, $A = (A \cap H) \oplus \bigoplus_{\alpha\in\Phi}(A \cap L_{\rho(\alpha)})$.

Suppose that $A \cap L_{\rho(\alpha)} \neq 1$ for some $\alpha \in \Phi$. Since $L_{\rho(\alpha)}$ is 1-dimensional we get $L_{\rho(\alpha)} \leq A$. Let $\beta \in \Pi$ be conjugate to α inder $W(\Phi)$. Tehn (9°) implies that both L_β and $L_{-\beta}$ are in A. Hence x_β and $x_{-\beta}$ are in A and sine A is abelian $h_\alpha = [x_\alpha, x_{-\alpha}] = 0$, a contradiction.

Hence $A \cap L_{\rho(\alpha} \neq 1$ for all $\alpha \in \Phi$ and so $A \leq H$. Moreover, $[A, L_\alpha] \leq A \cap L_\alpha = 0$ and so $A \leq \ker \rho(a)$. Since $(\cdot,\cdot)$ is non-degerate,(3°) implies that $\rho(\alpha), \alpha \in \Phi$ is a basis for H^*. Thus $\bigcap_{\alpha\in\Pi} \ker \rho(\alpha) = 0$, $A = 0$ and (12°) is proved.

13° [**15**] *H is a Cartan subalgebra of L.*

H is abelian and so nilpotent. By (4°)(c), $C_{L/H}(H) = 0$ and so $N_L(H) = 0$. Thus H is selfnormalizing.

14° [**16**] *Φ and $\Phi_H(L)$ are isomorphic rootsytem over $\mathbb{Q}$.*

By (10°), $\rho(\Phi) = \Phi_H(L)$ and so also $\rho(\mathbb{Q}\Phi) = \mathbb{Q}\Phi_H(L)$. So ρ is an isomorphism of root systems.

The Theorem now follows from (11°)-(14°). □

Corollary 4.2.9 [**isomorphism**] *For $\epsilon \in \{\bullet, \dagger\}$ let L^ϵ be a standard, perfect, semisimple Lie algebra over the field $\mathbb{K}$. Let H^ϵ be a cartan subalgebra for L^ϵ with root system Φ^ϵ and base Π^ϵ. For $\alpha \in \pm\Pi^\epsilon$ let $x^\epsilon_\alpha \in L^\epsilon_\alpha$ with $[x^\epsilon_\alpha, x^\epsilon_{-\alpha}] = h^\epsilon_\alpha$. Suppose that $\rho : \Phi^\bullet \to \Phi^\dagger$ is an isomomorphism of root systems with $\rho(\Pi^\bullet) = \Pi^\dagger$. Then there exists a unique isomorphism of Lie-algebras $\tau : L^\bullet \to L^\dagger$ with $\tau(x^\bullet_\alpha) = x^\dagger_{\rho(\alpha)}$ for all $\alpha \in \pm\Pi^\bullet$. Moreover, $\tau(h^\bullet_\beta) = h^\dagger_{\rho(\beta)}$ and $\tau(L^\bullet_\beta) = L^\dagger_{\rho(\alpha)}$ for all $\beta \in \Phi^\bullet$.*

Proof: Let $\overline{L}^\epsilon$ be the Lie algebra with generators $\overline{x}^\epsilon_\alpha, \overline{x}^\epsilon_{-\alpha}, \overline{h}^\epsilon_\alpha, \alpha \in \Pi^\epsilon$ and relations as in 4.2.1. Then by 4.2.1 there exists an Lie-homomorphism from $\tau^\epsilon : \overline{L}^\epsilon \to L^\epsilon$ with $\overline{r}^\epsilon_\alpha = r^\epsilon_\alpha$ for all $r \in \{x, h\}$ and $\alpha \in \pm\Pi^\epsilon$. Since L^ϵ is generated by the r^ϵ_α's, τ^ϵ is onto. By 4.2.8,

$\dim \overline{L} = |\Phi^\epsilon| + |\Pi^\epsilon| = \dim L$ and so τ^ϵ is onto. Since ρ is an isomorphism of roots systems, $c^\bullet_{\alpha\beta} = c^\dagger_{\rho(\alpha)\rho(\phi)}$ for all $\alpha, \beta \in \Pi^\bullet$. Thus $\overline{L}^\bullet$ and $\overline{L}^\dagger$ are defined by the same relations and there exists a Lie-isomorphism $\tau_0 : \overline{L}^\bullet \to \overline{L}^\dagger$ with $\tau_0(\overline{r}^bullet_\alpha) = \overline{r}^\dagger_{\rho(\alpha)}$. Put $\tau = \tau^\dagger \tau_0 \tau^{\bullet -1}$. Then $\tau(r^\bullet_\alpha) = r^\dagger_{\rho(\alpha)}$, as desired. Since the $x^\bullet_\alpha, x^\bullet_{-\alpha}$ generate $L^\bullet$, τ is uniquely determined.

Finally, $\tau(h^\bullet_\alpha) = \tau([x^\bullet_\alpha, x^\bullet_{-\alpha}] = [x^\dagger_{\rho\alpha}, x^\dagger_{-\rho\alpha}] = h^\dagger_{\rho(a)}$ □

Corollary 4.2.10 [**auto of order two**] *Retain the notations of 4.2.1. Then there exists a unique Lie-automorphism τ of L such that $\tau(x_\alpha) = -x_{-\alpha}$ for all $\alpha \in \pm\Pi$. For this τ, $\tau(h) = -h$ for all $h \in H$, $\tau(L_\alpha) = L_{-\alpha}$ for all $\alpha \in \Phi$ and τ has order 2.*

For $\epsilon = \{\bullet, \dagger\}$ let $L^\epsilon = L$ and $H^\epsilon = H$. Then also $\Phi^\epsilon = \Phi$ and $h^\epsilon_\alpha = h_\alpha$. Choose $\Pi^\bullet = \Pi$, $\Pi^\dagger = -\Pi$ and $\rho(\beta) = -\beta$ for all $\beta \in \Phi$. For $\alpha \in \pm\Pi$ put $x^\bullet_\alpha = x_\alpha$ and $x^\dagger_\alpha = -x_\alpha$. Then the assumptions of 4.2.9 are fulfilled. All but the very last assertion now follow from 4.2.9. Since τ^2 fixes x_α for all $\alpha \in \pm\Pi$ the uniqueness assertion in 4.2.9 implies that $\tau^2 = \mathrm{id}_L$. □

Chapter 5

Chevalley Lie Algebras and Groups

5.1 The Chevalley Basis

The goal of this section is to find a basis for a given standard, perfect and semisimple Lie algebras such that all of the structure constants are integers. This will allows has the construct Lie algebras over arbitray fields in arbitray characteristic.

We start with a lemma about root systems which helps in the computation of the structure constants.

Definition 5.1.1 [**def: pphi**] *Let Φ be a root system. If Φ is connected let $p_\phi = \frac{(\alpha,\alpha)}{(\beta,\beta)}$, where α is a long and β is a short root in Φ. Im general let p_Φ be the maximum of the p_Ψ, ψ a connected component of Φ.*

For $\alpha \in \Phi$ let Φ_α be the connected component of Φ containing α and $p_\alpha = p_{\Phi\alpha} = \frac{(\beta,\beta)}{(\alpha,\alpha)}$, where β is a long root in Φ_α.

Lemma 5.1.2 [**rab+1**] *Let Φ be a root system and $\alpha, \beta \in \Phi$ with $\alpha \neq \pm\beta$. Then*

(a) [**a**] $c_{\alpha\beta} = (\check{\alpha}, \beta) = r_{\alpha\beta} - s_{\alpha\beta}$.

(b) [**b**] *At most two different root lenghts occur in the α-string through β.*

(c) [**c**] *If $\alpha + \beta \in \Phi$, then $r_{\alpha\beta} + 1 = s_{\alpha\beta}\frac{(\alpha+\beta,\alpha+b)}{(\beta,\beta)}$.*

(d) [**d**] $r_{\alpha\beta} + s_{\alpha\beta} \leq p_\alpha$.

Proof: (a) This is 2.11.10 c.

(b) By 3.1.12(e) is a roots system in $\mathbb{F}\langle\alpha,\beta\rangle$ and so has a basis Π'. Since $\dim \mathbb{F} < \alpha\,\beta\rangle = 2$, $|\Pi'| = 2$. By 3.3.2(i), each elemet of Φ' is conjugate under $W(\Phi')$ to an element of Π'. Conjugate elements have the same length and so (b) holds.

(c) Put $r = r_{\alpha\beta}$ and $s = s_{\alpha\beta}$. Then using (a)

$$\begin{aligned}
A := r+1-s\frac{(\alpha+\beta,\alpha+b)}{(\beta,\beta)} &= s+(\check{\alpha},\beta)+1-s\frac{(\alpha+\beta,\alpha+b)}{(\beta,\beta)}\\
&= s+(\check{\alpha},\beta)+1-s\frac{(\alpha,\alpha)}{(\beta,\beta)}-2s\frac{(\alpha,\beta)}{(\beta,\beta)}-s\frac{(\beta,\beta)}{(\beta,\beta)}\\
&= (\check{\alpha},\beta)+1-s\frac{(\alpha,\alpha)}{(\beta,\beta)}(1+2\frac{(\alpha,\beta)}{(\alpha,\alpha)})\\
&= ((\check{\alpha},\beta)+1)(1-s\frac{(\alpha,\alpha)}{(\beta,\beta)})
\end{aligned}$$

Put $B=(\check{\alpha},\beta)+1)$ and $C=1-s\frac{(\alpha,\alpha)}{(\beta,\beta)}$. Then $A=BC$. We need to show that $A=0$ that is $B=0$ or $C=0$.

Case 1: $(\alpha,\alpha)\geq(\beta,\beta)$.

Then by 3.1.4 $(\check{\alpha},\beta)\in\{1,0,-1\}$. If $(\alpha,\beta)<0$ we conclude $(\check{\alpha},\beta)=1$ and $B=0$. So suppose that $(\alpha,\beta)\geq 0$.

Then $\alpha+\beta$ has length larger than α and than β. From (b) we conclude that $(\alpha,\alpha)=(\beta,\beta)$. Also $\alpha+2\beta$ has length larger $\alpha+\beta$ and (b) implies that $\alpha+2\beta$ is not a root. Therefore $s=1$ and $C=0$.

Case 2: $(\alpha,\alpha)<(\beta,\beta)$.

If $(\alpha,\beta)\geq 0$, then $\alpha+\beta$ has length larger than β, contradicting (b). Thus $(\alpha,\beta)<0$ and $\beta-\alpha$ has length larger than β. Thus $\beta-\alpha$ is not a root and $r=0$. Also by 3.1.4 $(\alpha,\check{\beta})=-1$. Hence by (a)

$$s=-(\check{\alpha},\beta)=\frac{(\beta,\beta)}{(\alpha,\alpha)}(\alpha,\check{\beta})=\frac{(\beta,\beta)}{(\alpha,\alpha)}.$$

and so $C=0$, completing the proof of (c).

(d) Note that $r_{\alpha\beta}+s_{\alpha\beta}+1$ is the size of the α string through α, and so we may replace β by the second to last term of the this α-string. Hence $s_{\alpha\beta}=1$ and so by (c),

$$r_{\alpha\beta}+s_{\alpha\beta}=r_{\alpha\beta}+1=s_{\alpha\beta}\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)}\leq p_\Phi.$$

Moreover, $r_{\alpha\beta}+s_{\alpha\beta}\leq 1$, unless $\alpha+\beta$ is long in Φ' and β is short. Consider the latter case. If α is long, then by 3.5.3 also $\beta=(-\alpha)+(\alpha+\beta)$ is long, a contradiction. Thus α is short and (d) holds. □

Lemma 5.1.3 [**xxx**] *Let L be a standard, perfect, semisimple Lie algebra with roots system Φ. Let $a,\beta\in\Phi$ with $\alpha\neq\pm\beta$ and $x_\alpha\in L_\alpha, x_{-\alpha}\in L_{-\alpha}$ and $x_\beta\in L_\beta$ with $[x_\alpha,x_{-\alpha}]=h_\alpha$. Then*

$$[x_{-\alpha},[x_\alpha,x_\beta]]=s_{\alpha\beta}(r_{\alpha\beta}+1)x_\beta.$$

Proof: If $\alpha+\beta$ is not a root, then both sides are zero and we may assume that $\alpha+\beta \in \Phi$. Put $r = r_{\alpha\beta}$, $s = s_{\alpha\beta}$ and $v = x_{-\alpha}^r * x_\beta$. Since $\beta - r\alpha$ is a root, but $\beta - (r+1)\alpha$ is not, we have $0 \neq v \in L_{\beta - r\alpha}$ and $x_{-a} * v = 0$. Also $0 \neq w := x_\alpha^r * v \in L_\beta$ and so x_β is a scalar multiple of w. By 2.5.2(b) we have

$$[x_{-\alpha}, [x_\alpha, w]] = x_{-\alpha} x_\alpha^{r+1} * v = x_\alpha^{r+1} x_{-\alpha} * v - (r+1) x_\alpha^r (h_\alpha + r) * v$$

Using 5.1.2(a), $(\check{a}, b - r\alpha) = (\check{\alpha}, \beta) - r(\check{a}, \alpha) = r - s - 2r = -r - s$ and so $(h_\alpha + r) * v = -sv$ and so

$$[x_{-\alpha}, [x_\alpha, w]] = (r+1)sw.$$

Since $x_\beta \in \mathbb{K}w$, the lemma is proved. □

Lemma 5.1.4 [**char of chev basis**] *Let L be a standard,perfect,semisimple Lie algebra with roots system Φ and s $0 \neq x_\alpha \in L_\alpha, \alpha \in \Phi$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$ for all $a \in \Phi$. Then the following are equivalent:*

(a) [**a**] *For $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$ define $k_{\alpha\beta} \in \mathbb{K}$ by $[x_\alpha, x_\beta] = k_{\alpha\beta} x_{\alpha+\beta}$. Then $k_{-\alpha-\beta} = -\kappa_{\alpha\beta}$.*

(b) [**b**] *The unique $\mathbb{K}$-linear map $\tau : L \to L$ with $\tau(x_\alpha) = -x_a$ for all $\alpha \in \Phi$ and $\tau(h) = -h$ for all $h \in H$ is an Lie-automorphism of L.*

Proof: Note first that τ is isomorphism of $\mathbb{K}$ spaces.

1° [**1**] $[\tau(h_1), \tau(h_2)] = \tau([h_1, h_2])$ *for all $h_1, h_2 \in H$.*

Clear since both side are zero.

2° [**2**] $[\tau(h), \tau(x_\alpha)] = \tau([h, x_\alpha])$ *for all $h \in H$ and $\alpha \in \Phi$.*

$[\tau(h), \tau(x_\alpha)] = [-h, -x_{-\alpha}] = (-h)(-\alpha) \cdot (-x_{-\alpha}) = h(\alpha)\tau(x_\alpha) = \tau(h(\alpha)x_a) = \tau([h, x_\alpha])$.

3° [**3**] $[\tau(x_\alpha), \tau(x_{-\alpha})] = \tau([x_\alpha, x_{-\alpha}])$ *for all $a \in \Phi$.*

$$[\tau(x_\alpha), \tau(x_{-\alpha})] = [-x_{-\alpha}, -x_\alpha] = h_{-\alpha} = -h_\alpha = \tau(h_\alpha) = \tau([x_\alpha, x_{-\alpha}].$$

4° [**4**] $[\tau(x_\alpha), \tau x_\beta] = \tau([x_\alpha, x_b])$ *for all $\alpha, \beta \in \Phi$ with $0 \neq \alpha + \beta \notin \Phi$.*

From $0 \neq \alpha + \beta \notin \Phi$ we have $0 \neq (-\alpha) + (-\beta) \notin \Phi$ and so both sides in (4°) are zero.

5° [**5**] *Let $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$. Then $[\tau(x_\alpha), \tau x_\beta] = \tau([x_\alpha, x_b])$ iff $k_{-\alpha-\beta} = -k_{\alpha\beta}$.*

$$[\tau(x_\alpha), \tau x_\beta] = [-x_{-\alpha}, -x_{-\beta}] = k_{-\alpha-\beta}x_{-\alpha-\beta}$$

and

$$\tau([x_\alpha, x_\beta]) = \tau(k_{\alpha\beta}x_{\alpha\beta} = -k_{\alpha\beta}x_{-\alpha-\beta}$$

So (5°) holds.

Clearly (1°) -(5°) imply the lemma. □

Definition 5.1.5 [**def:chev basis**] *Let L be a standard,perfect,semisimple Lie algebra with roots system Φ and $0 \neq x_\alpha \in L_\alpha, \alpha \in \Phi$ such that*

(a) [**a**] $[x_\alpha, x_{-\alpha}] = h_\alpha$ *for all* $a \in \Phi$.

(b) [**b**] *For* $\alpha, \beta \in \Phi$ *with* $\alpha + \beta \in \Phi$ *define* $k_{\alpha\beta} \in \mathbb{K}$ *by* $[x_\alpha, x_\beta] = k_{\alpha\beta}x_{\alpha+\beta}$. *Then* $k_{-\alpha-\beta} = -\kappa_{\alpha\beta}$.

Then the basis $\mathcal{C} = (x_\alpha, h_\beta \mid a \in \Phi, \beta \in \Phi)$ *for L is called a* Chevalley basis *for L.*

Proposition 5.1.6 [**existence of chev basis**] *Let L be a standard,perfect,semisimple Lie algebra with roots system Φ. Then L has a Chevalley basis.*

Proof: By 4.2.10 there exists an automorphism τ of L of order two such that $\tau(L_\alpha) = L_{-\alpha}$ and $\tau(h) = -h$ for all $\alpha \in \Phi$, $h \in H$. For $a \in \Phi^+$ pick $0 \neq z_\alpha \in L_\alpha$. Then $\tau(z_\alpha) \in L_\alpha$ and so $[z_\alpha, -\tau(z_\alpha)] = k_\alpha h_\alpha$ for some $k_\alpha \in \mathbb{K}$. Since $\mathbb{K}$ is algebrailcy closed there exists $c_\alpha \in \mathbb{K}$ with $c_\alpha^2 k_\alpha = 1$. Put $x_\alpha = c_\alpha z_\alpha$ and $x_{-\alpha} = -\tau(x_\alpha)$. Then

$$[x_\alpha, x_{-\alpha}] = c_\alpha^2[z_\alpha, -\tau(z_\alpha)] = c_\alpha^2 k_\alpha h_\alpha = h_\alpha$$

Hence also $[x_{-a}, x_\alpha] = -[x_\alpha, x_a] = -h_\alpha = h_{-\alpha}$. Note also that $\tau(-\alpha) = -\tau^2(\alpha) = -\alpha$ and hence $\tau(\beta) = -\beta$ for all $\beta \in \Phi$. 5.1.4 now implies that $(x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$ is a Chevalley basis. □

The next lemma shows that Chevalley bases are unique up to an automorphisms and $\pm$-signs.

Lemma 5.1.7 [**uniqueness of chev basis**] *For $i \in \{\bullet, \dagger\}$ let L^i be a standard, perfect, semisimple Lie algebra with roots system Φ^i and $0 \neq x^i_\alpha \in L^i_\alpha, \alpha \in \Phi^i$ such that $[x^i_\alpha, x^i_{-\alpha}] = h^i_\alpha$ for all $\alpha \in \Phi^i$. Put $\mathcal{C}^i := (x^i_\alpha t, h^i_\beta \mid \alpha \in \Phi^i, \beta \in \Phi^i)$. Suppose that $\mathcal{C}^\bullet$ is a Chevalley basis for $L^\bullet$ and that $\rho : \Phi^\bullet \to \Phi^\dagger$ is an isomorphism of root systems with $\rho(\Pi^\bullet) = \Pi^\dagger$. According to 4.2.9, let σ be the unique isomorphism from $L^\bullet$ to $L^\dagger$ with $\sigma(x^\bullet_\alpha) = x^\dagger_{\rho(\alpha)}$ for all $\alpha \in \pm\Pi^\bullet$. Then the following are equivalent:*

(a) [**a**] $\mathcal{C}^\dagger$ *is a Chevalley basis for* $L^\dagger$.

(b) [**b**] *There exist* $\epsilon_\alpha \in \{\pm 1\}, \alpha \in \Phi^\bullet$ *such that* $\epsilon_\alpha = \epsilon_{-\alpha}$ *and* $x^\dagger_{\rho(\alpha)} = \epsilon_\alpha \sigma(x^\bullet_\alpha)$ *for all* $\alpha \in \Phi^\bullet$.

Proof: Replacing α by $\rho^{-1}(\alpha)$, $x^\dagger_\alpha$ by $\sigma^{-1}(x^\dagger)$ for all $\alpha \in \Phi^\dagger$, $\Phi^\dagger$ by $\Phi^\bullet$ and $L^\dagger$ by $L^\bullet$ we may assume that ρ and σ are the identity map and $x^\bullet_\alpha = x^\dagger_\alpha$ for all $\alpha \in \pm\Pi^\bullet$. We drop the superscript $\bullet$. Since L_α is 1-dimensional there exists unique $\epsilon_\alpha \in \mathbb{K}^\sharp$ with $x^\dagger = \epsilon_\alpha x_\alpha$ for all $\alpha \in \Phi$. We have

$$h_\alpha = [x^\dagger_\alpha, x^\dagger_{-\alpha}] = [\epsilon_\alpha x_\alpha, e_{-\alpha} x_{-\alpha}] = \epsilon_\alpha \epsilon_{-\alpha} h_\alpha$$

and so

$$(*) \quad \epsilon_{-\alpha} = \epsilon_\alpha^{-1}.$$

Let τ be the unique automorphism of L with $\tau(x_\alpha) = -x_{-\alpha}$. Since $\mathcal{C}$ is a Chevalley basis 5.1.4 implies that

$$(**) \quad \tau(x_\alpha) = -x_{-\alpha} \text{ for all } \alpha \in \Phi$$

A second application of 5.1.4 shows that $\mathcal{C}^\dagger$ is a Chevalley basis iff $\tau(x^\dagger_\alpha) = -x^\dagger_{-\alpha}$ for all $\alpha \in \Phi$. Now from(*) and (**)

$$\tau(x^\dagger_\alpha) = \tau(\epsilon_\alpha x_\alpha) = -\epsilon_\alpha x_{-\alpha} = -\epsilon^2_\alpha x^\dagger_\alpha$$

If follows that $\mathcal{C}^\dagger$ is a Chevalley basis if and only if $e^2_\alpha = 1$ for all $\alpha \in \Phi$. So (a) and (b) are equivalent. □

Theorem 5.1.8 (Chevalley) [**chevalley**] *Let L be a standard, perfect, semisimple Lie-algebra with Chevalley* $\mathcal{C} = (x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$. *Then* $[a, b] \in \mathbb{Z}\mathcal{C}$ *for all* $a, b \in \mathcal{C}$. *More precisely*

(a) [**a**] $[h_\alpha, h_\beta]$ *for all* $\alpha, b \in \pi$.

(b) [**b**] $[h_\alpha, x_\beta] = (\check{a}, \beta\,)$ *for all* $a \in \Pi$, $b \in \Phi$.

(c) [**c**]

 (a) [**a**] $[x_\alpha, x_{-a}] = \sum_{n \in \Pi} n_{\alpha\beta} h_\beta$ *for all* $a \in \Pi$, *where* $n_{\alpha\beta} \in \mathbb{Z}$ *is defined by* $\check{a} = \sum_{\beta \in \pi} n_\beta \check{\beta}$.

 (b) [**b**] $[x_\alpha, x_\beta] = 0$ *for all* $\alpha, \beta \in \Phi$ *with* $0 \neq \alpha + \beta \notin \Phi$.

 (c) [**c**] $[x_\alpha, x_b] = \pm(r_{\alpha\beta} + 1) x_{\alpha+b}$ *for all* $\alpha, \beta \in \Phi$ *with* $\alpha + \beta \in \Phi$.

Proof: (a), (b) and (c:b) are obvious. For (c:a) observe that $h_\alpha = t_{\check{a}}$ and the linearity of t imply $h_\alpha = \sum_{n\in\Pi} n_{\alpha\beta} h_\beta$. So it remains to proof (c:b).

So let $a\,\beta \in \phi$ with $\alpha+\beta \in \Phi$. We will compute $h_{\alpha+b}$ in two different ways. First by 3.3.1 $\alpha \check{+} b = \frac{(\alpha,\alpha\,)}{(\alpha+\beta,\alpha+\beta\,)}\check{a} + \frac{(\beta,\beta\,)}{(\alpha+\beta,\alpha+\beta\,)}\check{\beta}$ and so

$$(*) \quad h_{\alpha+b} = \frac{(\alpha,\alpha\,)}{(\alpha+\beta,\alpha+\beta\,)}\,\mathrm{ht}_a + \frac{(\beta,\beta\,)}{(\alpha+\beta,\alpha+\beta\,)} h_\beta$$

Using the defining relation 5.1.6 we have $h_{\alpha+\beta} = [x_{\alpha+\beta}, x_{-(\alpha+\beta)}]$, $x_{\alpha+b} = k_{\alpha\beta}[x_\alpha, x_\beta]$ and $x_{-\alpha-\beta} = k_{-\alpha\beta}[x_{-\alpha}, x_{-\beta}] = -k_{\alpha\beta}[x_{-\alpha}, x_{-\beta}]$ and so

$$(**) \quad -k_{\alpha\beta}^2 h_{\alpha\beta} = [[x_\alpha, x_\beta], [x_{-\alpha}, x_{-\beta}]]$$

From the Jacobi identity applied with $a = [x_\alpha, x_\beta], b = x_{-\alpha}$ and $c = x_{-\beta}$)

$$\begin{aligned}(***)[[x_\alpha, x_\beta],[x_{-\alpha}, x_{-\beta}]] &= -[x_{-\alpha},[x_{-\beta},[x_\alpha,x_\beta]]] - [x_{-\beta},[[x_\alpha,x_\beta],x_{-\alpha}]] \\ &= [x_{-\alpha},[x_{-\beta},[x_\beta,x_\alpha]]] + [x_{-\beta},[x_{-\alpha},[x_\alpha,x_\beta]]]\end{aligned}$$

By 5.1.3 $[x_{-\beta},[x_{-\alpha},[x_\alpha,x_\beta] = [x_b, s_{\alpha\beta}(r_{\alpha\beta}+1)x_\beta] = -s_{\alpha\beta}(r_{\alpha\beta}+1)h_\beta$. Similary (with the roles of α and β interchanged: $[x_{-\alpha},[x_{-\beta},[x_\beta,x_\alpha]]] = -s\beta\alpha(r_{\beta\alpha}+1)h_\alpha$. Substituting into (***) and then into (**) gives:

$$(****) \quad k_{\alpha\beta}^2 h_{\alpha\beta} = s\beta\alpha(r_{\beta\alpha}+1)h_\alpha + s_{\alpha\beta}(r_{\alpha\beta}+1)h_\beta$$

Comparing the coefficent of h_β in (*) and (****) (and using that h_α and h_β are linearly independent) we get

$$k_{\alpha\beta}^2 \frac{(\beta,\beta\,)}{(\alpha+\beta,\alpha+\beta\,)} = s_{\alpha\beta}(r_{\alpha\beta}+1)$$

Hence using 5.1.2 we conclude

$$k_{\alpha\beta}^2 = s_{\alpha\beta}\frac{(\alpha+\beta,\alpha+\beta\,)}{(\beta,\beta\,)}(r_{\alpha\beta}+1) = (r_{\alpha\beta}+1)^2$$

Hence $k_{\alpha\beta} = \pm(r_{\alpha\beta}+1)$ as desired. □

Lemma 5.1.9 [**nab**] *Let Φ be a root system and $(x_\alpha, h_\beta \mid \alpha \in \Phi)$ a Chevalley basis for $L_{\mathbb{K}}(\Phi)$. Let $\alpha, \beta\,\gamma \in \Phi$ with $\alpha+\beta+\gamma = 0$ and define $\eta_{\alpha\beta} = \frac{k_{\alpha\beta}}{r_{\alpha\beta}+1}$ Then*

(a) [**a**] $\eta_{\alpha\beta} = \mathrm{sgn} k_{\alpha\beta} = \pm 1$.

(b) [**b**] $\eta_{\alpha\beta} = -\eta_{\beta\alpha}$

(c) [**c**] $\eta_{\alpha\beta} = -\eta_{-\alpha-\beta}$

(d) [**d**] $\eta_{\alpha\beta} = \eta_{\beta\gamma} = \eta_{\gamma\alpha}$

(e) [**e**] *Exactly one of the sets* $\{\alpha, \beta\}$, $\{-\alpha, -\beta\}$, $\{\beta, \gamma\}$, $\{-\beta, -\gamma\}$, $\{\gamma, \alpha\}$ *and* $\{-\gamma, -\alpha\}$ *consists of two positive roots.*

(a) follows from 5.1.8(c:c). (b) follows from $[x_\alpha, x_\beta] = -[x_\beta, x_\alpha]$. (c) follows from (a) and the definition of a Chevalley basis (see 5.1.5(b)).

(d) We have

$$[x_{-\alpha}, [x_\alpha, x_\beta]] = [x_{-\alpha}, k_{\alpha\beta} x_{-\gamma}] = k_{-\alpha-\gamma} k_{\alpha\beta} x_\beta$$

On the other hand by 5.1.3

$$[x_{-\alpha}, [x_\alpha, x_\beta]] = s_{\alpha\beta}(r_{\alpha\beta} + 1)x_\beta$$

and we concldue that $\eta_{-\alpha-\gamma}\eta_{\alpha\beta} = 1$. Thus using (b) and (c)

$$\eta_{\alpha\beta} = \eta_{-\alpha-\gamma} = -\eta_{\alpha\gamma} = \eta_{\gamma\alpha}.$$

By symmetry also $\eta_{\beta\gamma} = \eta_{\gamma\alpha}$ and so (d) holds.

(e) Replacing (α, β, γ) by $(-\alpha, -\beta, -\gamma)$ we may assume that at least two of the root $(\alpha, \beta\,\gamma)$ are positive. Say α, β are positive. Then γ, $-\alpha$ and b are negative and $-\gamma$ is positive. Thus $\{\alpha\,\beta\}$ is the unique positive pair of the six pairs in (e). □

The preceeding lemma shows that the $\eta_{\alpha\beta}$ and so also the $k_{\alpha\beta}$ are uniquely determined by the $\eta_{\alpha\beta}$'s for $\alpha, \beta \in \Phi^+$.

5.2 Chevalley algebras

Given a standard perfect, semisimple Lie algebra L with Chevalley basis $\mathcal{C}$. Define $= \mathbb{Z}\mathcal{C}$. By 5.1.8, $[a, b] \in \mathbb{Z}$ for all $a, b \in \mathbb{C}$ and so restriction of $[\cdot, \cdot]$ to $L_\mathbb{Z}$ gives a well defined $\mathbb{Z}$-bilinear map

$$[\cdot, \cdot] : L_\mathbb{Z} \times L_\mathbb{Z} \to L_\mathbb{Z}$$

Note that the Jacobi identity holds and so $L_\mathbb{Z}$ is a Lie-algebra over $\mathbb{Z}$. Also $\mathcal{C}$ is a $\mathbb{Z}$ basis for $L_\mathbb{Z}$.

Now let $\mathbb{E}$ be an arbitray field of arbitray characteristic and define $= \mathbb{E} \otimes_\mathbb{Z} L_\mathbb{Z}$. Note that map $\mathbb{E} \otimes \mathbb{Z} \to \mathbb{E}, (e \otimes m) \to me$ is well defined and has inverse $\mathbb{E} \to \mathbb{E} \otimes \mathbb{E}, e \to e \otimes 1$. Thus $\mathbb{E} \cong \mathbb{Z} \otimes \mathbb{E}$, in other words $\mathbb{Z} \otimes \mathbb{E}$ is 1-dimensional vectorspace over $\mathbb{E}$ with basis $1 \otimes 1$. Since tensor products behave well with respect to direct sums we conclude that $\mathcal{C}_\mathbb{E} = (1 \otimes c \mid c \in \mathcal{C})$ is a $\mathbb{E}$-basis for $L_\mathbb{E}$. Observe that the map

$$(\mathbb{E} \times L_\mathbb{Z}) \times (\mathbb{E} \times L_\mathbb{Z}) \to L_\mathbb{E}, ((e, a), (f, b)) \to ef \otimes [a, b]$$

is linear in each coordinate and so we obtain a unique bilinear map

$$[\cdot,\cdot] : L_{\mathbb{E}} \times L_{\mathbb{E}} \to L_{\mathbb{E}} \text{ with } [e \otimes a, f \otimes b] = ef \otimes [a,b]$$

for all $e, f \in \mathbb{E}$, $a, b \in\in L_{\mathbb{Z}}$.

Since the Jacobi identity holds for the subset $1 \otimes L_{\mathbb{Z}}$ of $L_{\mathbb{E}}$ and since $1 \otimes L_{\mathbb{Z}}$ spans $L_{\mathbb{E}}$ has an $\mathbb{E}$-space we conclude that the Jacobi identity holds on all of $L_{\mathbb{E}}$. Thus $L_{\mathbb{E}}$ is Lie-algebra over $\mathbb{E}$. Lie algebras of this form are called *Chevalley algebras.* Let $\mathfrak{U}_{\mathbb{Z}}(L)$ be the subring of $\mathfrak{U}(L)$ generated by the $\frac{x_\alpha^n}{n!}$, $\alpha \in \Phi$, $m \in \mathbb{N}$.

Lemma 5.2.1 [**hz**] *Let $H_{\mathbb{Z}} = \mathbb{Z}\langle h_\alpha \alpha \in \Pi \rangle$. Then*

(a) [**a**] $H_{\mathbb{Z}} = L_{\mathbb{Z}} \cap H$.

(b) [**b**] *There exists an $\mathbb{Z}$-isomorphism $g : \mathbb{Z}\check{\Phi} \to H_{\mathbb{Z}}$ with $g(\check{\alpha}) = h_\alpha$ for all $\alpha \in \Phi$.*

(c) [**c**] $h_\alpha \in H_{\mathbb{Z}}$ *for all* $\alpha \in \Phi$.

Proof: (a) Since $(x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$ is a $\mathbb{K}$ basis for L and a $\mathbb{Z}$ basis for $L_{\mathbb{Z}}$ we see that $H \cap L_{\mathbb{X}} = H_{\mathbb{Z}}$.

(b) Let g be the restriction of the K-isomomorphism $H^* \to H, \alpha \to t_\alpha$ to $\mathbb{Z}\Phi$ (See before 2.11.6). Then for $\alpha \in \Phi$, $f(\check{\alpha}) = t_{\check{a}} = h_\alpha$ and so f sends the $\mathbb{Z}$ basis $(\check{a} \mid \alpha \in \Pi)$ to the $\mathbb{Z}$-basis $(h_\alpha \mid \alpha \in \Pi)$ of $\mathbb{Z}$. Thus (b) holds.

(c) Let $\alpha \in \Phi$, then $h_\alpha = f(\check{\alpha}) \in H_{\mathbb{Z}}$. □

Proposition 5.2.2 [**lz invariant**] *Let L be a standard Lie algebra with Chevalley basis $\mathcal{C} = (x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$. Let $m \in \mathbb{Z}_+$ and $\alpha\, \beta \in \Phi$. Then*

(a) [**a**] $\frac{x_\alpha^m}{m!} * h_\beta = \begin{cases} -(\alpha, \check{\beta})x_\alpha & \text{if } m = 1 \\ 0 & \text{if } m > 1. \end{cases}$

(b) [**b**] $\frac{x_\alpha^m}{m!} * x_b = \begin{cases} h_\alpha & \text{if } \beta + m\alpha = 0 \\ -x_\alpha & \text{if } \beta + m\alpha = \alpha \\ \pm\binom{r_{\alpha\beta}+m}{m} x_{\beta+m\alpha} & \text{if } \alpha \neq \beta + m\alpha \in \Phi \\ 0 & \text{if } 0 \neq \beta + m\alpha \notin \Phi \end{cases}$

In particular, $L_{\mathbb{Z}}$ is invariant under $\mathfrak{U}_{\mathbb{Z}}(L)$.

Proof: (a) $x_\alpha * h_\beta = [x_\alpha, h_\beta] = -[h_\beta, x_\alpha] = -(\alpha, \check{\beta})x_\alpha$. Since $[x_\alpha, x_\alpha] = 0$ concldue $\frac{x_\alpha^m}{m!} * h_\beta =$ for all $m > 1$.

(b) If $\beta + m\alpha = 0$, then $\beta = -m\alpha$ and so $m = 1$. Also $[x_\alpha, x_{-\alpha}] = h_\alpha$.

If $\beta + m\alpha = \alpha$, then $\beta = (1 - m\alpha)$ and so $m = 2$ and $\beta = a$. Also

$$\frac{x_\alpha^2}{2!} * x_{-\alpha} = \frac{1}{2}[x_\alpha, h_\alpha] == \frac{1}{2}(\alpha, \check{\alpha})x_\alpha = -x_\alpha.$$

If $\alpha \neq b + m\alpha \in \Phi$, then $\alpha \neq \pm\beta$ and $m \leq (\check{,\alpha\beta})$. We claim that $\frac{x_\alpha^m}{m!} * x_\beta = \pm\binom{r_{\alpha\beta}+m}{m} x_{\beta+m\alpha}$. Clearly this is true for $m = 0$. Supppose $m > 0$ and that its true for $m-1$. Note that $r_{\alpha,\beta+m-1\alpha} = r_{\alpha\beta} + m - 1$ and so

$$\begin{aligned}
\frac{x_\alpha^m}{m!} * x_b &= \frac{1}{m}[x_\alpha, \pm\binom{r_{\alpha\beta}+m-1}{m-1} x_{\alpha+(m-1)\beta} \\
&= \pm\frac{1}{m}\binom{r_{\alpha\beta}+m-1}{m-1}(\pm(r_{\alpha,\beta+(m-1)\alpha}+1)x_{\beta+m\alpha}) \\
&= \pm\binom{r_{\alpha\beta+m}}{m} x_{\beta+m\alpha}.
\end{aligned}$$

Suppose finally that $0 \neq \beta + m\alpha \notin \Phi$. Then $\frac{x_\alpha^m}{m!} * x_b \in L_{\alpha+m\beta} = 0$. $\square$

Let $\mathbb{E}$ be a field, $e \in E$ and $\phi \in \mathrm{End}(L_\mathbb{Z})$. Then we denotes the unique endomorphism of $L_\mathbb{E}$ which sends $f \otimes l$ to $ef \otimes \phi(l)$ by $e \otimes \phi$. By 5.2.2 each $\frac{x_\alpha^m}{m!}$ gives rises to an endomorphsim $\phi_{\alpha,m}$ of $L_\mathbb{Z}$. Moreover, $\phi_{\alpha,m}$ is zero for almost all m and so we can define $\chi_\alpha(e) =:= \sum_{i=0}^\infty e^n \otimes \phi_{\alpha,m} \in \mathrm{End}(L_\mathbb{E})$. Also put $\mathcal{X}_\alpha = \{\chi_\alpha(e) \mid e \in E\}$. We will later proof that $\chi_\alpha(e)\chi_\alpha(f) = \chi_{e+f}$ and so each $\chi_\alpha(e)$ is invertible and $\mathcal{X}_\alpha$ is a subgroup of $GL(L_E)$ isomorphic to $(\mathbb{E}, +)$. Denote by $G_\mathbb{E}(L_\mathbb{Z})$ the subgroups of $GL_\mathbb{E}(L_\mathbb{E})$ generated by the $\mathcal{X}_\alpha, \alpha \in \Phi$. $G_\mathbb{E}(L_\mathbb{Z})$ is called a adjoint Chevalley group of type Φ over the field $\mathbb{E}$. The next lemma shows that the isomorphism class of the group $G_\mathbb{E}(L_\mathbb{Z})$ does not depend on the choice of the Chevalley basis $\mathcal{C}$. Moreover, it shows that any graph autmorphism of Π can be extended to an automorphism of $G_\mathbb{E}(L_\mathbb{Z})$.

Proposition 5.2.3 [**gel well defined**] *For $\epsilon \in \{\bullet, \dagger\}$ let L^ϵ be a standard perfect, semisimple Lie algebra with Chevalley basis $\mathcal{C}^\epsilon = (x_\alpha^\epsilon, h_\alpha^\epsilon \mid \alpha \in \Phi^\epsilon, \beta \in \Pi^\epsilon)$ over $\mathbb{K}$. Suppose that $\rho : \Phi^\bullet \to \Phi^\dagger$ is an isomorphism of root systems with $\rho(\Pi^\bullet) = \Pi^\dagger$ and let $\mathbb{E}$ be any field. Then there exist$\epsilon_\alpha \in \{\pm\}$, $\alpha \in \Phi^\bullet$, with $\epsilon_\alpha = \epsilon_{-\alpha}$ and $\epsilon_\alpha = 1$ for $\alpha \in \Pi^\bullet$ and an isomorphism $\sigma : G_\mathbb{E}(L_\mathbb{Z}^\bullet) \to G_\mathbb{E}(L_\mathbb{Z}^\dagger)$ such that*

$$\sigma(\chi_\alpha(t)) = \chi_{\rho(\alpha)}(\epsilon_\alpha t)$$

for all $\alpha \in \pm\Pi^\bullet$

Proof: Let ϵ_α, $\alpha \in \Phi^\bullet$ and $\tau : L^+ \to L^-$ be as in 5.1.7. Replacing $L^\dagger$ by $L^\bullet$ and $\mathcal{C}^\dagger$ by $\tau^{-1}(C^\dagger)$ we may assume $\rho = \mathrm{id}_{\Phi^\bullet}$, and $x_\alpha^\bullet = \epsilon_\alpha x_\alpha^\dagger$ for all $\alpha in \Phi^+$. But then $\chi_\alpha^\bullet(t) = \chi_\alpha^\dagger(\epsilon_\alpha t)$, $\mathcal{X}_\alpha^\bullet = \mathcal{X}_\alpha^\dagger$, and $G := G_\mathbb{E}(L_\mathbb{Z}^\bullet) = G_\mathbb{E}(L_\mathbb{Z}^\dagger)$. So the lemma holds with $\sigma = \mathrm{id}_G$. $\square$

5.3 Konstant Theorem

Let L be a standard, perfect semisimple Lie algebra with Chevalley basis $\mathcal{C} = (x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$. The goal of this section is to find a $\mathbb{Z}$-basis for $\mathfrak{U}_\mathbb{Z}(L)$.

Let $\mathbb{K}$ be a field with $\operatorname{char}\mathbb{K} = 0$ and A a $\mathbb{K}$-algebra. For $m \in \mathbb{N}$ and $a \in A$ define

$$\binom{a}{m} = \frac{\prod_{i=0}^{m-1} a + i - 1}{m!}$$

We claim $\binom{a}{m-1} + \binom{a}{m} = \binom{a+1}{m}$. Note that both sides are polynomials in a with rational coefficients. The difference of these polynomials has each positive integer as a root and therefore is 0.

Let I be a finite set. For $a = (a_i)_{i\in I} \in A^I$ and $m = (m_i) \in \mathbb{N}^I$ we define

$$\binom{a}{m} = \prod_{i\in I}^{n} \binom{a_i}{m_i}.$$

Fix some total ordering on Φ. For $\Psi \subseteq \Phi$ and $m = (m_\alpha)_{\alpha\in\Psi} \in \mathbb{N}^\Psi$ define

$$x^m = \prod_{\alpha\in\Psi} x_\alpha^{m_\alpha} \in \mathfrak{U}(L)$$

where the product is taken in the given order. Also define $\operatorname{ht} m = \sum_{\alpha\in\Psi} m_\alpha$, $m! = \prod_{\alpha\in\Psi} m_\alpha!$ and $\lambda_m = \sum_{a\in\Psi} m_\alpha \alpha$. Note that $\lambda_m \in \mathbb{Q}\Pi$ and $\operatorname{ht} m = \operatorname{ht}\lambda_m$. Since $\Phi \subseteq H^*$ we have $\lambda_m \in H^*$ and $\lambda_m(h_\beta) = \sum_{\alpha\in\Psi} m_\alpha(\alpha, \check{\beta})$. Put $h = (h_\beta)_{\beta\in Pi}$.

Let $\mathbb{K}[s_i \mid i \in I]$ the ring of polynomials in the variables $s_i, i \in I$ and coefficients in $\mathbb{K}$. We say that $f \in \mathbb{K}[s_i \mid i \in I]$ is *integral-valued* if $f(\mathbb{Z}^I) \subseteq \mathbb{Z}$. $\mathbb{K}_\mathbb{Z}[s_i \mid i \in I]$ is the set of integral valued polynomials. Let $s = (s_i)_I \in I$ and $m \in \mathbb{N}^I$. Then $\binom{s}{m}$ is an integral valued polynomial.

Lemma 5.3.1 [**basis for kz**] *Let I be a finite set, and $\mathbb{K}$ a field with $\operatorname{char}\mathbb{K} = 0$. Then $(\binom{s}{m} \mid m \in \mathbb{N}^I)$ is a $\mathbb{Z}$ basis for $\mathbb{K}_\mathbb{Z}[s_i \mid i \in I]$.*

Proof: By induction on I. If $I = \emptyset$ then $\mathbb{K}_\mathbb{Z}[s_i \mid i \in I] = \mathbb{Z}$ and has basis 1.

Suppose now that $|I| \geq 1$, pick $i \in I$, and put $J = I - i$. Let $f \in \mathbb{K}_\mathbb{Z}[s_i \mid i \in I]$. Then there exists $n \in \mathbb{N}$ and $f_k \in \mathbb{K}[s_i \mid i \in J]$, $0 \leq k \leq n$ with $f = \sum_{i=0}^n f_k \binom{s_i}{k}$. Let $g_0 = f$ viewed as a polynomial in $B[s_i]$. Inductively define $g_{r+1} = g_r(s_i + 1) - g_r(s_i)$. We claim $g_r = \sum_{k=r} f_k \binom{s_i}{k-r}$. Indeed this is true for $r = 0$ and by induction

$$g_{r+1} = \sum_{k=r}^{n} f_k(\binom{s_i+1}{k-r} - \binom{s_i}{k-r}) = \sum_{k=r+1}^{n} f_k \binom{s_i}{k-(r+1)}$$

Thus completes the proof of the claim. Also by induction each $g_r \in \mathbb{K}_\mathbb{Z}[s_i \mid i \in I]$. For $r = n$ we get

$$g_n = f_n$$

and so f_n is integral valued. By induction on

$$f_n \in \mathbb{Z}\langle \binom{(s_j)_{j\in J}}{m'}, m' \in \mathbb{N}^J . \rangle$$

Hence $f_n\binom{s_i}{n} \in \mathbb{Z}\langle\binom{s}{m} \mid m \in \mathbb{N}^I\rangle$. By induction on n, the same is true for $f - f_n\binom{s_i}{n}$. It is easy to see that $\binom{s}{m}, m \in \mathbb{N}^I$ are linearly independent over $\mathbb{Z}$ and the theorem is proved. □

Lemma 5.3.2 [**basis for u0**] *Let* $\mathfrak{U}^0_{\mathbb{Z}}(L) = \{f(h) \mid f \in \mathbb{K}_{\mathbb{Z}}[s_\alpha \mid \alpha \in \Pi]\}$. *Then*

(a) [**a**] $(\binom{h}{m} \mid m \in \mathbb{N}^\Pi)$ *is a* $\mathbb{Z}$ *basis for* $\mathfrak{U}^0_{\mathbb{Z}}(L)$.

(b) [**b**] $\mathfrak{U}^0_{\mathbb{Z}}(L)$ *is subring of* $\mathfrak{U}(L)$ *generated by the* $\binom{h_\beta}{m}$, $\beta \in \Pi, m \in \mathbb{N}$.

(c) [**c**] *For all* $\alpha \in \Phi$, $m \in \mathbb{N}$ *and* $n \in \mathbb{Z}$, $\binom{h_\alpha+n}{m} \in \mathfrak{U}^0_{\mathbb{Z}}(L)$.

Proof: By the PBW-theorem,

$$(\binom{h}{m} \mid m \in \mathbb{N}^\Pi),$$

is a $\mathbb{K}$ basis for $\mathfrak{U}(H)$. In particular, its linearly independent over $\mathbb{Z}$. (a) now follows from 5.3.1. Observe that (a) implies (b).

For (c) note that h_α is an intergral linear combination of the $h_\beta, \beta \in \Pi$. Thus $\binom{h_\alpha+n}{m}$ is an integral valued polynomial in the $h_\beta, \beta \in \Pi$ and (c) holds. □

Lemma 5.3.3 [**xa eigenvalue**] *Let* $m \in \mathbb{N}^\Phi$ *and* $f \in \mathbb{K}[s_\alpha \mid \alpha \in \Pi)$. *For* $e = (e_\alpha)_{\alpha\in\Pi} \in H^\Pi$ *and* $\lambda \in H^*$ *define* $\lambda^\Pi(e) = (\lambda(e_\alpha)_{\alpha\in\Pi}$. *Let* V *be an* L*-module. Also view* $\mathfrak{U}(L)$ *is a* L*-module via the adjoint action* $l * u = [l, u]$.

(a) [**y**] *Let* $\lambda \in H^*$ *and* $v \in V$. *Then* $f(h)v = f(\lambda^\Pi(h))v$.

(b) [**a**] *Let* $\lambda \in H^*$ *and* $u \in \mathfrak{U}$. *Then* $uf(h) = f(h - \lambda^\Pi(h))u$.

(c) [**z**] *Let* $\lambda, \mu \in H^*$, $v \in V_\mu$ *and* $u \in \mathfrak{U}_\mu$. *Then* $uv \in V_{\lambda+\mu}$.

(d) [**b**] *For* $i = 1, 2$ *let* $\lambda_i \in H^*l$. *Let* $u \in U_{\lambda_i}$. *Then* $u_1u_2 \in \mathfrak{U}_{\lambda_1+\lambda_2}$.

(e) [**c**] $x_m \in U_{\lambda(m)}$.

(f) [**d**] $x_m f(h) = f(h - \lambda_m(h))x_m$

Proof: (a) Note that $f(s) \to f(\lambda^\Pi(h))$ is the unique $\mathbb{K}$-linear ringhomorphism $\mathbb{K}[s] \to \mathbb{K}$ with $s_\beta \to \lambda(h_\beta)$. This implies (a).

(b) For a fixed $\lambda \in H^*$ and $u \in \mathfrak{U}$ let A be set of all $f \in \mathbb{K}[s] := \mathbb{K}[s_\alpha \mid \alpha \in \Phi$ for which (b) holds. We first show that A is a $\mathbb{K}$-subalgebra of $\mathbb{K}[s]$. Since (b) is linear in f, A is a $\mathbb{K}$-subspace of $\mathbb{K}[s]$. Now let $f_1, f_2 \in A$. Then

$u(f_1f_2)(h) = f_1(h - \lambda)uf_2(h) = f_1(h - \lambda(h))f_2(h - \lambda(a)u = (f_1f_2)(h - \lambda(h))$

and so $f_1f_2 \in A$.

Now let $\beta \in \Pi$ and $f = s_\beta$. Then $f(h) = h_\beta$ and so

$$uf(h) = uh_\beta = h_\beta u - \lambda(h_\beta)u = (h_\beta \lambda(h_\beta)u = f(h - \lambda(h))u$$

Thus $s_\beta \in A$. Clearly also $1 \in A$ and since A is a subalgebra, $\mathbb{K}[s] \leq A$. Hence (b) holds.

(c) This follows from 2.2.2(a) and 2.3.5.

(d) Apply (c) to $V = \mathfrak{U}$ views as an L-module by left multiplication.

(e) Since $x_\alpha \in L_\alpha \leq \mathfrak{U}(L)_\alpha$, (e) holds for $\operatorname{ht} m = 1$. Using (d) and induction on $\operatorname{ht} m$ we see that (e) holds.

(f) Follows from (e) and (b). □

Lemma 5.3.4 [**konstant for sl2**] *Let (x, y, h) be a Chevalley basis for $L \cong \mathfrak{sl}(\mathbb{K}^2)$. Then for all $i, j \in \mathbb{N}$.*

$$\frac{x^i}{i!}\frac{y^j}{j!} = \sum_{k+m=i, k+l=j} \frac{y^l}{l!}\binom{h-l-m}{k}\frac{x^m}{m!}$$

Proof: By 5.3.3(f) we have

1° [**1**] $\quad$ *$xf(h) = f(h-2)x$ for any $f \in \mathbb{K}[s]$.*

From 2.5.2(b) we have

2° [**2**] $\quad x\frac{y^j}{j!} = \frac{y^j}{j!}x + \frac{y^{j-1}}{j-1!}(h-(j-1))$

The proof of the lemma is by induction on i For $i = 0$ both sides are equal to $\frac{y^j}{j!}$. Suppose the lemma is true for i. Then using (1°) and (2°):

$$\begin{aligned}
x^{i+1}\frac{y^j}{j!} &= = \sum_{k+m=i,k+l=j} x\frac{y^l}{l!}\binom{h-l-m}{k}\frac{x^m}{m!} \\
&= \sum_{k+m=i,k+l=j} (\frac{y^l}{l!}x + \frac{y^{l-1}}{l-1!}(h-(l-1)))\binom{h-l-m}{k}\frac{x^m}{m!} \\
&= \sum_{k+m=i,k+l=j} \frac{y^l}{l!}\binom{h-l-m-2}{k}(m+1)\frac{x^{m+1}}{m+1!} \\
&\quad + \sum_{k+m=i,k+l=j} \frac{y^{l-1}}{l-1!}(h-(l-1)\binom{h-l-m-2}{k}\frac{x^m}{m!}
\end{aligned}$$

We shift the summation indices: In the first summand we replace $m+1$ by m and in the second $l-1$ by l and k by $k-1$. Thus

$$\begin{aligned}
x^{i+1}\frac{y^j}{j!} &= \sum_{k+m=i+1,k+l=j} \frac{y^l}{l!}\binom{h-l-m-1}{k} m\frac{x^m}{m!} \\
&= + \sum_{k+m=i+1,k+l=j} \frac{y^l}{l!}(h-l)\binom{h-l-m-1}{k-1}\frac{x^m}{m!} \\
& \sum_{k+m=i+1,k+l=j} \frac{y^l}{l!}\left(m\binom{h-l-m-1}{k} + (h-l)\binom{h-l-m-1}{k-1}\right)\frac{x^m}{m!}
\end{aligned}$$

Put $t = h-l-m-1$. Then it remains to show that

$$m\binom{t}{k} + (h-l)\binom{t}{k-1} = (i+1)\binom{t+1}{k}.$$

Since $\binom{t+1}{k} = \binom{t}{k} + \binom{t}{k-1}$ this is equivalent to

$$\frac{h-l-i+1}{i+1-m}\binom{t}{k-1} = \binom{t}{k}$$

From $k+m = i+1$ and the definiton of t we have $h-l-i-1 = t+m-i = t-k+1$ and $i+1-m = k$. So we need to show

$$(\frac{t-k+1}{k}\binom{t}{k-1} = \binom{t}{k}$$

But this follows directly from the definition of $\binom{t}{k}$. $\square$

Corollary 5.3.5 [**hb in uz**] $\mathfrak{U}^0_{\mathbb{Z}} \leq \mathfrak{U}_{\mathbb{Z}}$.

Proof: By 5.3.2 is suffices to show that $\binom{h_\beta}{i} \in \mathfrak{U}_{\mathbb{Z}}$ for all $\beta \in \Pi$ and $i \in \mathbb{N}$. This is clearly true for $i = 0$. Apply 5.3.4 to $\langle L_\alpha, L_{-\alpha}\rangle_{Lie}$ with $i = j$. So

$$(*) \quad \frac{x_\alpha^i}{i!}\frac{x_{-\alpha}^j}{j!} = \binom{h_\alpha}{i} + \sum_{m=1}^{i} \frac{x_{-\alpha}^i}{i!}\binom{h_\alpha - 2m}{i-m}\frac{x_\alpha^m}{m!}$$

By 5.3.2, each $\binom{h_\alpha - 2m}{i-m}$ is an integral linear combinations of $\binom{h_\alpha}{k}$ with $k \leq i-m > i$ and so is by induction on i contained in $\mathfrak{U}_{\mathbb{Z}}$. So by (*) also $\binom{h_\alpha}{i} \in \mathfrak{U}_{\mathbb{Z}}$.

Lemma 5.3.6 [**xm and tensor**] *Let I be a finite set and for $i \in I$, let V_i be an L-module. Let $\Psi \subseteq \Phi$, $m \in \mathbb{N}^\Psi$ and $v_i \in V_i$. Then*

$$\frac{x^m}{m!} \cdot \bigotimes_{i\in I} v_i = \sum\{\bigotimes_{i\in I} \frac{x^{m_i}}{m_i!} \cdot v_i \mid m_i \in \mathbb{N}^\Psi, \sum_{i\in I} m_i = m.\}$$

Proof: See Exercise II.2.1. □

Corollary 5.3.7 [**tensor invariant**] *Let I be a finite set and for $i \in I$, let V_i be a L-module and M_i and $\mathfrak{U}_\mathbb{Z}$ invarinat subgroup of V_i. Then $\bigotimes_{i\in I} M_i$ is a $\mathfrak{U}_\mathbb{Z}$-invaraint subgroup of $\bigotimes_{i\in I} V_i$.*

Proof: Follows immediately from 5.3.6 □

Lemma 5.3.8 [**relatively prime**] *For $1 \leq i \leq d$ let $\alpha_i \in \Phi$. Let $H_\mathbb{Z} = \mathbb{Z} < h_\alpha \mid \alpha \in \Phi\rangle$ and let $k \in \mathbb{K}$. Put $h = \otimes h_{\alpha_i} \in \bigotimes^d L$ and suppose that $kh \in \bigotimes^d H_\mathbb{Z}$. Then $k \in \mathbb{Z}$*

Proof: For $1 \leq i \leq d$ choose a base Π_i of Φ with $a_i \in \Pi_i$. Then $\check{\Pi}_i$ is a $\mathbb{Z}$-basis for $\mathbb{Z}\check{\Phi}$ and so $\mathcal{H}_i = \{h_\alpha \mid \alpha\Pi\}$ is a $\mathbb{Z}$-basis for $H_\mathbb{Z}$ and a $\mathbb{K}$-basis for H. Thus $\bigotimes \mathcal{H}_i$ is a $\mathbb{Z}$ basis for $\bigotimes^d H_\mathbb{Z}$ and a $\mathbb{K}$-basis for $\bigotimes^H$. Since h is one of these basis vectors we get $m \in \mathbb{Z}$. □

Lemma 5.3.9 [**konstant for nilpotent**] *Let Ψ be a closed subset of Φ with $\Psi \cap -\Psi = \emptyset$. Let $\mathfrak{U}_\mathbb{Z}(\Psi)$ be the subring of $\mathfrak{U}_\mathbb{Z}(L)$ generated by the $\frac{x_\alpha^m}{m!}, \alpha \in \Psi, m \in \mathbb{N}$. Then*

$$(\frac{x^m}{m!} \mid m \in \mathbb{N}^\Psi)$$

is a $\mathbb{Z}$ basis for $\mathfrak{U}_\mathbb{Z}(\Psi)$.

Proof: Let $L(\Psi)$ be the Lie subalgebra of L generated by the $x_\alpha, \alpha \in \Psi$ and note that since Ψ is closed and $\Psi \cap -\Psi = \emptyset$, $(x_\alpha, \alpha \in \Psi)$ is a basis for $L(\Psi)$. Also $\mathfrak{U}_\mathbb{K}(\Psi) := \mathbb{K}U_\mathbb{Z}(\Psi)$ is the K-subalgebra of $\mathfrak{U}(L)$ generated by $L(\Psi)$ and using the PBW-theorem we see that $(x^m \mid m \in \mathbb{N}^\Psi)$ is a $\mathbb{K}$ basis for $\mathfrak{U}_\mathbb{K}(\Psi)$. Thus also

$$(\frac{x^m}{m!} \mid m \in \mathbb{N}^\Psi)$$

is a $\mathbb{K}$-basis of $\mathfrak{U}_\mathbb{K}(\Psi)$. Let $u \in \mathfrak{U}_\mathbb{Z}(\Psi)$. Then $u = \sum_{m\in\mathbb{N}^\psi} k_m \frac{x^m}{m!}$, where almost all $k_m = 0$. We need to show that $k_m \in \mathbb{Z}$ for all m. Define $d = \operatorname{ht}(u) = \{\max \operatorname{ht} m \mid k_m \neq 0\}$. By induction on d it sufficed to show that $k_m \in \mathbb{Z}$ for all m with $\operatorname{ht} m = d$. So fix $n \in \mathbb{N}^\Psi$ with $k_n \neq 0$ and $\operatorname{ht} n = d$. Let $I = \{(\alpha, k) \mid \alpha \in \Psi, 1 \leq k \leq n_\alpha\}$ and for $i = (\alpha, k) \in I$ define $\alpha(i) = \alpha$. Put

$$v := \bigotimes_{i\in I} x_{-\alpha(i)} \in \bigotimes^I L_\mathbb{Z}, \text{ and } h := \bigotimes_{i\in I} h_{\alpha(i)}$$

Note that $\bigotimes^I \mathcal{C}$ is a $\mathbb{Z}$ basis for $\bigotimes^I L_\mathbb{Z}$ and $\mathbb{K}$-basis for $\bigotimes^I L$. Let $\mathcal{H} = \{h_\beta \mid \beta \in \Pi\}$ and $A = \bigotimes^I H$. So A has $\mathbb{K}$ basis $\bigotimes^I \mathcal{H}$. Let B be the $\mathbb{K}$-subspace spanned by the remaining basis vectors $\bigotimes^I \mathcal{C} \setminus \bigotimes^I \mathcal{H}$. We compute the projection w of $u*v$ onto A. For this we first investigate $\frac{x^m}{m!} * v$ for $m \in \mathbb{N}^\Psi$ with $\operatorname{ht} m \leq d$. By 5.3.6

$$\frac{v^m}{m!} * v = \frac{x^m}{m!} * \bigotimes_{i \in I} x_{-\alpha(i)} = \sum \{ \bigotimes_{i \in I} \frac{x^{m_i}}{m_i!} * x_{-\alpha} \mid m_i \in \mathbb{N}^\Psi, m = \sum_{i \in I} m_i \}$$

Consider the summand $z := \bigotimes_{i \in I} \frac{x^{m_i}}{m_i!} * x_{-\alpha(i)}$. If $m_i = 0$ for some $i \in I$, then $\frac{x^{m_i}}{m_i!} * x_{-\alpha(i)} = 1 * x_{-a(i)} = x_{-\alpha(i)}$ and so $z \in B$. Suppose that $m_i \neq 0$ for all $i \in i$. Since $d \geq \operatorname{ht} m = \sum_{i \in I} \operatorname{ht} m_i$ and $|I| = \operatorname{ht} n = d$ we see that $\operatorname{ht} m_i = 1$ for all $i \in I$. Hence there exists $\beta(i) \in \Psi$ with $m_i = (\delta_{\beta(i)\beta})_{\beta \in \Psi}$. That is $\frac{x^{m_i}}{m_i!} = x_{\beta(i)}$ and so

$$z = \bigotimes_{i \in I} [x_{\beta(i)}, x_{-\alpha(i)}]$$

Since $[x_{\beta(i)}, x_{-\alpha(i)}] \in L_{\beta(i)-\alpha(i)}$ we conclude that $z \in B$ unless $\beta(i) = \alpha(i)$ for all $i \in I$. Now if $\beta(i) = \alpha(i)$ for all $i \in I$, then $z = h$ and

$$m = \sum_{i \in I} m_i = \sum_{i \in I} (\delta_{\alpha(i)\beta})_{\beta \in \Psi} = (n_\beta)_{\beta \in \Psi} = n.$$

Therefore $\frac{x^m}{m!} * v \in B$ for all $m \neq n$, while for $m = n$ the projection of $\frac{x^m}{m!} * v$ onto A is h. As $u = \sum k_m \frac{x^m}{m!}$ we conclude that the projection of $u * v$ onto A is $k_n h$.

By 5.3.7 $\bigotimes^I L_\mathbb{Z}$ is invariant under $\mathfrak{U}_\mathbb{Z}(L)$ and so $u*v$ and the projection $k_n h$ are in $L_\mathbb{Z}$. Thus 5.3.8 implies $k_n \in \mathbb{Z}$. □

We call any product in any order of elements of the forms $\frac{x_\alpha^m}{m!}$ and $f(h)$, where $m \in \mathbb{Z}$, $\alpha \in \Phi$ and $f \in \mathbb{K}_\mathbb{Z}[s]$ a monomial of height the sum of the $m's$.

Lemma 5.3.10 [**xaxb**] *Let $\alpha, \beta \in \Phi$ and $i, j \in \mathbb{N}$. Then*

$$\frac{x_\alpha^i}{i!} \frac{x_\beta^j}{j!} = \frac{x_b^j}{j!} \frac{x_\alpha^j}{j!} + \text{ an integral linear combination of monomials of height less than } i{+}j.$$

Proof: If $\alpha = \beta$, this is obvious, if $\alpha = -\beta$ this follows from 5.3.4. So suppose $\alpha \neq \pm\beta$ and let $\Psi = (\mathbb{N}\alpha + \mathbb{N}\beta) \cap \Psi$. Then Ψ is clearly closed and since α and β are linearly independent, $\Psi \cap -\Psi = 0$. Choose the ordering on Φ such that $\beta < \alpha$. By 5.3.9 $\frac{x_\alpha^n}{n!} \frac{x_\beta^m}{m!} = \sum_{m \in \mathbb{N}^\Psi} k_m \frac{x^m}{m!}$ with $k_m \in \mathbb{Z}$. From the proof of the PBW theorem we have $k_m = 0$ for all m with $\operatorname{ht} m > i{+}j$ and there exists a unique summand of height $i+j$, namely $\frac{x_b^j}{j!} \frac{x_\alpha^i}{i!}$. □

Theorem 5.3.11 (Konstant) [**konstant**] *Let L be standard,perfect, simisimple Lie algebar with Chevalley basis $\mathcal{C} = (x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$. For $\epsilon = \pm$ put $\mathfrak{U}_\mathbb{Z}^\epsilon = \mathfrak{U}_\mathbb{Z}(\Phi^\epsilon)$. Then* *(a)* [**a**] $(\frac{x^{m_+}}{m_+!}\binom{h}{m_0}\frac{x^{m_-}}{m_-!} \mid m_+ \in \mathbb{N}^{\phi^+}, m_0 \in \mathbb{N}^{\Pi}, m_- \in \mathbb{N}^{\Phi^-}$ *is a* $\mathbb{Z}$ *basis for* $\mathfrak{U}_{\mathbb{Z}}$ *and a* $\mathbb{K}$*-basis for* $\mathfrak{U}$.

(b) [**b**] $\mathfrak{U}_{\mathbb{Z}} = \mathfrak{U}^+_{\mathbb{Z}}\, U^0_{\mathbb{Z}}\mathfrak{U}^-_{\mathbb{Z}}$.

Proof: (a) By 5.3.5 all monomials are in $\mathfrak{U}_{\mathbb{Z}}$. So we just need to show that every monomial is a an integral linear combination of the monomials $\frac{x^{m_+}}{m_+!}\binom{h}{m_0}\frac{x^{m_-}}{m_-!}$. Monomials of height 0 are handle by 5.3.2(a). By 5.3.10 and 5.3.3(f) two monomials with the same factors but different order of multiplication only differ by a integral combination of monomials of smaller height. Also $\frac{x^i_\alpha}{i!}\frac{x^j_\alpha}{j!} = \binom{i+j}{i}\frac{x^{i+j}_\alpha}{i+j!}$ and so we can combine factors. Induction on the height of the monomials now completes the proof of (a).

(b) Follows immediately from (a).

5.4 Highest weight modules

In this section L continues to be a standard, semisimple Lie-algebra with Chevalley basis $\mathcal{C} = \{x_\alpha, h_\beta \mid \alpha \in \Phi, b \in \Pi)$. Let $L^\epsilon = \sum_{a\in\Phi^\epsilon} L_\alpha$ and $B = L^+ \oplus H$. By the PBW-theorem we may view $\mathfrak{U}(B)$ as subalgebra of $\mathfrak{U}(L)$.

Lemma 5.4.1 [**structure of l**]

(a) [**a**] L^+ *is nilpotent.*

(b) [**b**] $L^+ = [L^+, H] = [L^+, B] = [B, B]$.

(c) [**c**] B *is solvable.*

Proof: (a) For $i \in \mathbb{Z}$ let $L_i = \sum\{L_\lambda \mid \lambda \in \Phi \cup \{0\}, \operatorname{ht}\lambda \leq i$. Since $[L_\lambda, L_\alpha] \leq L_{\lambda+\alpha}$ we have $[L_i, L^+] \leq L_{i+1}$. Also $L_i = 0$ if $i > \operatorname{ht}\alpha$ for all $\alpha\Phi$ and so (a) holds.

(b) $[L_\alpha, H] = L_\alpha$ for al $a \in \Phi$ and so $[L^+, H] = 0$. Also $[H.H] = 0$ and so (b) holds.

(c) Follows from (a) and (b). □

Definition 5.4.2 [**def:maximal vector**] *Let V be an L-module.*

(a) [**a**] *A* maximal vector *for L on V with weight $\tilde\lambda \in \Lambda_V(B)$ is a non-zero vector $v^+ \in L_{\tilde\lambda}$.*

(b) [**b**] *V is called a cyclic L-module with heighest weight $\tilde\lambda$ provided that there exists a maximal vector with weight $\tilde\lambda$ which is not conatined in any proper L-submodule of V.*

Let $\tilde\lambda \in \Lambda(L) = \operatorname{Hom}_{Lie}(B, \mathfrak{gl}(\mathbb{K}))$. Since $\mathfrak{gl}(\mathbb{K})$ is abelian, $L^+ \leq \ker\tilde\lambda$ and so $\tilde\lambda$ is uniquely determined by its restriction λ to H. Conversely, if λ is a weight for H, then $\tilde\lambda : L \to \mathfrak{gl}(\mathbb{K}), l + h \to \lambda(h)$ is a well defined weight for H. In particular, a maximal vector for L on V is just a weight vector v^+ for H on V with $L^+v^+ = 0$.

Lemma 5.4.3 [**weight for b**] *Let V be a standard L-module. Then V contains a maximal vector. In particular, any standard, simple L-module is cylic and V is the direct sum of cylic, simple L-modules.*

By 2.1.6 there exists a 1-dimensional B submodule $V_1 \leq V$. Then any non-zero vector in V_1 is a maximal weight. This clearly implies that standard simple L-module is cyclic. By Weyl's Theorem 2.9.3 V is the direct sum of simple L-modules and so all parts of the lemma are proved. □

Lemma 5.4.4 [**structure of cyclic modules**] *Let V be a cyclic L-module with maximal vector v^+ of heighest weight λ. Then*

(a) [**a**] $V = \mathfrak{U}v^+ = \mathfrak{U}^- v^+$.

(b) [**b**] $V = \bigoplus_{\mu \in \Lambda_V(H)} V_\mu$.

(c) [**c**] *Let* $\mu \in \Lambda_V(H)$. *Then* $\mu = \lambda - \delta$ *for some* $\delta \in \mathbb{N}\Pi$ *and* $V_\mu = \mathbb{K}\langle \frac{x^m}{m!} v^+ \mid m \in \mathbb{N}^{\Phi^-}, \delta = -\lambda(m) \rangle$

(d) [**d**] $V_\lambda = \mathbb{K}v^+$ *and* $\dim V_{\lambda-\delta} \leq |\{m \in N^{\Phi^-} \mid \delta = -\lambda(m)\}| < \infty$.

(e) [**e**] *Let W be an H-submodule in V. Then* $W = \bigoplus_{\mu \in \Lambda_V(H)} W \cap V_\mu$.

(f) [**f**] *Any nontrival quotient of V is cylic with highest weight λ.*

(g) [**g**] *V has a unique maximal L-submodule W, W contains all proper L-submodules and V/W is a simple.*

Proof: (a) $\mathfrak{U}v^+$ is a L-submodule of V containing v^+ and so the definition of a cyclic module implies that $V = \mathfrak{U}v^+$. Since $\mathbb{K}v^+$ is invariant under B, it is also invariant under $\mathfrak{U}(B) = \mathfrak{U}^0\mathfrak{U}^+$. So

$$V = \mathfrak{U}v^+ = \mathfrak{U}^-\mathfrak{U}^0\mathfrak{U}^+ v^+ = \mathfrak{U}^- v^+ =$$

and so (a) holds.

For $\delta \in \mathbb{N}\Pi$ put $V(\delta) = \mathbb{K}\langle \frac{x^m}{m!} v^+ \mid m \in \mathbb{N}^{\Phi^-}, \delta = -\lambda(m) \rangle$. Then by 5.3.3(f), a, $V(\delta) \leq V_{\lambda-\delta}$. Thus using (a) we have

$$V = \sum_{\delta \in \mathbb{N}\Pi} V(\delta) \leq \sum_{\delta \in \mathbb{N}\pi} V_{\lambda-\delta} \leq \bigoplus_{\mu \in \Lambda_V(H)} V_\mu$$

Thus (b) and (c) hold. If $m \in \mathbb{N}^\phi$ with $\lambda(m) = 0$, then $m = 0$. So $V_\lambda = V(0) = \mathbb{K}v^+$ and (d) holds.

(e) By (b), V is a semisimple L-module. So by 1.7.10(c) also W is a semisimple H-module and so (e) holds.

(f) Let $W \neq V$ be a L-submodule. Then $v^+ + W$ is a maximal vector in V/W with weight λ and is not contain in any prober L-submodule of V/W.

(g) Let U be any proper L-submodule. Then $v^+ \notin U$ and sicne $V_\lambda = \mathbb{K}v^+$, λ is not a weight for H on U. Hence by 1.7.10(d), λ is also not a weight on the sum W of all proper L-submodules. Thus $W \neq V$ and (g) holds. □

Corollary 5.4.5 [**unique maximal vector**] *Let V be a L module. For $i =, 1, 2.$ let v_i be a maximal vectors with weight λ_i. If V is simple or if V is cyclic with respect to v_1 and $v+2$, then $\lambda_1 = \lambda_2$ and $\mathbb{K}v_1 = \mathbb{K}v_2$.*

Proof: If V is simple, then V is cylic with respect to v_1 and v_2. So we may assume the latter. Then by 5.4.4(c) applied to $v^+ = v_1$, $\lambda_1 - \lambda_2 \in \mathbb{N}\Pi$. By symmetry, also $\lambda_2 - \lambda_1 \in \mathbb{N}\Pi$. Thus $\lambda_1 = \lambda_2$ and $\mathbb{K}v_1 = V_{\lambda_1} = \mathbb{K}v_2$. □

Lemma 5.4.6 [**ul free for ub**] *View $\mathfrak{U}(L)$ as a right-$\mathfrak{U}(B)$-module via right multiplication. Then $\mathfrak{U}(L)$ is a free $\mathfrak{U}(B)$-module with basis $(x^m \mid m \in \mathbb{N}^{\Phi^-})$.*

Proof: By the PBW-Theorem 1.5.7 $x^{m^+}h^{m_0}x^m, m^+ \in \mathbb{N}^{\Phi^+}, m_0 \in \mathbb{N}^\pi, m \in \mathbb{N}^{\Phi^-}$ is a $\mathbb{K}$-basis for $\mathfrak{U}(L)$. Also $x^{m^+}h^{m_0}, m^+ \in \mathbb{N}^{\Phi^+}, m_0 \in \mathbb{N}^\Pi$ is a $\mathfrak{U}(B)$. Thus every element in $\mathfrak{U}(L)$ can be written as $\sum_{m \in \mathbb{N}^{\Phi^-}} b_m x^m$ for uniquely determined $b_m \in \mathfrak{U}(B)$. So $(x^m \mid m \in \mathbb{N}^{\Phi^-})$ is an $\mathfrak{U}(B)$-basis for $\mathfrak{U}(L)$. □

Lemma 5.4.7 [**induced**] *Let A be a ring (with 1), $1 \in B \leq A$ and W a B-module. Put $V = A \otimes_B W$ and define $f : W \to A, w \to 1 \otimes w$. Then V is an A-module via $a(e \otimes w) = ae \otimes w$. f is B-invariant and if $\tilde{V}$ is an A-module and $\tilde{f} : W \to \tilde{V}$ is B-invariant, then there exists a unique A-invarinat $g : V \to \tilde{V}$ with $g \circ f = \tilde{f}$.*

Proof: Fix $a \in A$. Then map $(e, w) \to ae \otimes w$ is $\mathbb{Z}$-bilinear and B-balanced and so we obtain a welldefined map $A \times A \otimes_B B \to A \otimes B$ with $(a, e \otimes w) \to ae \otimes w$. It is readily verfied that this is an action. Also for $b \in B$ and $w \in W$:

$$bf(w) = b(1 \otimes w) = b \otimes w = 1b \otimes w = 1 \otimes bw = f(bw)$$

and so f is B-invariant.

Given $\tilde{f} : W \to \tilde{V}$. Define $A \times W \to \tilde{V}, (a, w) \to a\tilde{f}$. Thus is $\mathbb{Z}$-bilinear and B-balanced and so we obtain a unique $g : V \to \tilde{V}, a \otimes w \to a\tilde{f}(w)$. Clearly g is A-invarinat, and $g \circ f = \tilde{f}$. □

Proposition 5.4.8 [**highest weight modules**] *Let $\lambda \in H^*$. Let $\mathrm{D}(\lambda)$ be the B-module, such that $\mathrm{D}(\lambda) = \mathbb{K}$ as a $\mathbb{K}$-space, $lk = 0$ and $hk = \lambda(h)k$ for all $l \in L^+$, $h \in H$ and $k \in K$. Put $M(\lambda) = \mathfrak{U}(L) \otimes_{\mathfrak{U}(B)} D(\lambda)$.*

(a) [**a**] $x^m \otimes 1, m \in \mathbb{N}^{\Phi^-}$ *is an* $\mathbb{K}$*-basis for* $M(\lambda)$*.*

(b) [**b**] $M(\lambda)$ *is a cyclic* L*-module with maximal vector* $v^+ = 1 \otimes 1$ *and highest weight* λ*.*

(c) [**c**] *Let* V *be any cylic* L*-module with maximal vector* $\tilde{v}^+$ *and heighest weight* λ*. Then there exists a unique* L*-invariant map* $g : M(\lambda) \to V$ *with* $g(v^+) = \tilde{v}^+$*. Moreover* g *is onto.*

(d) [**d**] *Let* $W(\lambda)$ *be the unique maximal* L*-submodule in* $M(\lambda)$*. Then any simple* L*-module with a maximal vector of highest weight* λ *is isomorphic to* $V(\lambda) := M(\lambda)/W(\lambda)$*.*

(a) follows from 5.4.6.

(b) Since $I(\lambda) = \mathfrak{U}v^+$, v^+ is not contained in any proper L-submodule. $\mathbb{K}v^+ = 1 \otimes D(\lambda)$ is ismomoprhic to $D(\lambda)$ as B-module. So v^+ is a maximal vector with weight λ.

(c) The existence and uniqueness of g follows from 5.4.7. Since $g(I(\lambda)$ is a submodule of V containing $\tilde{v}^+$, g is onto.

(d) Let V be any simple L-module with a maximal vector of heighest weight λ. Then V is cyclic and so by (a) there exists an onto L-invaraint maps $g : I(\lambda) \to V$. So $I(\lambda)/\ker g \cong V$. As V is simple, $\ker g$ is a maximal L-submodule. So by 5.4.4(g), $\ker g = W(\lambda)$. □

Definition 5.4.9 [**def:integral**] *Let* $\lambda \in H^*$*. We say that* λ *is* integral *if* $\lambda(h_\alpha) \in \mathbb{Z}$ *for all* $\alpha \in \Phi$*.* Λ *is dominant integral if an only if* $\lambda(h_\alpha) \in \mathbb{N}$ *for all* $\alpha \in \Phi^+$*.*

Note that the roots system $t(\check{\Phi}) = \{h_\alpha \mid a \in \Phi\} \leq \mathbb{Q}\langle h_\alpha \mid \alpha \in \Phi\rangle$ is isomorphic to $\check{\Phi}$. In particular we have natural bijection between the (domminat) integral weights for $\check{\Phi}$ and the (dominant) integral weights for H.

Theorem 5.4.10 [**standard simple**] *Let* $\lambda \in H^*$*. Then the following are equivalent*

(a) [**a**] $V(\lambda)$ *is finite dimensional.*

(b) [**b**] λ *is dominant integral.*

(c) [**c**] $\dim V_\mu = \dim V_{w(\mu)}$ *for all* $w \in W_\Phi$ *and* $\mu \in H^*$*.*

(d) [**d**] $\Lambda_V(H)$ *is invariant under* W_Φ*.*

(e) [**e**] $\Lambda_V(H)$ *is finite.*

Proof: Let v^+ be a maximal vector with weight λ in $V := V(\lambda)$.

(a)$\Longrightarrow$ (b): Let $\alpha \in \Phi^+$ and put $S_\alpha = \langle L_\alpha, L_{-\alpha}\rangle_{Lie} \cong \mathfrak{sl}_2(\mathbb{K})$. Let $T_1 \leq T_2 \leq V(\lambda)$ be S_α-submodules such that $v^+ \in T_2 \setminus T_1$ and T_1 is a maximal S_α-submodule. Let $t = v^+T_1 \leq T := T_1/T_2$. Then t is a highest maximal vector for S_α on T with weight $\lambda\mid_{H_\alpha}$. Since T is a finite dimensional simple S_α-module 2.5.4 implies that $\lambda(h_\alpha) = \dim V - 1 \in \mathbb{N}$. So λ is dominant integral.

(b)$\Longrightarrow$ (c): Suppose now that λ is dominant integral. Let $\beta \in \Phi^+$ define $m_\beta = \lambda(h_\beta)$. So m_β is a positive integer.

1° [**1**] $x_{-\beta}^{m_\beta+1}v^+ = 0.$

Suppose $w := x_{-\beta}^{m_\beta+1}v^+ \neq 0.$. Then by 5.3.3(a), w is a weightvector for H with weight $\lambda - 2m_{\beta+1}\beta \neq \lambda$. Let $\beta \neq \alpha \in \Pi$. Then x_α commutes with $x_{-\beta}$ and annihilates v^+. Thus x_α also annihilates w. By 2.5.4(b:d) (applied with $i = m_\beta + 1$) also x_β annihilates Since L^+ is generate by the $x_\alpha, \alpha \in \Pi$ as a Lie-algebra, we have $L^+w = 0$ and w is a maximal vector of weight unequal to λ. This contradiction to 5.4.5 proves (1°).

2° [**2**] *Let $S_\alpha = \langle L_\alpha, L_{-\alpha}\rangle_{Lie}$. Then V is the union of the finite dimensioal S_α-submodules.*

Let W be the union of the finite dimensioanl L-submodules in V. From (1°) and 2.5.4(b:f) we have that v^+ is a contained in a finite dimensional S_α module and so $v^+ \in W$. Let T be any finite dimensional S_α-submodule. Since L is finite dimensional, also LT is finite dimensional Since $S_\alpha LT \leq (LS_\alpha + [L, S_\alpha])T \leq LT$, LT is S_α invariant and $LT \leq W$. Thus W is an L-submodule. Since V is cylic and $v^+ \in W$, $V = W$.

So (2°) hold. Since $W(\Phi)$ is generated by the $\omega_a, \alpha \in \Pi$, we may assume that $w = \omega_\alpha$ for $\alpha \in \Pi$. From (2°) and 4.2.5, x_α and x_a are locally nilpotent. 4.2.6 now implies that (c) holds.

(c)$\Longrightarrow$ (d): is obvious.

(d)$\Longrightarrow$ (e):

Let $\mu \in \Lambda_V(H)$. By 3.5.5(a) there exists $w \in W_\Phi$ such that $w(\mu)$ is dominant. From (d) we have $w(\mu) \in \Lambda_V(H)$. Thus by 5.4.4(c), $w(\mu) \prec \lambda$. 3.7.19(c) implies that there are only finitely many possiblities for $w(\mu)$. Since W_Φ is finite (e) holds.

(e)$\Longrightarrow$ (a):

From (e) and 5.4.4 implies that $\dim V = \sum_{\mu\in\Lambda_V(H)} \dim V_\mu$ is finite. □

Proposition 5.4.11 [**all standard l-modules**] *Let $\check{L}ambda^+$ be the set of dominant integral weights for H.*

(a) [**a**] *For each standard simple L-module V there exists a unique $\lambda \in \check{\Lambda}^+$ with $V \cong V(\Lambda)$.*

(b) [**b**] *For each standard L-module V there exists a unique $m \in \bigoplus_{\check{\Lambda}^+} \mathbb{N}$ with*

$$V \cong \bigoplus_{\lambda\in\check{\Lambda}^+} V(\lambda)^{m_\lambda}$$

Proof: (a) By 5.4.3 V is cylic. So by 5.4.8 $V \cong V(\lambda)$ for some $\lambda \in H^*$. By 5.4.5 λ is unique and by 5.4.10 λ is dominant integral.

(b) By Weyl Theorem 2.9.3 V is the direct sum of simple L-modules. So by (a) theres exist $m \in \bigoplus_{\check{\Lambda}^+} \mathbb{N}$ with $V \cong \bigoplus_{\lambda\in\check{\Lambda}^+} V(\lambda)^{m_\lambda}$.

Let $\lambda \in \check{\Lambda}^+$. From 5.4.5, m_λ is the dimension of $C_V(L^+)_\lambda$. So m is unique. □

Lemma 5.4.12 [**symmetric powers**] *Suppose $L \cong \mathfrak{sl}(\mathbb{K}^2)$, $\Phi = \{\alpha\}$ and λ the dominant integral weight with $\lambda = (\check{\alpha})^*$. Then for all $n \in \mathbb{N}$,*

$$V(n\lambda) \cong S^n(V(\lambda)).$$

Proof: Let v be a maximal vector in $V := V(\lambda)$. Then $v^n \in S^n(V)$ is a maximal vector with weight $n\lambda$. Let $X = \mathfrak{U}(L)v^n$. Then by 5.4.8, X has a unique maximal L-submodule Y and $X/Y \cong V(n\lambda)$. From 2.5.4, V is 2-dimensional and $V(n\lambda)$ is $n+1$-dimensional. Thus $\dim S^n(V) = \binom{2+n-1}{n} = n+1$-dimensional. Hence $X = S^n(V)$, $Y = 0$ and $V(n\lambda) \cong S^n(V)$. □

5.5 Modules over arbitary fields

Lemma 5.5.1 [**delta polynomial**] *Let I be finite set, D a finite subset of $\mathbb{Z}^I$ and $d \in D$. Then there exists $f \in \mathbb{Q}_{\mathbb{Z}}[s_i \mid i \in I]$ such that for all $e \in D$*

$$f_d(e) = \delta_{de}.$$

Proof: Choose $k \in \mathbb{N}$ such that $|d_i - e_i| leqk$ for all $e \in D$ and $i \in I$. Define

$$f = \prod_{i \in I} \binom{s_i - d_i + k}{k} \binom{-s_i + d_i + k}{k}$$

Then clearly $f(d) = 1$. Let $d \neq e \in D$ and choose $i \in I$ with $d_i \neq e_i$. If $e_i < d_i$ then $0 \leq -e_i + d_i + k < k$ and so $\binom{-e_i+d_i+k}{k} = 0$. If $d_i < e_i$, then $0 \leq e_i - d_i + k \leq k$ and so $\binom{s_i-d_i+k}{k} = 0$. In any case $f(e) = 0$. □

Lemma 5.5.2 [**decompose m**] *Let V be a finite dimensional L-module and M a $\mathfrak{U}^0\mathbb{Z}$-invariant subgroup of V. Then*

$$M = \bigoplus_{\mu \in \lambda_V(H)} (M \cap V_\mu)$$

Proof: Let $m \in M$. Then $m = \sum_{\mu \in \Lambda_V(H)} m_\mu$ with $m_\mu \in V_\mu$. We need to show that $m_\rho \in M$ for all $\rho \in \Lambda_V(H)$. Let $D = \{\mu^I(h) \mid \mu \in \Lambda_V(H)\} \subseteq \mathbb{Z}^\Pi$. By 5.5.1 there exists $f {\in} \mathbb{K}_{\mathbb{Z}}[s_\alpha \mid \alpha\Pi]$ with $f(\mu^I(h)) = \delta_{\rho\mu}$. Then by 5.3.3(b)

$$f(h)m = \sum_\mu f(h)m_\mu = \sum_\mu f(\mu^I(h))m_\mu = \sum_\mu \delta_{\rho\mu} m_\mu = m_\rho.$$

Since $f(h) \in \mathbb{U}^0_{\mathbb{Z}}$ this imples $m_\rho \in M$ as desired. □

Definition 5.5.3 [**def:lattice**] *Let V be a vector space over the field $\mathbb{E}$ and R a subring of $\mathbb{E}$. Then an R-*lattice *in V is a free R-submodule M in V such that the map $\mathbb{E} \otimes RM \to V, e \otimes m \to em$ is an isomorphism. A lattice in V is a $\mathbb{Z}$-lattice.*

Observe that M is an R-lattice in V iff there exists a $\mathbb{E}$-basis $\mathcal{B}$ for V with $M = R\mathcal{B}$ and iff M is a free R-submodule of V such that any R-basis for M is an $\mathbb{E}$-basis for V.

Lemma 5.5.4 [**invariant lattice**] *Let V be a standard L-module and that L is perfect and semisimple. Then there exists an $\mathfrak{U}_{\mathbb{Z}}$-invariant lattice in V.*

Proof: By Weyl's Theorem 2.9.3 V is the direct sum of simple L-modules an so we may assume that V is a simple L-module. Let v^+ be a maximal vector with highest weight λ. Put $M = \mathfrak{U}_{\mathbb{Z}} v^+$. Let $f \in \mathbb{K}_{\mathbb{Z}}[s]$. Then $f(h)v^+ = f(\lambda(h))v^+ \in \mathbb{Z}v^+$. Also $\mathfrak{U}^+_{\mathbb{Z}} v^+ = 0$. This implies that $M = \mathfrak{U}^-_{\mathbb{Z}} v^+$ and

1° [**1**] *the projection of M onto V_λ is $\mathbb{Z}v^+$.*

Since $V_\mu = 0$ for almost all μ we have $x(m)v^+ = 0$ for almost all $m \in \mathbb{N}^{Phi^-}$ and so M is a finitely generated $\mathbb{Z}$-module and so is free. Since V is simple

2° [**2**] $V = \mathfrak{U}v = (\mathbb{K}\mathfrak{U}_{\mathbb{Z}})v = \mathbb{K}(\mathfrak{U}_{\mathbb{Z}} v)$ *for all* $0 \neq v \in V$.

In particular, $\mathbb{K}M = V$. Thus the $\mathbb{K}$-span of any $\mathbb{Z}$-basis of M is V. It remains to show that any $\mathbb{Z}$-linearly independent subset is also linearly independent over $\mathbb{K}$. Suppose not. Then there exists a $\mathbb{Z}$-linearly independent subset $\mathcal{M}$ in M of minimal size such that $\sum_{m \in caM} k_m m = 0$ for some $k_m \in \mathbb{K}$ not all zero. Next we show

3° [**3**] $(k_m, m \in \mathcal{M})$ *is linearly independent over $\mathbb{Z}$.*

Suppose not. Then $\sum n_m k_m = 0$ for some $n_m \in \mathbb{Z}$, not all zero. Pick $a \in \mathcal{M}$ with $n_a \neq 0$. Then

$$0 = n_a(\sum k_m m) - (\sum n_m k_m)a = \sum_{a \neq m \in \mathcal{M}} k_m(n_a m - n_m a)$$

But the $n_a m - n_m a, a \neq m \in \mathcal{M}$ are linear independent over $\mathbb{Z}$, contradicting the minimal choice of $\mathcal{M}$. Thus (3°) holds.

For $v \in \mathcal{V}$ and let v_λ be the projection of v onto V_λ.let $m \in \mathcal{M}$. By (3°), $(um)_\lambda \neq 0$ for some $u \in \mathfrak{U}_{\mathbb{Z}}$ and by (1°), $(um)_\lambda = n_m v^+$ for some $0 \neq n_m \in \mathbb{Z}$. Thus

$$0 = 0_\lambda = (\sum k_m m)_\lambda = \sum k_m m_\lambda = \sum k_m n_m v^+$$

and $\sum k_m n_m = 0$, a contradiction to (3°). $\square$

5.6 Properties of the exponential map

Lemma 5.6.1 [**e ad y**] *Let $\mathbb{K}$ be field with* $\operatorname{char}\mathbb{K} = 0$ *and let A be an associative $\mathbb{K}$ algebra with a* 1. *Let $x, y \in A$*

(a) [**a**]

(a) [**a**] $\dfrac{\operatorname{ad} x^n}{n!}(y) = \displaystyle\sum_{i+j=n} \frac{x^i}{i!} y \frac{(-x)^j}{j!}$

(b) [**b**] $yx^n = \displaystyle\sum_{i+j=k} (-i)^j \binom{n}{i} x^i \cdot \operatorname{ad}(x)^j(y)$

(b) [**b**] *Suppose that x is nilpotent.*

(a) [**a**] $\operatorname{ad} x \in \operatorname{End}(A)$ *is nilpotent.*

(b) [**b**] $e^{\operatorname{ad} x}(y) = e^x y e^{-x}$.

(c) [**c**] *If y is invertible, then* $ye^x y^{-1} = e^{yxy^{-1}}$

(c) [**c**] *Suppose that x, y and $[x, y]$ are nilpotent and that $[x, y]$ commutes with x and y. Then*

(a) [**a**] $x + y$ *is nilpotent.*

(b) [**b**] $e^{x+y} = e^x e^y e^{-\frac{1}{2}[x,y]}$

(a) Let l_x and r_x be endomorphism of A obtained by left and right multiplication by x. Then $\operatorname{ad}(x) = l_x - r_x$. Since A is associative, l_x and r_x commute. So (a:a) follows from the binomial formula. Similarly $r_x = l_x - \operatorname{ad}(x)$ implies (a:b)

(b:a) and (b:b) follows from (a). (b:c) follows from $yx^n y^{-1} = (yxxy^{-1})^n$.

(c) Put $z = -\frac{1}{2}[x, y]$. The first prove that

$$(*) \qquad \frac{(x+y)^n}{n!} = \sum_{i+j+2k=n} \frac{x^i}{i!} \frac{y^j}{j!} \frac{z^k}{k!}.$$

This is true for $n = 0$. Since $[x, y]$ commutes with x we have $\operatorname{ad}(x)^j(y) = 0$ for all $j > 1$. So by (a:b) $yx^i = x^i y - ix^{i-1}[x, y] = x^i y + 2x^{i-1} z$. Assume (*) is true for $n - 1$. Then

$$
\begin{aligned}
\frac{(x+y)^n}{n!} &= \frac{x+y}{n} \cdot \sum_{i+j+2k=n-1} \frac{x^i}{i!}\frac{y^j}{j!}\frac{z^k}{k!} \\
&= \frac{1}{n} \sum_{i+j+2k=n-1} \frac{x^{i+1}}{i!}\frac{y^j}{j!}\frac{z^k}{k!} \\
&\quad + \frac{1}{n} \sum_{i+j+2k=n-1} \frac{x^i}{i!}\frac{y^{j+1}}{j!}\frac{z^k}{k!} \\
&\quad + \frac{2}{n} \sum_{i+j+2k=n-1} \frac{x^{i-1}}{i!}\frac{y^j}{j!}\frac{z^{k+1}}{k!} \\
&= \sum_{i+j+2k=n} \frac{i+j+2k}{n} \cdot \frac{x^i}{i!}\frac{y^j}{j!}\frac{z^k}{k!} \\
&= \sum_{i+j+2k=n} \frac{x^i}{i!}\frac{y^j}{j!}\frac{z^k}{k!}
\end{aligned}
$$

So (*) holds. But (*) implies both (c:a) and (c:b). □

5.7 Rational functions

Definition 5.7.1 [**def:rational**] *Let J be a finite set, V, W be standard $\mathbb{K}$-spaces, M and N lattices lattice in V and W, respectively and $\mathcal{A}$ and $\mathcal{B}$ an $\mathbb{Z}$- basis for M and N, respectively. Let $f : V \times \mathbb{K}^{\sharp J} \to W$ be a function. We say that f is rational with respect to M and N if there exists integral polynomials*

$$f_b \in \mathbb{Z}[s_a, a \in \mathcal{A}; s_j, j \in J; \tilde{s}_j, j \in J],$$

for $b \in \mathcal{B}$ such that

$$f(\sum_{a\in\mathcal{A}} k_a a; k_j, j \in J) = \sum_{b\in\mathcal{B}} f_b(k_a, a \in \mathcal{A}; k_j, j \in J; k_j^{-1}, j \in J)\, b$$

for all $k_a \in \mathbb{K}, a \in \mathcal{A}$ and $k_j \in \mathbb{K}^\sharp, j \in J$.

Definition 5.7.2 [**def:fe**] *Let $\mathbb{E}$ be a field.*

Let $f : A \to W$ be a rational function as in 5.7.1 Let $V^{\mathbb{E}} = \mathbb{E} \otimes_{\mathbb{Z}} M$, $A^{\mathbb{E}} = V^{\mathbb{E}} \times \mathbb{E}^{\sharp J}$, $W^{\mathbb{E}} = \mathbb{E} \otimes_{\mathbb{Z}} \mathbb{K}N$. Then$f^{\mathbb{E}} : A^{\mathbb{E}} \to W^{\mathbb{E}}$ is defined by

$$f^{\mathbb{E}}(\sum_{a\in\mathcal{A}} e_a \otimes a; e_j, j \in J) = \sum_{b\in\mathcal{B}} f_b(e_a \mid a \in \mathcal{A}; e_j, j \in J) \otimes b$$

for all $e_a \in \mathbb{K}, a \in \mathcal{A}$,$e_j \in \mathbb{K}^\sharp, j \in J$

Lemma 5.7.3 [**basic rational**] *Let V be a standard $\mathbb{K}$-space with $\mathbb{Z}$-lattice M.*

(a) [**a**] *Let $m_i \in M$, $0 \le i \le n$. Then the function $\mathbb{K} \to V, t \to \sum_{i=0}^n t^i v_i$ is rational.*

(b) [**b**] *The map $\mathrm{End}(V) \times V \to V, (g, v) \to g(v)$ is rational with respect to the lattices $\mathrm{End}_{\mathbb{Z}}(M) \times M$ in $\mathrm{End}(V) \times V$ and M in V.*

(c) [**c**] *The map $\mathrm{End}_{\mathbb{K}}(V) \times \mathrm{End}_{\mathbb{K}}(V) \to \mathrm{End}_{\mathbb{K}}(V), (f, g) \to fg$ is rational.*

(d) [**d**] *Suppose V is an L-module and M is $\mathfrak{U}_{\mathbb{Z}}$ invariant. Then the map $\mathbb{K} \to \mathrm{End}(V), t \to \chi_\alpha(t)$, is rational for all $\alpha \in \Phi$.*

(e) [**e**] *The map $L \times L \to L, (a, b) \to [a, b]$ is rational.*

(f) [**f**] *Compositions of rational function are rational.*

Proof: Readily verified. (d) for example follows from (a) applied to $m_i = \frac{x_\alpha^i}{i!}$, $''M = \mathrm{End}_{\mathbb{Z}}(M)''$ and $''V = \mathrm{End}(V)''$. □

Proposition 5.7.4 [**rational equation**] *Let $f, g : A \to W$ be a rational function. Suppose that $f(a) = g(a)$ for all $a \in A$. Then also $f^{\mathbb{E}}(e) = g^{\mathbb{E}}(e)$ for all $e \in \mathbb{E}$.*

Proof: Put $h = f - g$. Then h is rational and $h(a) = 0$ for all $a \in A$. We can choose $n_j \in \mathbb{N}, j \in J)$ such that for all $b \in \mathcal{B}$,

$$\tilde{h}_b(s_l, l \in \mathcal{A} \cup J) := h(s_a, a \in \mathcal{A}; s_j, j \in J; s_j^{-1}) \prod_{j \in j} s_j^{n_j}$$

is in a integral polynomial in the $s_l, l \in \mathcal{A} \cup J$. From $h(a) = 0$ for all $a \in A$ we get that $\tilde{h}_b(k_l, l \in \mathcal{A} \cup J) = 0$ for all $0 \neq k_l \in \mathbb{K}$. Since $\mathbb{K}$ is infinite, this implies that $\tilde{h}_b = 0$. Let $\tilde{h} = h \prod_{j \in J} k_j^{n_j}$. From $\tilde{h}_b = 0$ we get $\tilde{h}^{\mathbb{E}} = 0$. Thus $0 = \tilde{h}^{\mathbb{E}}(e) = h^{\mathbb{E}}(e) \prod_{j \in J} e_j^{n_j}$ for all $e = (v; e_j, j \in J) \in A^{\mathbb{E}}$. Hence also $h^{\mathbb{E}}(e) = 0$ and $f^{\mathbb{E}}(e) = g^{\mathbb{E}}(e)$. □

We call an equation $f(a) = g(a)$ as in the preceeding lemma a *rational* equation.

5.8 Relations in Chevalley groups

[**sec:relations**]

In this section $\mathbb{K}$ is a standard field and $\mathbb{E}$ is any field. Let $V^{\mathbb{K}}$ be a finite dimensional vector space over $\mathbb{K}$ and $L^{\mathbb{K}}$ a perfect, semisimple Lie-subalgebra of $\mathfrak{gl}(V^{\mathbb{K}})$. Let $\mathcal{C}^{\mathbb{K}} = (x_\alpha^{\mathbb{K}}, h_\beta^{\mathbb{K}} \mid \alpha \in \Phi, \beta \in \Pi)$ be a Chevalley basis for $L^{\mathbb{K}}$. Let M be an $\mathfrak{U}_{\mathbb{Z}}$ invariant lattice in $V^{\mathbb{K}}$. Let $V = \mathbb{E} \otimes_{\mathbb{Z}} M$, $L = \mathbb{E} \otimes M$, $\frac{x_\alpha^i}{i!} = 1 \otimes \frac{x_\alpha^i}{i!}^{\mathbb{K}} \in \mathrm{End}_{\mathbb{E}}(V)$. For $\alpha \in \Phi$ and $t \in \mathbb{E}$ let $\chi_\alpha(t) = \chi_\alpha^{M,\mathbb{E}}(t) = \sum_{i=0}^\infty \frac{x_\alpha^i}{i!} \in \mathrm{End}(V)$. Also let $\chi^{\mathrm{ad}}{}_\alpha(t) = \chi_\alpha^{L_{\mathbb{Z}},\mathbb{E}}(t)$. $\mathcal{X}_\alpha = \{\chi_\alpha(t) \mid t \in \mathbb{E}\}$ and $G = \langle \mathcal{X}_\alpha \mid a \in \Phi \rangle$. We view $\Lambda(\check{\Phi})$ as a subset of $(H^{\mathbb{K}})^*$. For $\mu \in \Lambda(\check{\Phi})$ let $V_\mu = \mathbb{E} \otimes M_\mu$.

In this section we establish some relations between the various $\chi_\alpha(t)$.

Lemma 5.8.1 [**automorphism**]

(a) [**a**] $\chi_\alpha(t)l\chi_\alpha(-t) = \chi^{\mathrm{ad}}(l)$ *for all* $\alpha \in \Phi$, $t \in \mathbb{E}$ *and* $l \in L$.

(b) [**b**] L *is invariant under* G *in the action of* G *on* $\mathrm{End}(V)$ *by conjugation.*

Proof: (a) If $\mathbb{E} = \mathbb{K}$ this follows from 5.6.1(b:a). Observe that the equation is rational in t and l. So 5.7.4 implies that (a) holds for arbitrary $\mathbb{E}$.

(b) By (b), L is invariant under all $\chi_\alpha(t)$ and so also under G. $\square$

For $g \in G$ and $l \in \mathrm{End}(V)$ we write $g * l$ for glg^{-1}. We have $e^x * l = e^{-x}le^x = e^{\mathrm{ad}\,x}l = \sum_{i=0}^{\infty} \frac{\mathrm{ad}\,x^i}{i!}(l) = \sum_{i=0}^{\infty} \frac{x^i}{i!} * l$ so this is consistent with our '*"-notation for the adjoint action of $\mathfrak{U}(L)$ on L. Also observe that $\chi_\alpha(t) * l = \chi_\alpha^L(l)$.

For $\alpha, \beta \in \Phi$ and $i \in \mathbb{N}$ with $\beta + i\alpha \in \Phi$ define $m_{\alpha\beta i}$ by $\frac{x_\alpha^i}{i!} * x_\beta = m_{\alpha\beta i}x_{\beta+i\alpha}$. Note that by 5.2.2, $m_{\alpha\beta i}$ is an integer.

For a, b in a groups G define $\lceil a, b\rceil = a^{-1}b^{-1}ab$.

Theorem 5.8.2 (Chevalley Commutator Formula) [**commutator formula**] *There exist integers* $c_{\alpha\beta ij}$ *such that for all* $\alpha, \beta \in \Phi$ *with* $\alpha \neq \beta$ *and all* $t, u \in \mathbb{E}$

$$\lceil \chi_\beta(u), \chi_\alpha(t)\rceil = \prod_{i,j>0} \chi_{i\alpha+j\beta}(c_{\alpha\beta ij}(-t)^i u^j)$$

where the product is taking over all postive integers i *and* j *for which* $i\alpha + j\beta$ *is a root, in any order which is non-decreasing in* $i + j$. *Moreover,*

$$\begin{aligned} C_{\alpha\beta i1} &= m_{\alpha\beta i} \\ C_{\alpha\beta 1j} &= (-1)^j m_{\beta\alpha j} \\ C_{\alpha\beta 32} &= \frac{1}{3} m_{\alpha+\beta,\alpha 2} \\ C_{\alpha\beta 23} &= -\frac{2}{3} m_{\alpha+\beta,\alpha 2} \end{aligned}$$

Proof: Both sides of the commutator formula are rational functions in t and u and so by 5.7.4 we may assume that $\mathbb{E} = \mathbb{K}$.

1° [**1**] $\quad \chi_\alpha(t) * x_\beta = \sum_{i=0}^{s_{\alpha\beta}} m_{\alpha\beta i}\, t^i\, x_{\beta+i\alpha}$

This follows immediately from the definitions.

2° [**2**] $\quad \chi_\alpha(t) * \chi_s(u) = \exp\left(\sum_{i=0}^{s_{\alpha\beta}} m_{\alpha\beta i}\, t^i u\, x_{\beta+i\alpha}\right).$

By 5.6.1(b:c) $\chi_\alpha(t) * \chi_s(u) = \chi_\alpha(t)e^{ux_\beta}\chi_\alpha^{-1} = \exp(\chi_\alpha(t)ux_\beta\chi_\alpha(t)^{-1})$ and so (2°) follows from (1°).

Let $\Psi = \langle\alpha,\beta\rangle_{\mathbb{Z}}$. Then Ψ is of type $A_1 \times A_1$, A_2, B_2 or G_2. The easiest case is A_1 and A_2. Then $\alpha+\beta$ is not a root and x_α and x_β commute. Then also $\chi_\alpha(t)$ and $\chi_\beta(u)$ commute. Actually this is just a special case of

3° [**3**] *Suppose that neither $\alpha + 2\beta$ nor $3\alpha + 2\beta$ are roots. Then*

$$\chi_\alpha(t) * \chi_s(u) = \prod_{i=0}^{s_{\alpha\beta}} \chi_{i\alpha+\beta}(m_{\alpha\beta i}t^i u).$$

By inspection of Ψ we see that the condition on α and β just says that $i\alpha + j\beta \in \Phi$ for $i, j > 0$ implies that $j = 1$. In particular, the $x_{i\alpha+\beta}$, $i \in \mathbb{N}$ commute with each other and so (3°) follows from (2°).

4° [**4**] *Suppose that $\alpha + 2\beta$ is a root and Ψ is of type B_2. Then*

$$\chi_\alpha(t) * \chi_\beta(u) = \chi_s(u)\chi_{\alpha+\beta}(m_{\alpha\beta}tu)\chi_{\alpha+2\beta}(m_{\beta\alpha 2}tu^2).$$

From (2°), $\chi_\alpha(t) * \chi_b(u) = \exp(ux_\beta + k_{\alpha\beta}tux_{\alpha+\beta})$.
Put $x = ux_\beta$ and $y = k_{\alpha\beta}tux_{\alpha+\beta}$. Then

$$[x, y] = k_{\alpha\beta}k_{\beta,\alpha+\beta}tu^2x_{\alpha+2b}.$$

So $[x, y]$ is nilpotent. Also neither $\alpha+3\beta$ nor $2\alpha+3\beta$ is a root and $[x, y]$ commutes with x and y. So by 5.6.1(c:b)

$$\begin{aligned}
\chi_\alpha(t) * \chi_\beta(u) &= e^{x+y} \\
&= e^x e^y e^{-\frac{1}{2}[x,y]} \\
&= e^{ux_\beta} e^{k_{\alpha\beta}tu} e^{-\frac{1}{2}k_{\alpha\beta}k_{\beta,\alpha+b}tu^2x_{\alpha+2\beta}} \\
&= \chi_\beta(u)\chi_{\alpha+\beta}(k_{\alpha\beta}tu)\chi_{\alpha+2\beta}(m_{\beta\alpha 2}tu^2)
\end{aligned}$$

5° [**5**] *Suppose $3\alpha + 2\beta$ is a root (and so Ψ is of type G_2). Then $\chi_\alpha(t) * \chi_\beta(u) =$*

$$\chi_\beta(u)\chi_{\alpha+\beta}(k_{\alpha\beta}\, tu)\chi_{2\alpha+\beta}(m_{\alpha\beta 2}\, t^2u)\, \chi_{3\alpha+\beta}(m_{\alpha\beta 3}\, t^3u)\, \chi_{3\alpha+2\beta}(\frac{1}{3}m_{\alpha+\beta,\beta,2}\, t^3u^2).$$

First observe that

$$\Phi \cap \mathbb{Z}^+\{\alpha, \beta\} = \{\alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$$

In particular, $s_{\alpha\beta} = 3$. Put $x = \sum_{i=0}^2 m_{\alpha\beta i}\, t^i ux_{\beta+i\alpha}$ and $y = m_{\alpha\beta 3}\, t^3u\, x_{3\alpha+\beta}$. Then

$$[x, y] = k_{\beta,3\alpha+\beta}\, m_{\alpha\beta 3}\, t^3u^2\, x_{3\alpha+2\beta}$$

Thus $[x, y]$ commutes with x and y and so by 5.6.1(c:b) $\chi_\alpha(t) * \chi_t(u) = e^{x+y} = e^x e^y e^{-\frac{1}{2}[x,y]}$. Hence

$$(*) \quad \chi_\alpha(t) * \chi_t(u) = e^x \chi_{3\alpha+b}(m_{\alpha\beta 3}\, t^3 u)\chi_{3\alpha+2\beta}(-\frac{1}{2} k_{\beta,3\alpha+\beta}\, m_{\alpha\beta 3} t^3 u^2).$$

Put $\tilde{x} = \sum_{i=0}^{1} m_{\alpha\beta i}\, t^i u x_{\beta+i\alpha}$ and $\tilde{y} = m_{\alpha\beta 2}\, t^2 u\, x_{2\alpha+\beta}$. Then $x = \tilde{x} + \tilde{y}$ and

$$[\tilde{x}, \tilde{y}] = k_{\alpha+\beta,2\alpha+b} m_{\alpha\beta 1}\, m_{\alpha\beta 2} t^3 u^2\, x_{3\alpha+2\beta}$$

So $[\tilde{x}, \tilde{y}]$ commutes with $\tilde{x}$ and $\tilde{y}$. So by 5.6.1(c:b) allows us to compute $e^x = e^{\tilde{x}} e^{\tilde{y}} e^{-\frac{1}{2}[\tilde{x}\tilde{y}]}$ and (*) gives

$$\begin{aligned}
\chi_\alpha(t) * \chi_\beta(u) \quad &= \quad \exp(\tilde{x})\, \chi_{2\alpha+\beta}(m_{\alpha\beta 2}\, t^2 u)\, \chi_{3\alpha+2\beta}(-\frac{1}{2} k_{\alpha+\beta,2\alpha+\beta}\, m_{\alpha\beta 1}\, m_{\alpha\beta 2} t^3 u^2) \\
&\quad \cdot \quad \chi_{3\alpha+b}(m_{\alpha\beta 3}\, t^3 u)\chi_{3\alpha+2\beta}(-\frac{1}{2} k_{\beta,3\alpha+\beta}\, m_{\alpha\beta 3}\, t^3 u^2) \\
(**) \qquad\qquad &= \quad \chi_\beta(u)\chi_{\alpha+\beta}(m_{\alpha\beta 1}\, tu)\chi_{3\alpha+b}(m_{\alpha\beta 3}\, t^3 u) \\
&\quad \cdot \quad \chi_{3\alpha+2\beta}(-\frac{1}{2}(k_{\alpha+\beta,2\alpha+\beta}\, m_{\alpha\beta 1}\, m_{\alpha\beta 2} + k_{\beta,3\alpha+\beta}\, m_{\alpha\beta 3})\, t^3 u^2)
\end{aligned}$$

We now compute each of the summand in the parameter for $\chi_{3\alpha+2\beta}$. Using $k_{\alpha\beta}^2 = (\pm 1)^2 = 1$ we have

$$\begin{aligned}
(***) \qquad k_{\alpha+\beta,2\alpha+\beta}\, m_{\alpha\beta 1} m_{\alpha\beta 2} \quad &= \quad k_{\alpha+\beta,2\alpha+\beta} k_{\alpha\beta} \frac{1}{2} k_{\alpha\beta} k_{\alpha,\alpha+\beta} \\
&= \quad -\frac{1}{2} k_{\alpha+\beta,\alpha} k_{\alpha+\beta,2\alpha+\beta} \\
&= \quad -m_{\alpha+\beta,\alpha,2}
\end{aligned}$$

Also $x_{\alpha+\beta}^2 = (k_{\alpha\beta}[x_\alpha, x_\beta])^2 = [x_\alpha, x_\beta]^2$ and thus

$$\begin{aligned}
2m_{\alpha+\beta,\alpha\, 2} x_{3\alpha+2\beta} \quad &= \quad [x_\alpha, x_\beta]^2 * x_\alpha \\
&= \quad [[x_\beta, x_\alpha], [[x_\beta, x_\alpha], x_\alpha]]
\end{aligned}$$

Put $u = [[x_\beta, x_\alpha], x_\alpha] \in L_{2\alpha+\beta}$. Then $[x_\beta, u] \in L_{2\alpha+2\beta} = 0$ and so

$$[x_\beta, x_\alpha], u] = [x_\beta, [x_\alpha, u]] = (x_b x_\alpha) * u.$$

Since $u = [x_\alpha, [x_\alpha, x_b]] = x_\alpha^2 * x_b$ and so

$$2m_{\alpha+\beta,\alpha,2}\, x_{3\alpha+2\beta} = (x_b x_\alpha^3) * x_\beta = 6x_b * m_{\alpha\beta 3}\, x_{3\alpha+\beta} = 6k_{\beta,3\alpha+\beta}\, m_{\alpha\beta 3}$$

Hence

$$(****) \quad k_{\beta,3\alpha+\beta}\, m_{\alpha\beta 3} = \frac{1}{3} m_{\alpha+\beta,\alpha,2}.$$

Together with (***) we see that the parameter of $\chi_{3\alpha+2\beta}$ in (**) is

$$-\frac{1}{2}(-1+\frac{1}{3})\, m_{\alpha+\beta,\alpha,2}\, t^3u^2 = \frac{1}{3}m_{\alpha+\beta}\, t^3u^2$$

Hence (5°) follows from (**).

6° [**6**] *Suppose that $2\alpha+3\beta \in \Phi$ (and so Φ is of type G_2). Then $\chi_\alpha(t) * \chi_\beta(u)$ equals*

$$\chi_\beta(u)\chi_{\alpha+\beta}(k_{\alpha\beta}\, tu)\, \chi_{\alpha+2\beta}(m_{\alpha\beta 2}\, tu^2)\, \chi_{\alpha+3\beta}(-m_{\beta\alpha 3}\, tu^3)\chi_{2\alpha+3\beta}(-\frac{2}{3}m_{\alpha+\beta,\beta,2}t^2u^3)$$

This time we have

$$\Phi \cap \mathbb{Z}^+\{\alpha,\beta\} = \{\alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}.$$

By (5°) with the roles of α and β interchanged

$$\begin{aligned}
\chi_\beta(u) * \chi_\alpha(t) &= \chi_\alpha(t)\chi_{\alpha+\beta}(k_{\beta\alpha}ut)\chi_{2\beta+\alpha}(m_{\beta\alpha 2}\, u^2t) \\
&\quad \cdot \ \chi_{3\beta+\alpha}(m_{\beta\alpha 3}\, u^3t)\chi_{3\beta+2\alpha}(\frac{1}{3}m_{\beta+\alpha,\beta,2}\, u^3t^2) \\
&= \chi_{\alpha+\beta}(k_{\beta\alpha}ut)\chi_{2\beta+\alpha}(m_{\beta\alpha 2}\, u^2t) \\
&\quad \cdot \ (\chi_\alpha(t) * \chi_{3\beta+\alpha}(m_{\beta\alpha 3}\, u^3t))\ \chi_{3\beta+2\alpha}(\frac{1}{3}m_{\beta+\alpha,\beta,2}\, u^3t^2)\, \chi_\alpha(t) \\
&= \chi_{\alpha+\beta}(k_{\beta\alpha}ut)\chi_{2\beta+\alpha}(m_{\beta\alpha 2}\, u^2t) \\
&\quad \cdot \ \chi_{3\beta+\alpha}(m_{\beta\alpha 3}\, u^3t)\, \chi_{3\beta+2\alpha}((k_{\alpha,3\beta+\alpha}\, m_{\beta\alpha 3}\, \frac{1}{3}m_{\beta+\alpha,\beta,2})\, u^3t^2)\, \chi_\alpha(t)
\end{aligned}$$

By (****) $k_{\alpha,3\beta+\alpha}\, m_{\beta\alpha 3} = \frac{1}{3}m_{\beta+\alpha,\beta,2}$ and so the parameter of $\chi_{3\beta+2\alpha}$ in the above equation is $\frac{2}{3}m_{\beta+\alpha,\beta,2}$. Multiplication with $\chi_\beta(-u)$ from the left, with $\chi_\alpha(-t)$ from the right and replacing u by $-u$ gives (6°).

7° [**7**] *Suppose that $2\alpha+\beta \in \Phi$, Φ is of type G_2 and neither $2\alpha+3\beta$ nor $3\alpha+2\beta$ are in Φ. Then*

$$\chi_\alpha(t) * \chi_\beta(u) = \chi_\beta(u)\chi_{\alpha+\beta}(k_{\alpha\beta}\, tu)\, \chi_{2\alpha+\beta}(m_{\alpha\beta 2}\, t^2u)\, \chi_{\alpha+2\beta}(m_{\beta\alpha 2}\, tu^2)$$

Observe that $\Phi \cap \mathbb{Z}^+\{\alpha,\beta\} = \{\alpha+\beta, 2\alpha+\beta, \alpha+2\beta.\}$ So $s_{\alpha\beta} = 2$. Let

$$x = \sum_{i=0,2} m_{\alpha\beta i}t^iux_{i\alpha+b} = x_\beta + m_{\alpha\beta 2}t^2ux_{2\alpha+\beta}$$

and

$$y = m_{\alpha\beta 1}tux_{\alpha+\beta} = \kappa_{\alpha\beta}tux_{\alpha+\beta}.$$

Then

$$[x,y] = k_{\beta,\alpha+\beta}k_{\alpha\beta}tu^2x_{\alpha+2\beta} = -2\frac{1}{2}k_{\beta,\alpha+b}k_{\alpha\beta}x_{\alpha+2\beta} = -2m_{\beta\alpha 2}x_{\alpha\beta}$$

So $[x, y]$ commutes with x and y. Also the two summands of x commute. Thus (7°) follows from (2°) and 5.6.1(c:b). □

Put $\Lambda = \breve{\Lambda}(\breve{\Phi})$. Let Φ° be the set of weights for H^K on $V^{\mathbb{K}}$, viewed as a subset of $\Lambda \leq \mathbb{Q}\Pi$. Let

$$\breve{\Phi}^\circ = \{\rho \in \mathbb{Q}\Phi \mid (\rho, \mu) \in \mathbb{Z}\,\forall \mu \in \Phi^\circ\}$$

and note that $\mathbb{Z}\breve{\Phi} \subseteq \breve{\Phi}^\circ$.

For $\rho \in \breve{\Phi}^\circ$ and $k \in \mathbb{E}^\sharp$ define $t_\rho(k) \in GL_{\mathbb{E}}(V)$ by

$$t_\rho(k)v = k^{(\rho,\mu)}v$$

whenever $\mu \in \Phi^\circ$ and $v \in V_\mu$. For $\alpha \in \Phi$ put $h_\alpha(k) = t_{\breve{\alpha}}(k)$. Also put

$$\omega_\alpha(k) := \chi_\alpha(k)\chi_{-\alpha}(-k^{-1})\chi_\alpha(k).$$

Lemma 5.8.3 [**chi and tensor**] *For $i = 1, 2$ let $V_i^{\mathbb{K}}$ be a standard $L^{\mathbb{K}}$ modules with a $\mathfrak{U}_{\mathbb{Z}}$-invariant lattice M_i. Then*

(a) [**a**] $\chi_\alpha^{M_1 \otimes_{\mathbb{Z}} M_2, \mathbb{E}}(t) = \chi_\alpha^{M_1,\mathbb{E}}(t) \otimes_{\mathbb{E}} \chi_\alpha^{M_2,\mathbb{E}}(t)$ *for all* $\alpha \in \Phi$, $t \in \mathbb{E}$.

(b) [**b**] $t_\rho^{M_1 \otimes_{\mathbb{Z}} M_2, \mathbb{E}}(k) = t_\rho^{M_1,\mathbb{E}}(k) \otimes_{\mathbb{E}} t_\rho^{M_2,\mathbb{E}}(k)$ *for all* $\rho \in \mathbb{Z}\breve{\Phi}$, $k \in \mathbb{E}^\sharp$.

(a) The equation is rational and so we may assume that $\mathbb{E} = \mathbb{K}$. Since x_α acts as $x_\alpha \otimes 1 + 1 \otimes x_\alpha$ on $V_1 \otimes V_2$ we have

$$\begin{aligned}
\chi_\alpha^{M_1 \otimes M_2}(t) &= e^{t(x_\alpha \otimes 1 + 1 \otimes x_\alpha)} \\
&= e^{t x_\alpha \otimes 1} e^{1 \otimes t x_\alpha} \\
&= (\chi_\alpha^{M_1}(t) \otimes 1)(1 \otimes \chi_\alpha^{M_2}(t)) \\
&= \chi_\alpha^{M_1}(t) \otimes \chi_\alpha^{M_2}(t)
\end{aligned}$$

(b) This follows from $(V_1^{\mathbb{K}})_{\mu_1} \otimes (V_2^{\mathbb{K}})_{\mu_2} \leq (V_1^{\mathbb{K}} \otimes V_2^{\mathbb{K}})_{\mu_1+\mu_2}$. □

Theorem 5.8.4 [**sl2 relations**] *Let $\alpha, \beta \in \Phi$, $\rho \in \breve{\Phi}^\circ$ and $k \in \mathbb{E}^\sharp$.*

(a) [**a**] *Let $\mu \in \Phi^\circ$ and $v \in V_\mu$. Then there exists $\tilde{v} \in V_{\omega_\alpha(\mu)}$ independent from k such that*

$$\omega_\alpha(k)v = k^{(\mu,\breve{\alpha})}\tilde{v}$$

(b) [**b**] $h_\alpha(k) = \omega_\alpha(k)\omega_\alpha(1)^{-1}$.

(c) [**c**] $\omega_\alpha(k) * h = \omega_\alpha(h)$ *for all* $h \in H$.

(d) [**d**] $\omega_\alpha(k) * x_\beta = c_{\alpha\beta}\, k^{(\beta,\breve{\alpha})}\, x_{\omega_\alpha(\beta)}$ *where $c_{\alpha\beta} = \pm 1$ is independent from $\mathbb{E}$, $V^{\mathbb{K}}$ and k. Moreover $c_{\alpha\beta} = c_{\alpha,-\beta}$ and $c_{\alpha\alpha} = -1$*

(e) [**e**] $t_\rho(k) * x_\alpha = t_\rho^{\mathrm{ad}}(k)(x_\alpha) = k^{(\rho,\alpha)}x_\alpha$

Proof: All the equations are rational. So we may assume $\mathbb{K} = \mathbb{E}$.

(a) - (c): Note that $H = H_\alpha \oplus \ker\alpha$. From $x_\alpha * \ker\alpha = 0$ we conldue that $\chi_\alpha(k)$ and $\omega_\alpha(k)$ fix each element in $\ker\alpha$. So we may assume that $L = \mathfrak{sl}(\mathbb{K}^2)$. V is a direct sum of simple L-modules and we may assume that V is simple. From 5.8.3 if (a) and (b) hold for V_1 and V_2 then also for $V_1 \otimes V_2$. So if we (a) - (c) hold for $V(\lambda)$, where $\lambda \in \check{\Lambda}(\check{\Phi})$ with $\lambda(\check{a}) = 1$, they also hold for $\bigotimes^n V(\lambda)$ and $V(n\lambda) \cong S^n(V(\lambda))$. So we may assume that $V = V(\lambda)$. By 2.5.4 $V(\lambda)$ has a basis v_0 and v_1 such that the matrices for x_α, h_α and $x_{-\alpha}$ are

$$x_\alpha \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad h_\alpha \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad x_{-\alpha} \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

In particular $x_\alpha^2 = 0$ and so $\chi_\alpha(t) = 1 + tx_\alpha$ and $\chi_{-\alpha} = 1 + tx_{-\alpha}$. Hence

$$\chi_\alpha(t) \leftrightarrow \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \qquad \chi_{-\alpha}(t) \leftrightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$$

and

$$\begin{aligned} \omega_\alpha(k) \quad &\leftrightarrow \quad \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -k^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \\ &= \quad \begin{pmatrix} 0 & k \\ -k^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \\ &= \quad \begin{pmatrix} 0 & k \\ -k^{-1} & 0 \end{pmatrix} \end{aligned}$$

The weight spaces for H on Φ° are $\mathbb{K}v_0$ with weight λ and $\mathbb{K}v_1$ with weight $-\lambda$. Thus (a) holds. Also this shows

$$h_\alpha(k) = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$

Now

$$\omega_\alpha(k)\omega_\alpha(1)^{-1} = \begin{pmatrix} 0 & k \\ -k^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$

So (b) holds. As

$$\omega_\alpha(k)h_\alpha \leftrightarrow \begin{pmatrix} 0 & -k \\ -k^{-1} & 1 \end{pmatrix} \leftrightarrow h_{-\alpha}\omega_\alpha(k)$$

(c) holds.

(d) From 5.8.1 we have $\omega_\alpha(k) * x_\alpha = \omega_\alpha^L(k)(x_\alpha)$. So from by (a) applied to the adjoint module

$$\omega_\alpha(k) * x_\beta = c_{\alpha\beta}k^{(\beta,\check{\alpha})}x_{\omega_\alpha(\beta)}$$

for some $c_{\alpha\beta} \in \mathbb{K}$. Observe that $\mathfrak{U}_{\mathbb{Z}}$ is invariant under $\chi_\alpha(1)$, $\chi_{-\alpha}(-1)$ and $\omega_\alpha(1)$. Thus $c_{\alpha\beta} \in \mathbb{Z}$.

Conjugating the equation $[x_\beta, x_{-\beta}] = h_\beta$ by $\omega_\alpha(1)$ and using (c) we get

$$c_{\alpha\beta}c_{\alpha,-\beta}h_{\omega_\alpha(\beta)} = h_{\omega_\alpha(\beta)}$$

Hence $c_{\alpha\beta}c_{\alpha,-\beta} = 1$. As $c_{\alpha\beta}$ and $c_{\alpha,-\beta}$ are both integers, this implies $c_{\alpha\beta} = c_{\alpha,-\beta} = \pm 1$. Now

As

$$\omega_\alpha(k)x_\alpha \leftrightarrow \begin{pmatrix} 0 & 0 \\ 0 & -k^{-1} \end{pmatrix} \leftrightarrow -x_{-\alpha}\omega_\alpha(k)$$

we have $\omega_\alpha(k) * x_\alpha = -x_{-\alpha}$ and so $c_{\alpha\alpha} - 1$.

(e) Let $\mu \in \Phi^\circ$ and $v \in V_\mu$. Then $x_\alpha v \in V_{\mu+\alpha}$ and so $t_\rho x_\alpha v = k^{(\rho,\mu+\alpha)}x_\alpha v$. Also $x_\alpha t_\rho(k)v = x_\alpha k^{r\rho\mu}v$. This implies (e). □

Lemma 5.8.5 [**h*chi**] *Let $\rho, \sigma \in \check{\Phi}^\circ$, $\alpha, \beta \in \Phi$, $k, u \in \mathbb{E}^\sharp$ and $t \in \mathbb{E}$. Then*

(a) [**a**] $t_\rho(k) * \chi_\alpha(t) = \chi_\alpha(k^{(\rho,\beta)}t)$.

(b) [**d**] $\omega_\alpha(k) * \chi_\beta(t) = \chi_{\omega_\alpha(\beta)}(c_{\alpha\beta}t^{(\beta,\check{\alpha})})$.

(c) [**b**] $\omega_\alpha(k) * t_\rho(u) = t_{\omega_\alpha(\rho)}(u)$.

(d) [**c**] $t_{\rho+\sigma}(k) = t_\rho(k)t_\sigma(k)$

Proof: (a) and (b). The equations are rational and we may assume $\mathbb{K} = \mathbb{E}$. From 5.8.4(d) and e we have

$$\omega_\alpha(k) * x_\beta = c_{\alpha\beta}t^{(\beta,\check{\alpha})}x_{\omega_\alpha(\beta)} \quad \text{and} \quad t_\rho(k) * x_\alpha = k^{(\rho,\check{\alpha})}x_\alpha.$$

Exponentiating (see 5.6.1(b:c)) gives (a) and (b).

(c) Let μ, v and $\tilde{v}$ be as in 5.8.4(c). Then

$$\omega_\alpha(k)t_\rho(u)v = \omega_\alpha(k)u^{(\rho,\mu)}v = c_{\alpha\beta}\, u^{(\rho,\mu)}k^{(\mu,\alpha)}\tilde{v}$$

and

$$t_{\omega_\alpha(\rho)}(u)\omega_\alpha(k)v = t_{\omega_\alpha(\rho)}(u)c_{\alpha\beta}\, k^{(\mu,\alpha)}\tilde{v} = c_{\alpha\beta}\, k^{(\mu,\alpha)}u^{(\omega_\alpha(\rho),\omega_\alpha(\mu))}\tilde{v}$$

Since ω_α is an isometry we see that (c) holds.

(d) follows immediately from the definition of $t_\rho(k)$.

Lemma 5.8.6 [**ge(m) non-degenerate**] *$\mathcal{X}_\alpha \neq \mathcal{X}_{-\alpha}$ for all $a \in \Phi$.*

Proof: Suppose that $\chi_\alpha(t) = \chi_{-\alpha}(s)$ for some $t, sin\mathbb{E}$ with $t \neq 0$. Let $\mu \in \Phi^\circ$ and $v \in V_\mu$. Note that $\frac{x_\alpha^i}{i!} v \in V_{\mu+i\alpha}$ and so the projection of $\chi_\alpha(t)v$ onto $V_{\mu+i\alpha}$ is $t^i \frac{x_\alpha^i}{i!} v$. On the otherhand $\frac{x_{-\alpha}^i}{i!} v \in V_{\mu-i\alpha}$ and so for $i \geq 1$, the projection of $\chi_{-\alpha}(s)(v)$ onto $V_{\mu+i\alpha}$ is 0. Thus $\frac{x_\alpha^i}{i!} v = 0$ for all $i \geq 1$. Thus $\mathcal{X}_\alpha = 1$ and so also $\mathcal{X}_{-\alpha} = 1$ and $w_\alpha = 1$. Thus $V_\mu = w_\alpha(V_\mu) = V_{\omega_\alpha(\mu)}$. But V is the direct sum of the V_μ and so $\mu = \omega_\alpha(\mu)$ and $(\alpha, \mu) = 0$ for all $\mu \in \Phi^\circ$. Thus implies that $h_\alpha^{\mathbb{K}} V_\mu^K = 0$ and $h_\alpha^{\mathbb{K}} V^K = 0$, a contradiction to $L^K \leq \mathfrak{sl}(V^{\mathbb{K}})$. $\square$

Chapter 6

Steinberg Groups

6.1 The Steinberg Relations

Definition 6.1.1 [**def:steinberg**] *Let $\mathbb{E}$ be a field and Φ a root system. Let G be group generated by elements $\chi_\alpha(t), \omega_\alpha(k), h_\alpha(k), w_\alpha, \alpha \in \Phi, t \in \mathbb{E}, k \in \mathbb{E}^\sharp$ such that*

(St: 1) [**1**] $\chi_\alpha(t+u) = \chi_\alpha(t)\chi_\alpha(u)$.

(St: 2) [**2**] $\lceil \chi_\beta(u), \chi_\alpha(t) \rceil = \prod_{i,j>0} \chi_{i\alpha+j\beta}(c_{\alpha\beta ij}(-t)^i u^j)$ *for $\alpha \neq \pm b$, where the $c_{\alpha\beta ij}$ and the order of multiplication are as in 5.8.2.*

(St: 3) [**3**] $\omega_\alpha(k) = \chi_\alpha(k)\chi_{-\alpha}(-k^{-1})\chi_\alpha(k)$.

(St: 4) [**5**] $w_\alpha = \omega_\alpha(1)$.

(St: 5) [**4**] $h_\alpha(k) = \omega_\alpha(k) w_\alpha^{-1}$.

(St: 6) [**6**] $w_\alpha h_\beta(k) w_\alpha^{-1} = h_{\omega_\alpha(\beta)}(k)$

(St: 7) [**7**] $w_\alpha \chi_\beta(t) w_\alpha^{-1} = \chi_{\omega_\alpha(\beta)}(c_{\alpha\beta} t)$, *where $c_{\alpha\beta}$ is as in 5.8.4.*

(St: 8) [**8**] $h_\alpha(k)\chi_\beta(t)h_\alpha(k)^{-1} = \chi_\beta(k^{(\beta,\check{\alpha})}(t))$.

(St: 9) [**9**]

- *(a)* [**1**] $h_\alpha(k)h_\alpha(k) = h_\alpha(kl)$
- *(b)* [**2**] $h_{-\alpha}(k) = h_\alpha(k)^{-1}$
- *(c)* [**3**] $h_\gamma(k) = h_\alpha(k)h_\beta(k)$, *whenever $\check{\alpha} + \check{\beta} = \check{\gamma} \in \check{\Phi}$.*

for all $\alpha, \beta \in \Phi$, $t, u \in \mathbb{E}$ and $k, l \in \mathbb{E}^\sharp$.

Then G is called a Steinberg group *of type Φ over the field $\mathbb{E}$. Let $\mathcal{X}_\alpha = \{\chi_\alpha(t) \mid t \in \mathbb{E}\}$. G is called* degenerate *if $\mathcal{X}_\alpha = \mathcal{X}_\beta$ some $\alpha \neq b \in \Phi$. The group $G_{\mathbb{E}}(\Phi)$* defined by the above generators and relations is called the *universal* Steinberg group of type Φ over $\mathbb{E}$.

In this chapter G is always a Steinberg group. The goal is to determine the structure of G. Also let $L^{\mathbb{K}}$ be a standard, perfect semisimple Lie algebra with roots sytems isomorphic to Φ and M a $\mathfrak{U}_{\mathbb{Z}}$ invariant lattice in the faithful $L^{\mathbb{K}}$ module $V^{\mathbb{K}}$.

For $\Psi \subseteq \Phi$ let $G_\Psi = \langle \mathcal{X}_\alpha \mid \alpha \in \Psi \rangle \leq G$. Define $T = \langle h_\alpha(k) \mid \alpha \in \Phi, k \in \mathbb{E}^\sharp \rangle$ and $N = \langle \omega_\alpha(k) \mid \alpha \in \Phi, k \in \ E^\sharp \rangle$. T is called the *Cartan subgroup* of G.

Lemma 6.1.2 [**positive closed**] *Let $\Psi \subseteq \Phi$ with $\mathbb{N}\Psi \cap \Phi \subseteq \Phi$ and $\Psi \cap -\Psi =$. Then there exists a regular $e \in \mathbb{Q}\Phi$ with $\Psi \subseteq \Phi^+_e$.*

Proof: Let $f := \sum \Psi$. We claim that $(f, \alpha)0$ for all $\alpha \in \Psi$. Let $\beta \in \Psi$ with $(\beta, \alpha\,) < 0$. Then $\omega_\alpha(\beta) = \beta - (\beta, \check{\alpha}\,)\alpha \in \mathbb{N}\Psi \subseteq \Psi$ and $(\beta + \omega_\alpha(\beta), \alpha\,) = 0$. Let $\Psi_1 = \{\alpha\}$, $\Psi_2 = \{\beta, \omega_\beta(\alpha) \mid (\beta, \alpha\,) < 0, \beta \in \Psi\}$, $\Psi_3 = \Psi \cap \alpha^\perp$ and $\Psi_4 = \{\beta \in \Psi \mid (\beta, \alpha\,) > 0, \omega_\beta(a) \notin \Psi\}$. Then $(\Psi_i \mid 1 \leq i \leq 4)$ is a partion of Ψ, $(\sum \Psi_1, \alpha\,) > 0$, $(\sum \Psi_2, \alpha\,) = 0$, $(\sum \Psi_3, \alpha\,) = 0$ and $(\Psi_4, \alpha\,) \geq 0$. So indeed $(f, \alpha\,) > 0$. Let $g \in \mathbb{Q}\Phi$ be regular and choose $k \in \mathbb{Q}$ with

$$0 < k < \frac{(f, \alpha\,)}{|(g, \alpha\,)|} \text{ for all } \alpha \in \Phi \setminus f^\perp.$$

Put $e = f + kg$. □

Lemma 6.1.3 [**product xi**] *Let X be a groups generated by subgroups $X_1, X_2, \ldots X_n$. Put $Z_m = \langle X_i \mid m \leq i \leq n \rangle$.*

(a) [**a**] *Suppose that $X_i X_j Z_k = X_j X_i Z_k$ for all $1 \leq i, j \leq n$ and $k = 1 + \max(i, j)$. Then for all $\pi \in \mathrm{Sym}(n)$, $X = X_{\pi(1)} X_{\pi(2)} \ldots X_{\pi(n)}$*

(b) [**b**] *Suppose that $[X_i, X_j] \leq Z_k$ for all $1 \leq i, j \leq n$ and $k = 1 + \max(i, j)$, Then*

  (a) [**a**] *$X_i X_j Z_k = X_i X_j Z_k$ for all $1 \leq i, j \leq n$ and $k = 1 + \max(i, j)$.*

  (b) [**b**] *X is nilpotent.*

  (c) [**c**] *Suppose that there exists $\mu \in \mathrm{Sym}(n)$ such that $x_i \in X_i$ and $\prod_{i=1}^n x_{\mu(i)} = 1$ implies $x_i = 1$ for all $1 \leq i \leq n$. Then for each $x \in X$ and $\pi \in \mathrm{Sym}(n)$ there exists uniquely determined $x_i \in X_i$ with $x = \prod_{i=1}^n x_{\pi(i)}$.*

Proof: (a) Since $Z_{n+1} = 1$, we have $X_i X_n = X_n X_i$ for all i In particular, $X_{n-1} X_n$ is a subgroup of X. Let $X^*_i = X_i$ for all $1 \leq i \leq n-2$ and $X^*_{n-1} = X_{n-1} X_n$. Note that for $1 \leq i < m$, $Z_m = \langle X^*_i \mid 1 \leq m-1 \rangle$.

Let $a = \pi^{-1}(n)$ and $b = \pi^{-1}(n-1)$. Since $X_i X_n = X_n X_i$ for all i,

$$\begin{aligned} X_{\pi(1)} X_{\pi(2)} \ldots X_{\pi(n)} &= X_{\pi(1)} \ldots X_{\pi(b-1)} X_{n-1} X_n X_{\pi(b+1)} \ldots X_{\pi(a-1)} X_{\pi a+1} \ldots X_n \\ &= X^*_{\pi(1)} \ldots X^*_{\pi(b-1)} X^*_{n-1} X^*_{\pi(b+1)} \ldots X^*_{\pi(a-1)} X^*_{\pi(a+1)} \ldots X^*_{\pi(n-1)} \end{aligned}$$

By induction on n the last product equals X.

(b:a) is obvious.

(b:b) We have $[X_i, X_n] \leq Z_{n+1} = 1$ and so $X_n \leq Z(X)$. By induction on n, X/X_n is nilpotent and so also X is nilpotent.

(b:c) Put $\overline{X} = X/X_n$ and $a = \mu^{-1}(n)$. For $1 \leq i \leq n-1$ let $x_i \in X_i$ with $\prod_{a\neq i=1}^{n} \overline{x}_{\mu(i)} = 1$. Let $x_n = (\prod_{a\neq i=1}^{n} x_{\mu(i)})^{-1}$. Then $x_n \in X_n \leq Z(X)$ and so $\prod_{i=1}^{n} x_{\mu(i)} = 1$. By assumptions this implies $x_i = 1$ for all $1 \leq i \leq m$ and in particular, $\overline{x}_i = 1$. This also shows that for $1 \leq i < n$, the map $X_i \to \overline{X}_i, z \to \overline{z}$ is a bijection.

Let $x \in X$ and $\pi \in \mathrm{Sym}(n)$. Put $b = \pi^{-1}(n)$. By induction there exists uniquely determined $\overline{x}_i \in \overline{X}_i$, $1 \leq i < n-1$ with $\overline{x} = \prod_{b\neq i}^{n} \overline{x}_{\pi(i)}$. Let x_i be the unique element in x_i with $x_i X_n = \overline{x}_i$ and put $x_n = x \cdot (\prod_{a\neq i=1}^{n} x_{\mu(i)})^{-1}$. Then $x_n \in X_n \leq Z(X)$, $x = \prod_{i=1}^{n} x_{\pi i}$ and the x_i are the uniquely determined. $\square$

Lemma 6.1.4 [**closed implies nilpotent**] *Let $\Psi \subseteq \Phi$ with $\mathbb{N}\Psi \cap \Phi \subseteq \Phi$ and $\Psi \cap -\Psi = \emptyset$. Then*

(a) [**a**] *$G_\Psi = \prod_{\alpha\in\Psi} \mathcal{X}_\alpha$, where the product is taken in any given order.*

(b) [**b**] *G_Ψ is nilpotent.*

Proof: By 6.1.2 we may assume that $\Psi \subseteq \Phi^+$. Let $\Psi = \{a_1, \dots \alpha_n\}$ with $\mathrm{ht}\,\alpha_1 \leq \mathrm{ht}\,\alpha_2 \leq \dots \leq \mathrm{ht}\,\alpha_n$. Put $X_i = \mathcal{X}_{\alpha_i}$ and $Z_m = \langle X_i \mid 1 \leq i \leq n\rangle$. Let $1 \leq i, j \leq n$ and $k = \max(i, j)+1$. If $n_i, n_j\mathbb{Z}^+$ with $n_i\alpha_i + n_j\alpha_j \in \Phi$, then $\mathrm{ht}(n_i\alpha_i + n + j\alpha_j) = n_i \,\mathrm{ht}\, n\rangle\alpha_i + n_j\alpha_j \geq i + j \geq k$. Thus by Relation 6.1.1(2), $[X_i, X_j] \leq Z_k$.

The lemma now follows from 6.1.3 $\square$

Lemma 6.1.5 [**nilpotent**] *Let X be a nilpotent group and $Y \leq X$ with $X = \langle Y^X\rangle := \langle x^{-1}Yx \mid x \in X\rangle$. Then $X = Y = Y^x$ for all $x \in X$*

Proof: If $X = 1$, the lemma holds. By induction on the nilpotency class of X we have $X/Z(X) = YZ(X)/Z(X)$ and so $X = YZ(X)$. Hence Y is normal in X and so $Y = \langle Y^X\rangle = X$. $\square$

Lemma 6.1.6 [**chi-a**] *Let $k \in \mathbb{E}^\sharp$. Then $\chi_a(k) = \chi_\alpha(k^{-1})\chi_\alpha(-k^{-1})w_\alpha\chi_\alpha(k^{-1})$.*

Proof: By 6.1.1(3), applied to $-k^{-1}$

$$\omega_\alpha(-k^{-1}) = \chi_\alpha(-k^{-1})\chi_{-\alpha}(k)\chi_\alpha(-k^{-1})$$

and so using 6.1.1(1)

$$\chi_{-\alpha}(k) = \chi_\alpha(k^{-1})\omega_\alpha(-k^{-1})\chi_\alpha(k^{-1})$$

and thus by 6.1.1(5)

$$\chi_{-\alpha}(k) = \chi_\alpha(k^{-1})h_\alpha(-k^{-1})w_\alpha\chi_\alpha(k^{-1}).$$

□

Lemma 6.1.7 [**char degenerate**] *Let $\alpha \in \Phi$. We say that α is* degenerate *with respect to G if $\mathcal{X}_\alpha = \mathcal{X}_\beta$ for some $\alpha \neq \beta \in \Phi$. Then the following statements are equivalent*

(a) [**a**] *α is degenerate with respect to G.*

(b) [**b**] *$\langle \mathcal{X}_\alpha, \mathcal{X}_\alpha^{\chi_{-\alpha}(k)}\rangle$ is nilpotent for some $k \in \mathbb{E}^\sharp$.*

(c) [**d**] *$\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha}\rangle$ is nilpotent.*

(d) [**c**] *$\mathcal{X}_\alpha = \mathcal{X}_{-\alpha}$.*

Proof: (a)$\Longrightarrow$ (b): Let $\alpha \neq \beta \in \Phi$ with $\mathcal{X}_\alpha = \mathcal{X}_\beta$. Let $\Psi = \mathbb{N}\{-\alpha, \beta\} \cap \Phi$. Then 6.1.4 implies that G_Ψ is nilpotent. Since $\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha}\rangle = \langle \mathcal{X}_\beta, \mathcal{X}_{-\alpha}\rangle \leq G_\Psi$ we see that (b) holds for all $k \in \mathbb{E}^\sharp$.

(b)$\Longrightarrow$ (d): By 6.1.1(8), $h_\alpha(-k^{-1})$ normalizes $\mathcal{X}_\alpha$. Also by 6.1.1(7), $\mathcal{X}_\alpha^{w_\alpha} = \mathcal{X}_{-\alpha}$ and so by 6.1.6

$$\mathcal{X}_\alpha^{\chi_{-\alpha}(k)} = \mathcal{X}_\alpha^{\chi_\alpha(k^{-1})\chi_\alpha(-k^{-1})w_\alpha\chi_\alpha(k^{-1}} = \mathcal{X}_{-\alpha}^{\chi_\alpha(k^{-1})}$$

Thus $\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha}^{\chi_\alpha(k^{-1})}\rangle$ is nilpotent and conjugation by $\chi_\alpha(k^{-1})^{-1}$ gives that

$$\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha}\rangle \quad \text{is nilpotent}$$

Thus (c) hold.

(c)$\Longrightarrow$ (d): By 6.1.1(3), $w_\alpha \in< \mathcal{X}_\alpha, \mathcal{X}_{-\alpha}\rangle$ and so $\mathcal{X}_\alpha$ and $\mathcal{X}_\alpha$ are conjugate in $\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha}\rangle$. Thus by 6.1.5, $\mathcal{X}_\alpha = \mathcal{X}_{-\alpha}$.

(d)$\Longrightarrow$ (a): Obvious. □

Lemma 6.1.8 [**chia neq chib**] *Let $V^{\mathbb{K}}$ be a faithful L^K-module and M a $\mathfrak{U}_{\mathbb{Z}}$ invarinat lattice in V^K. $G_{\mathbb{E}}(M)$ is a non-denerate Steinberg group. In particular, the universal Steinberg groups $G_{\mathbb{E}}(\Phi)$ is non degenerate.*

Proof: By the results in section 5.8 $G_{\mathbb{E}}(M)$ is a quotient of $G_{\mathbb{E}}(\Phi)$. By 5.8.6 $\mathcal{X}_\alpha^{\mathbb{E},M} \neq \mathcal{X}_{-\alpha}^{\mathbb{E},M}$. So by 6.1.7 $G_{\mathbb{E}}(M)$ is non-degenerate. Hence also $G_{\mathbb{E}}(\Phi)$ is non-degenerate. □

6.2 The degenerate Steinberg groups

Proposition 6.2.1 [**degenerate steinberg**] *Let G be a non-trivial degenerate Steinberg groups of type Φ over the field $\mathbb{E}$. Suppose that Φ is connected. Then $|\mathbb{E}| = |G| \in \{2, 3\}$ and one of the following holds.*

1. [**a**] $|\mathbb{E}| = 2$, $\Phi \cong A_1$ *or* B_2 *and* $\chi_\alpha(1) \neq 1 \neq w_\alpha$ *for all* $a \in \Phi$
2. [**b**] $|\mathbb{E}| = 2$, $\Phi \cong G_2$, $\chi_\alpha(1) \neq 1 \neq w_\alpha$ *all short roots* α, *but* $\chi_\alpha(1) = 1 = w_\alpha$ *for all the long roots* $\alpha \in \Phi$.
3. [**c**] $|\mathbb{E}| = 3$, $\Phi \cong A_1$, $\chi_\alpha(1) = \chi_{-\alpha}(-1) \neq 1$ *and* $w_\alpha(k) = h_\alpha(k) = 1$ *for all* $k \in \mathbb{K}^\sharp$, $\alpha \in \Phi$.

Proof: By 6.1.7 $\mathcal{X}_\alpha = \mathcal{X}_{-\alpha}$ for some $\alpha \in \Phi$. Let $r = (\alpha, \alpha)$. Since Φ is connected, W_Φ acts tranistively on Φ_t, the set of roots of length r. So 6.1.1(7) implies that $\mathcal{X}_\alpha = \mathcal{X}_{-\alpha}$ for all $\alpha \in \Phi_r$. Let $\alpha \in \Phi_r$ and put $R_\alpha = \mathcal{X}_\alpha = \langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha} \rangle$. Then R_α is abelian. Since $h_\alpha(k) \in R_\alpha$, we conclude that $\lceil \mathcal{X}_\alpha, h_\alpha(k) \rceil = 1$. By 6.1.1(8) ,

1° [**1**] $$\lceil h_\alpha(k), \chi_\beta(t) \rceil = \chi_\beta(\,(k^{(\beta,\check{\alpha})} - 1)\,t)$$

and so either $\mathcal{X}_\alpha = 1$ or $k^2 = 1$ for all $k \in \mathbb{E}^\sharp$.

By 5.8.4(d), $c_{\alpha\alpha} = -1$ and so by 6.1.1(7) (and since R_α is abelian)

$$\chi_\alpha(t) = w_\alpha \chi_\alpha(t) w_\alpha^{-1} = \chi_{-\alpha}(-t)$$

Let $\epsilon \in \pm 1$. Then $w_\alpha(\epsilon) = \chi_\alpha(\epsilon)\chi_{-\alpha}(\epsilon)\chi_\alpha(\epsilon) = \chi_\alpha(3\epsilon)$ and $h_\alpha(-1) = \chi_\alpha(3)\chi_\alpha(3)^{-1} = \chi_\alpha(6)$

Suppose that $\mathcal{X}_\alpha \neq 1$. Then $|\mathbb{E}| = 2$ or $|\mathbb{E}| = 3$. So we conclude

2° [**2**] *$h_\alpha(k) = 1$ for all $k \in \mathbb{E}^\sharp$ and one the following holds*

1. [**2:a**] $\mathcal{X}_\alpha = 1$ *and* $w_\alpha = 1$.
2. [**2:b**] $\mathbb{E} = 2$, $\chi_\alpha(1) = \chi_{-\alpha}(1) = w_\alpha \neq 1$.
3. [**2:c**] $\mathbb{E} = 3$, $\chi_\alpha(1) = \chi_{-\alpha}(-1) \neq 1$ *and* $w_\alpha = 1$.

In particular,

3° [**3**] *If $|\Pi| = 1$ then (1) or (3) holds.*

So suppose from now on that $|\Pi| > 1$. Next we prove

4° [**4**] $$\chi_{\alpha+\beta}(1) = \prod_{ij>1} \chi_{i\alpha+j\beta}(\pm c_{\alpha\beta ij})$$ *for all $\beta \in \Phi$ with $\beta \in \Phi$ with $\alpha + \beta \in \Phi$ and $-\alpha + \beta \notin \Phi$*

Since $-\alpha+\beta$ is not a root, 6.1.1(2) gives $\lceil \mathcal{X}_{-\alpha}, \mathcal{X}_\beta \rceil = 1$. Since $\mathcal{X}_\alpha = \mathcal{X}_{-\alpha}$ another application of 6.1.1(2) gives

$$1 = \lceil \chi_\alpha(1), \chi_\beta(1) \rceil = \prod_{i,j>0} \chi_{i\alpha+j\beta}(\pm c_{\alpha\beta ij})$$

Also $c_{\alpha\beta 11} = k_{\alpha\beta} = \pm(r_{\alpha\beta}+1) = \pm 1$ and so (4°) holds.

5° [**6**] *If $\langle \alpha, \beta \rangle_{\mathbb{Z}}$ is of type A_2 for some $\beta \in \Phi$, then $\mathcal{X}_\alpha = 1$. Inparticular, Φ has two root lengths.*

Without loss $\alpha+\beta \in \Phi$ and so $-\alpha+\beta \notin \Phi$. So we can apply (4°). But the product in (4°) has no factors and so $\chi_{\alpha+b}(1) = 1$. Since $\alpha+\beta$ and α have equal length we get $\mathcal{X}_\alpha = 1$.

Since Φ is connected all roots in Π of the same length as α are conjugate to α under W_Φ. So we can choose $\alpha \in \Phi$ and such that there exists $\beta \in \Pi$ with $(\alpha, \beta) \neq 0$ and α and β of different lengths.

6° [**5**] *Suppose $|\mathbb{E}| = 3$ and either α is long or $|(\beta, \check{\alpha})|$ is odd. Then $\mathcal{X}_\beta = 1$.*

By (2°), $h_\alpha(-1) = 1$. If α is long, $(\beta, \check{\alpha}) = 1$. So in any case $(\beta, \check{\alpha})$ is odd and and (6°) follows from (1°) applied with $k = -1$.

7° [**7**] *Suppose $\langle \alpha, \beta \rangle_{\mathbb{Z}}$ is of type B_2. Then (1) holds.*

Note that $\alpha+\beta$ is short and the the product on the right side of (4°) has just one factor, namely $\chi_\delta(\pm 1)$, where δ is long (and $\delta \in \{\alpha+2\beta, 2\alpha+\beta\}$.) Thus $\mathcal{X}_{\alpha+\beta} = \mathcal{X}_\delta$. Hence both $\alpha+\beta$ and δ are degenerate and it follows easily that $\mathcal{X}_\phi = \mathcal{X}_\alpha$ for all $\phi \in \langle \alpha, \beta \rangle$. Inparticular $\mathcal{X}_\alpha \neq 1$ and $|\mathbb{E}| \in 2, 3$. Suppose that $|\mathbb{E}| = 3$. We may assume that α is long. Then (6°) implies $\mathcal{X}_\beta = 1$ and so $\mathcal{X}_\phi = 1$ for all $\phi \in \Phi$, a contradiction.

So $|\mathbb{E}| = 2$. Suppose that $|\Pi| \geq 3$. Then there exists $\alpha_1, \alpha_2 \in \Pi$ such that $\langle \alpha_1, \alpha_2 \rangle_{\mathbb{Z}}$ is of type A_2 and so by (5°), $\mathcal{X}_{\alpha_1} = 1$, a contradiction. Thus $|\Pi| = 2$ and (1) holds.

8° [**8**] *Suppose $\langle \alpha, \beta \rangle_{\mathbb{Z}}$ is of type G_2. Then (2) holds.*

Suppose that $|\mathbb{E}| = 3$. Then $(\alpha, \beta) \in \{-1, -3\}$ and (6°) implies $\mathcal{X}_\beta = 1$. Thus also β is degenerate and by symmetry, $\mathcal{X}_\alpha = 1$, a contradiction.

Thus $|\mathbb{E}| \neq 3$.

Suppose that $\mathcal{X}_\alpha \neq 1$. Then (2°) implies $|\mathbb{E}| = 2$ The long roots in G_2 form a $\mathbb{Z}$-closed root system of type A_2. So if α is long, (5°) shows that $\mathcal{X}_\alpha = 1$, a contradiction. Thus α is short. Pick a short root δ with $(\alpha, \delta) > 0$. Then $\alpha+\delta$ is a long root and from 6.1.1(2),

$$\lceil \chi_\alpha(1), \chi_\delta(1) \rceil = \chi_{\alpha+\delta}(\pm 3) = \chi_{\alpha+\delta}(1).$$

As $\chi_\alpha(1) = \chi_{-\alpha}(1)$ and $\chi_\delta(1) = \chi_{-\delta}(1)$ we conclude that

$$\chi_{\alpha+\delta}(1) = \chi_{-\alpha-\delta}(1).$$

So $\alpha+\delta$ is degenerate long root. By (5°), $\mathcal{X}_{\alpha+\delta} = 1$. Thus $\mathcal{X}_\mu = 1$ for all the long roots μ. Hence $w_\mu = 1$ and so $\mathcal{X}_{-\alpha} = \mathcal{X}_\alpha = w_\mu \mathcal{X}_\alpha w_\mu^{-1} = Chi_{\omega_\mu(\alpha)} = \mathcal{X}_{-\omega_\mu(\alpha)}$. Thus $Chi_\alpha = \mathcal{X}_\rho$ for all short roots ρ and so $G = \mathcal{X}_\alpha$. Hence (2) holds in this case.

Suppose finally that $\mathcal{X}_\alpha = 1$, then $w_\alpha = 1$ and so $\mathcal{X}_\beta = w_\alpha \mathcal{X}_\beta w_\alpha^{-1} = Chi_{\omega_\alpha(\beta)}$. Thus also β is degenerate and $\mathcal{X}_\beta \neq 1$ and we are done by the previous case. □

6.3 Generators and Relations for Weyl Groups

Lemma 6.3.1 [**generators for w**] *Let Φ be a root system. Let R the group defined by the generators $r_\alpha, \alpha \in \Phi$ and relations $r_\alpha^2 = 1$ and $r_\alpha r_\beta r_\alpha^{-1} = r_{\omega_\alpha(b)}$, $\alpha, b \in \Phi$. Then there exists an isomorphism $f : R \to W(\Phi)$ with $f(r_\alpha) = \omega_\alpha$ for all $\alpha \in \Phi$.*

Proof: Since $W(\Phi)$ is generated by the $\omega_\alpha, a \in \Phi$ and the ω_α fulfill the above relations, there exists an onto homomorphism $f : R \to W(\Phi)$ with $f(r_\alpha) = \omega_\alpha$. In particular, there exists an action of R on Φ via $g(\alpha) = f(g)(\alpha)$ for all $g \in R, \alpha \in \Phi$. Then $r_\alpha r_\beta r_\alpha^{-1} = r_{r_\alpha(\beta)}$ for all $\alpha\beta \in \phi$. The set of all $g \in R$ with $g r_\beta g^{-1} = r_{g(\beta)}$ cleary forms a su group of R and so

1° [**1**] *$g r_\beta g^{-1} = r_{g(\beta)}$ for all $g \in R$.*

Let R_0 be the subgroup of R generated by the $r_\alpha, a \in \Pi$. Then $f(R_0) = W(\Pi) = W(\Phi)$. Let $\delta \in \Phi$ then $\delta = w(\alpha)$ for some $w \in W(\Phi)$ and $\alpha \in \Pi$. Pick $g \in R_0$ with $f(g) = w$. Then $g(\alpha) = \delta$ and so by (1°) $r_\delta = r_{g(\alpha)} = g r_\alpha g^{-1} \in R_0$. Thus

2° [**2**] $R = R_0$

To show that f is one to one let $g \in R$ with $f(g) = 1$. We need to show that $g = 1$. By (2°) we can choose $\alpha_1, \alpha_2, \ldots \alpha_k \in \Pi$ with

$$g = r_{\alpha_1} r_{\alpha_2} \ldots r_{\alpha_k}$$

If $k = 0$, $g = 1$. So suppose that $k > 0$. From $f(g) = 1$ we have

$$1 = \omega_{\alpha_1} \omega_{\alpha_2} \ldots \omega_{\alpha_k}$$

Since $l(1) = 0 < k$ we conclude that there exists $j < k$ with

$$l(\omega_{\alpha_1} \omega_{\alpha_2} \ldots \omega_{\alpha_{j+1}}) = l(\omega_{\alpha_1} \omega_{\alpha_2} \ldots \omega_{\alpha_j}) - 1$$

and so by 3.4.4(c) $\omega_{\alpha_1} \omega_{\alpha_2} \ldots \omega_{\alpha_j}(\alpha_{j+1}) \in \Phi^-$. Hence ther e exits $i \leq j$ such that

$$\begin{aligned} \omega_{\alpha_{i+1}} \dots \omega_{\alpha_j}(\alpha_{j+1}) &\in \Phi^+ \\ \omega_{\alpha_i}\omega_{\alpha_{i+1}} \dots \omega_{\alpha_j}(\alpha_{j+1}) &\in \Phi^- \end{aligned}$$

Since $\alpha_i \in \Pi$, 3.3.2(e) implies that $\omega_{\alpha_{i+1}} \dots \omega_{\alpha_j}(\alpha_{j+1}) = \alpha_i$. From (1°)

$$(r_{\alpha_{i+1}} \dots r_{\alpha_j})r_{\alpha_{j+1}}(r_{\alpha_{i+1}} \dots r_{\alpha_j})^{-1} = r_{\alpha_i}$$

and so

$$r_{\alpha_{i+1}} \dots r_{\alpha_j} r_{\alpha_{j+1}} = r_{\alpha_i} r_{\alpha_{i+1}} \dots r_{\alpha_j}$$

But this implies

$$g = (r_{\alpha_1} r_{\alpha_2} \dots r_{\alpha_i})(r_{\alpha_i} r_{\alpha_{i+1}} \dots r_{\alpha_j})(r_{\alpha_{j+2}} \dots r_{\alpha_k})$$

Since $r_{\alpha_i}^2 = 1$, induction on k gives $g = 1$. □

6.4 The structure of non-degenerate Steinberg groups

In this section G is a non-degenerate Steinberg Group. Let $T_{\mathbb{Z}} = \langle h_\alpha(-1) \mid \alpha \in \Phi \rangle$ and $N_{\mathbb{Z}} = \langle \omega_\alpha \mid \alpha \in \Phi \rangle$.

Lemma 6.4.1 [**n**]

(a) [**a**] *T is normal in N and $T_{\mathbb{Z}}$ is normal in $N_{\mathbb{Z}}$.*

(b) [**b**] *There exists a unique homomorphism $f : W \to N_{\mathbb{Z}}/T_{\mathbb{Z}}$ with $f(\omega_\alpha) = \omega_\alpha(k)T$ for all $\alpha \in \Phi$ and $k \in \mathbb{E}^\sharp$.*

(c) [**c**] *f is an isomorphism and $N_{\mathbb{Z}}/T_{\mathbb{Z}} \cong N/T \cong W$.*

(d) [**e**] *$N = N_{\mathbb{Z}}T$ and $N_{\mathbb{Z}} \cap T = T_{\mathbb{Z}}$.*

(e) [**d**] *There exists a well defined action of $N \times \Phi \to \Phi, (n, \phi) \to n(\phi)$ with $w_\beta(\alpha) = \omega_\beta(\alpha)$ for all $\alpha, \beta \in \Phi$. Moreover, $C_N(\Phi) = T$ and $n\mathcal{X}_\alpha n^{-1} = \mathcal{X}_{n(\alpha)}$ for all $n \in N, \alpha \in \Phi$.*

Proof: (a) By Relation 6.1.1(6), each w_α normalizes T and $T_{\mathbb{Z}}$. By 6.1.1(5),

$$(*) \quad \omega_\alpha(t) = h_\alpha(t)w_\alpha \in Tw_\alpha$$

and so (a) holds.

(b) From (a) and (*), $\omega_\alpha(t)T = w_\alpha T$ and so $N = N_{\mathbb{Z}}T$. Also by Relation 6.1.1(3) and 1 , $w_\alpha^{-1} = \omega_\alpha(-1)$ and so $h_\alpha(-1) = \omega_\alpha(-1)w_\alpha^{-1} = w_\alpha^{-2} \in N_{\mathbb{Z}}$. Hence $T_{\mathbb{Z}} \leq N_{\mathbb{Z}}$. For $n \in N_{\mathbb{Z}}$ let $\overline{n} = nT_{\mathbb{Z}}$. We will verify that the $\overline{w}_\alpha$ fullfill the relations in 6.3.1. First observe that

$$(**)\quad \overline{w}_\alpha^2 = \overline{h_\alpha(-1)}^{-1} = 1$$

Let $c = c_{\alpha\beta} = c_{\alpha,-\beta} = \pm 1$(see 5.8.4(d)). Then from Relations 6.1.1(3) and (7)

$$w_\alpha w_\beta w_\alpha^{-1} = w_\alpha \chi_\beta(1)\chi_{-\beta}(-1)\chi_\beta(1)w_\alpha^{-1} = \chi_{\omega_\alpha(\beta)}(c)\chi_{-\omega_\alpha(\beta)}(-c)\chi_{\omega_\alpha(\beta)}(c) = \omega_{\omega_\alpha(\beta)}(c).$$

From (*), $\omega_{\omega_\alpha(\beta)}(c) = w_{\omega_\alpha(b)}$ and so

$$(***)\quad \overline{w}_\alpha \overline{w}_\beta \overline{w}_\alpha^{-1} = \overline{w}_{\omega_\alpha(\beta)}.$$

From (**), (***) and 6.3.1 we see that (b) holds.

(c) Since $N_{\mathbb{Z}}$ is generated by the ω_α, f is onto.

Let $I = \{\not\asymp_\alpha \mid \alpha \in \Phi\}$. By 6.1.8 and 6.1.1(7), (W, Φ) and $(N/C_N(I), I)$ are isomorphic permutation groups. Hence $W \cong N/C_N(I)$. Also by 6.1.1(8), $T \leq C_N(I)$ and so

$$|W| \leq |f(W)| = |N_{\mathbb{Z}}/T_{\mathbb{Z}}| \leq |N_{\mathbb{Z}}/N_{\mathbb{Z}} \cap T| = |N_{\mathbb{Z}}T/T| = |N/T| \leq |N/C_N(I)|.$$

Thus f is one to one and $N_{\mathbb{Z}} \cap T = T_{\mathbb{Z}}$. So (b) and (d) holds.

(e) follows from (b) and (c) □

For a subset Σ of Π, let $\Phi_\Sigma := \langle \Sigma \rangle$ be the root subsystem of Φ generated by Σ. Let $U_\Sigma = G_{\Phi^+ \setminus \Phi_\Sigma}$, $K_\Sigma = G_{\Phi_\Sigma}$, and $P_\Sigma = \langle U_\Sigma, K_\Sigma, T \rangle$. Put $U = G_{\Phi_+} = U_\emptyset$ and $B = P_\emptyset = \langle U, T \rangle$.

Lemma 6.4.2 [**psigma**] *Let $\Sigma \subseteq \Pi$.*

(a) [**a**] *T normalizes U_Σ and K_Σ.*

(b) [**b**] *U_Σ is normal in P_Σ.*

(c) [**e**] *$P_\Sigma = U_\Sigma K_\Sigma T$*

Proof: (a) Follows from Relation 6.1.1(8)

(b) Let $\Delta = \Phi^+ \setminus \Phi_\Sigma$. Let $\delta \in \Delta$ and $\sigma \in \Phi_\Sigma$. Let $\mu = \sum_{\alpha \in \Pi \setminus \Sigma} \alpha^*$. Since $\delta \notin \Phi_\Sigma$ and δ is positive, $(\mu, \delta\,) > 0$. Let $i, j \in \mathbb{Z}^+$, with $i\delta + j\sigma \in \Phi$. Then $(\alpha, i\delta + j\sigma\,) = i(\alpha, \delta\,) > 0$ and so $i\delta + j\sigma \in\in \Delta$. Relation 6.1.1(2) implies now that K_Σ normalizes U_Σ. The same is true for T and U_Σ and so (b) holds.

(c) Follows from (a) and (b). □

Lemma 6.4.3 [**double cosets**] *Let $n \in N$, $\alpha \in \Phi^+$ and $k \in \mathbb{E}^\sharp$*

(a) [**a**] *$Bw_\alpha^{-1} = Bw_\alpha = Bw_\alpha(k)$ and $w_a^{-1}B = w_\alpha B = \omega_\alpha(k)B$.*

(b) [**b**] $w_\alpha^2 \in B$.

(c) [**c**] $n\mathcal{X}_\alpha^\sharp w_\alpha \subseteq Bnw_\alpha B \cdot Bw_\alpha B$.

(d) [**d**] $w_\alpha^{-1}\mathcal{X}_\alpha^\sharp w_\alpha \subseteq Bw_\alpha B$.

Proof: (a) follows from $w_\alpha T = \omega_\alpha(k)T$, $T \trianglelefteq N$ and $T \leq B$.
(b) follows from (a).
(c) Put $c = c_{\alpha\alpha} = c^{-1}$

$$\begin{array}{rcl} n\chi_\alpha(t)w_\alpha & = & nw_\alpha \cdot w_\alpha^{-1}\chi_\alpha(1)w_\alpha \\ & = & nw_\alpha \cdot \chi_{-\alpha}(ct) \\ & = & nw_\alpha\chi_\alpha((-ct)^{-1})\omega_\alpha((-ct)^{-1})))\chi_\alpha((-ct)^{-1})) \\ & \in & Bnw_\alpha B \cdot Bw_\alpha B \end{array}$$

Lemma 6.4.4 [**double cosets for sl2**] *Let $a \in \Pi$. Then $P_\alpha = B \cup Bw_\alpha B$.*

Proof: By 6.1.1(3), $w_\alpha \in P_\alpha$ and so by 6.4.2, w_α normalizes U_a. By 6.4.1(a), w_α normalizes T. Also from 6.4.1, $B = UK_\emptyset T = UT = U_\alpha\mathcal{X}_\alpha T$ and so using 6.4.3(d) we compute

$$\begin{array}{rcl} w_\alpha^{-1}Bw_\alpha & = & w_\alpha^{-1}U_\alpha w_\alpha \cdot w_\alpha\mathcal{X}_\alpha w_\alpha^{-1}w_\alpha T w_\alpha^{-1} \\ & \subseteq & U_\alpha(B \cup Bw_\alpha B)T \\ & = & B \cup Bw_\alpha B. \end{array}$$

Thus $Bw_\alpha Bw_\alpha B \subseteq B \cup Bw_\alpha B$. Put $A = B \cup Bw_\alpha B$. It follows that $AA \subseteq A$. Also $A = A^{-1}$ and so A is a subgroup of P_α and $A = \langle B, w_\alpha\rangle$. Since $\mathcal{X}_{-\alpha} = \mathcal{X}_{w_\alpha(a)} = w_\alpha\mathcal{X}_\alpha w_\alpha^{-1} \in \langle B, w_\alpha\rangle$ we conclude that $A = P_\alpha$. $\square$

Lemma 6.4.5 [**double coset multiplication**] *Let $\alpha \in Pi$ and $n \in N$. Then*

(a) [**a**] *If $n(\alpha) \in \Phi^+$, then $BnB \cdot Bw_\alpha B = Bnw_\alpha B$.*

(b) [**c**] *If $n(\alpha) \in \Phi^-$, then $BnB \cdot Bw_\alpha B = Bnw_\alpha B \cup BnB$*

(c) [**b**] $BnB \cdot Bw_\alpha B \subseteq Bnw_\alpha B \cup BnB$.

Proof: (a)

$$\begin{array}{rcl} BnB \cdot Bw_\alpha B & = & Bn\mathcal{X}_\alpha U_\alpha T w_\alpha B \\ & = & Bn\mathcal{X}_\alpha n^{-1} \cdot nw_\alpha w_\alpha^{-1}U_\alpha w_\alpha \cdot w_\alpha^{-1}Tw_\alpha \cdot B \\ & = & B\mathcal{X}_{n(\alpha)} \cdot nw_\alpha U_\alpha TB \\ & = & Bnw_\alpha B \end{array}$$

(c) Put $m = nw_\alpha$. Then $m(\alpha) = n(-\alpha) = -n(-\alpha) \in \Phi^+$. Thus by (a)

$$BmB \cdot Bw_\alpha B = Bmw_\alpha B = BnB$$

and so using 6.4.4

$$\begin{aligned} BnB \cdot Bw_\alpha B &= BmB \cdot Bw_a B \cdot Bw_a B \\ &\subseteq BmB \cdot (B \cup Bw_a B) \\ &= BmB \cup BmBw_a B \\ &= Bnw_\alpha \cup BnB \end{aligned}$$

Clearly $Bnw_\alpha B \subseteq BnB \cdot Bw_\alpha B$. From 6.4.3(c),

$$n\mathcal{X}^\sharp_\alpha w_\alpha \subseteq Bnw_\alpha B \cdot Bw_\alpha B = BnB$$

Note that the left hand side of this equation is contained in $BnB \cdot Bw_\alpha B$. Also $BnB \cdot Bw_\alpha B$ is the union of double cosets. Since double cosets are either equal or disjoint we get $BnB \subseteq BnB \cdot Bw_\alpha B$. Thus (c) holds.

(b) follows from (a) and (c). $\square$

Corollary 6.4.6 [**more double cosets**] *Let $\alpha \in \Pi$ and $n \in N$. Then $Bw_\alpha B \cdot BnB \subseteq Bw_\alpha nB \cup BnB$.*

Proof: $(Bw_\alpha B \cdot BnB)^{-1} = Bn^{-1}B \cdot Bw_\alpha B \subseteq Bn^{-1}w_\alpha B \cup Bn^{-1}B = (Bw_\alpha nB \cup BnB)^{-1}$. $\square$

Lemma 6.4.7 [**generators**] *Let X be a group, $Y \subseteq X$ with $X = \langle Y \rangle$ and $Y = Y^{-1}$ and $\emptyset \neq Z \subseteq X$ with $ZY \subseteq Z$. Then $Z = X$.*

Proof: Let $y \in Y$. Then $Zy \subset Z$ and $Zy^{-1} \subseteq Z$. Thus $Zy = Z$. Let $D = \{d \in X \mid Zd = Z\}$. Clearly D is a subgroup of X. Also $Y \subseteq D$ and so $X = \langle Y \rangle = D$. Let $z \in Z$. Then $X = zX = zD \subseteq Z$ and $Z = X$. $\square$

Corollary 6.4.8 [**g=bnb**] *Let $\Sigma \subseteq \Pi$ and define $N_\Sigma = T\langle w_\alpha \mid \alpha \in \Sigma \rangle$. Then $P_\Sigma = \langle B, N_\Sigma \rangle = BN_\Sigma B$. In particular, $G = BNB$.*

Proof: Note that $BN_\Sigma B \leq \langle B, N_\Sigma \rangle \leq P_\Sigma$. So it suffices to prove that $P_\Sigma \leq BN_\Sigma B$. Put $Z = BN_\Sigma B$, $Y = B \cup_{\alpha \in \Sigma} Bw_\alpha B$ and $X = \langle Y \rangle$. Then $Z \subseteq X \leq P_\Sigma$. Also $Y^{-1} = Y$ and by 6.4.5(c), $ZY \subseteq Z$. Thus by 6.4.7, $X = Z$ and it remains to show that $P_\Sigma \leq X$. From 6.4.2

$P_\Sigma = U_\Sigma T K_\Sigma = \langle B, \mathcal{X}_\alpha \mid \alpha \in \Phi_\Sigma \rangle$. Let $\alpha \in \Phi_\Sigma$. Then there exists $n \in N_\Sigma$ and $\beta \in \Sigma$ with $\alpha = n(\beta)$. Thus using 6.4.1(e),

$$\mathcal{X}_\alpha = \mathcal{X}_{n(\beta)} = n\mathcal{X}_\alpha n^{-1} \leq X$$

and so $P_\Sigma \leq X$. □

Note that for $n \in N$ there exists a unique $w_n \in W$ with $f(w_n) = nT$. We define $l(n) = l(w_n)$, that is $l = l(n)$ is minimal such that there exists $\alpha_1, \alpha_2, \dots \alpha_l$ with

$$n \in w_{\alpha_1} w_{\alpha_2} \dots w_{\alpha_n} T.$$

Let $\alpha \in \Pi$. From 3.4.4 we have $l(nw_a) > l(n)$ if and only if $n(\alpha) \in \Phi^+$. Also since $nw_\alpha T = nw_\alpha^{-1} T$, $l(nw_a) = l(nw_\alpha^{-1}$.

Lemma 6.4.9 [**b cap nbn**] *For $n \in N$ and $\epsilon \in \{\pm\}$ let $\Phi_n(\epsilon) = \{\alpha \in \Phi^+ \mid n(\alpha) \in \Phi^\epsilon\}$. Also put $U_n^\epsilon = \prod_{\alpha \in \Phi_n(\epsilon)} \mathcal{X}_\alpha$. Then*

(a) [**a**] *Let $t_\alpha \in \mathbb{E}, \alpha \in \Phi^+$, with $\prod_{\alpha \in \Phi^+} \chi_\alpha(t_\alpha) \in n^{-1}Bn$. Then $t_\alpha = 0$ for all $a \in \Phi_n(-)$.*

(b) [**c**] *$U = U_n^+ U_n^-$.*

(c) [**d**] *$U \cap n^{-1}Bn = U_n^+$ and $B \cap n^{-1}Bn = U_n^+ T$*

(d) [**e**] *Let $n_0 \in N$ with $n_0(\Phi^+) = \Phi^-$. Then $U \cap n^{-1}Bn = 1$ and $B \cap n^{-1}Bn = T$.*

(e) [**f**] *$U \cap T = 1$.*

(f) [**b**] *For every $u \in U$ there exists uniquely determined $t_\alpha \in \mathbb{E}$, $\alpha \in \Phi^+$ with $u = \prod_{\alpha \in \Phi^+} \chi_\alpha(t_a)$.*

Proof: (a) By induction on $l(n)$. If $l(n) = 0$, then $n \in T$, $\Phi_n(-) = \emptyset$ and $n^{-1}Bn = B$. So (a) holds in this case. So suppose $l(n) > 0$ and pick $\beta \in \Pi$ with $l(nw_\beta^{-1}) < l(n)$. Put $m = nw_\beta^{-1}$. Then

$$(*) \qquad n = mw_\alpha \quad \text{and} \quad m(\beta) \in \Phi^+$$

and

$$w_\beta \cdot \prod_{\alpha \in \Phi^+} \chi_\beta(t) \cdot w_\beta^{-1} \in w_\beta n^{-1} B n w_\beta^{-1}$$

and so

$$(**) \quad v := \prod_{\alpha \in \Phi^+} \chi_{\omega_\beta(\alpha)}(c_{\alpha\beta} t_\alpha) \in m^{-1} B m$$

Suppose for a contradiction that $t_\beta \neq 0$ and put $k = c_{\beta\beta} t_\beta$. Then $k \in \mathbb{E}^\sharp$. Note that $U_\beta \trianglelefteq P_\beta$, and $\mathcal{X}_{\omega_\beta(\alpha)} \leq U_\beta$ for all $\beta \neq \alpha \in \Phi^+$ and that $\omega_\beta(\beta) = b$. Let $\overline{P}_\beta = P_\beta / U_\beta$. The definition of v now implies

$$(***) \quad \overline{v} = \overline{\chi_{-\beta}(k)}$$

From (*), $m\mathcal{X}_\beta m^{-1} = \mathcal{X}_{m(\beta)} \le U \le B$ and so $\mathcal{X}_\beta \le m^{-1}Um$. Since $v \in m^{-1}Bm \le N_G(m^{-1}Um)$ we get $\mathcal{X}_\alpha^v \in m^{-1}Um$. As $m^{-1}Um$ is nilpotent, also $\langle \mathcal{X}_\beta, \mathcal{X}_\beta^v \rangle$ is nilpotent. (***) implies that

$$\langle \overline{\mathcal{X}_\beta}, \overline{\mathcal{X}_\beta}^{\overline{v}} \rangle = \langle \overline{\mathcal{X}_\beta}, \overline{\mathcal{X}_\beta}^{\overline{\chi}_\beta}(k) \rangle$$

and so $\langle \overline{\mathcal{X}_\beta}, \overline{\mathcal{X}_\beta}^{\overline{\chi}_\beta(k)} \rangle$ is nilpotent. 6.1.7 applied to the Steinberg group $\langle \overline{\mathcal{X}_\beta}, \overline{\mathcal{X}_{-\beta}} \rangle$ gives $\overline{\mathcal{X}_\beta} = \overline{\mathcal{X}_{-\beta}}$. Thus $\mathcal{X}_{-\beta} \le \mathcal{X}_\beta U_\beta = U$ and $\langle \mathcal{X}_\beta, \mathcal{X}_{-\beta} \rangle$ is nilpotent. So by 6.1.7, β is degenerate with respect to G, a contradiction.

Thus $t_\beta = 0$. For $\alpha \neq \beta$, $\omega_\alpha(b) \in \Phi^+$. Since $l(m) < l(n)$ we conclude from (**) and induction that $t_\alpha = 0$ for all $\beta \neq \alpha \in \Phi^+$ with $mw_\beta(\alpha) \in \Phi^-$. But $n = mw_\alpha$ and so (a) holds.

(b) Follows from 6.1.4(a).

(c) The first statement follows from (a). Since $T \le B \cap n^{-1}Bn$ we have

$$B \cap n^{-1}Bn = (UT) \cap n^{-1}Bn = (U \cap n^{-1}Bn)T = U_n^+ T.$$

(d) is a special case of (c).

(e) $U \cap T \le U \cap n_0^{-1}Bn_0 = 1$.

(f) If $\prod_{a \in \Phi} \chi_\alpha(t_\alpha) = 1$, then (a) applied with $n = n_0$ gives $t_\alpha = 0$ for all $\alpha \in \Phi$. (f) now follows from 6.1.3(b:c).

Lemma 6.4.10 [**xa in psigma**] *Let $a \in \Phi$ and $\Sigma, \tilde{\Sigma} \subseteq \Pi$.*

(a) [**a**] *Let $\mathcal{X}_\alpha \le N_G(U_\Sigma)$ if and only if $\alpha \in \Phi^+ \cup \Phi_\Sigma$.*

(b) [**b**] *$P_\Sigma \le P_{\tilde{\Sigma}}$ if and only if $\Sigma \subseteq \tilde{\Sigma}$.*

(c) [**c**] *$P_\Sigma = P_{\tilde{\Sigma}}$ if and only if $\Sigma = \tilde{\Sigma}$.*

(d) [**d**] *$\langle P_\Sigma, P_{\tilde{\Sigma}} \rangle = P_{\Sigma \cup \tilde{\Sigma}}$.*

Proof: (a) Let $\Delta = \Phi^+ \setminus \Phi_\Sigma$. Then $\Phi^+ \cup \Phi_\Sigma = \Delta \cup \Phi_\Sigma$ and so the definition of P_σ implies $\mathcal{X}_\alpha \le P_\Sigma \le N_G(U_\Sigma)$ for all $\alpha \in \Phi^+ \cup \Phi_\Sigma$. Now suppose that $\mathcal{X}_\alpha \le N_G(U_\Sigma)$ but $\alpha \notin \Phi^+ \cup \Phi_\Sigma$. Note that $\Phi \setminus (\Delta \cup \Phi_\Sigma) = -\Delta$ and so $\alpha = -\delta$ for some $d \in \Delta$. We conclude that $\langle \mathcal{X}_\delta, \mathcal{X}_{-\delta} \rangle \le U_\Sigma$ and so is nilpotent. Thus 6.1.7 implies that δ is degenerate with respect to G, a contradiction.

(b) Follows from (a) and the definition of P_Σ.

(c) follows immediately from (d).

(d) We have $N_{\Sigma \cup \tilde{\Sigma}} = \langle N_\Sigma, N_{\tilde{\Sigma}} \rangle$ and so

$$P_{\Sigma \cup \tilde{\Sigma}} = \langle B, N_{\Sigma \cup \tilde{\Sigma}} \rangle = \langle B, N_\Sigma \cup N_{\tilde{\Sigma}} \rangle = \langle P_\Sigma, P_{\tilde{\Sigma}} \rangle$$

□

Lemma 6.4.11 [**wa in bnbnb**] *Let $n \in N$ and $\alpha \in \Phi$ with $l(nw_\alpha^{-1}) \leq l(n)$. Then $w_\alpha \in Bn^{-1}BnB$.*

Proof: We have $n(\alpha) \in \Phi^-$ and so by 6.4.5(a) , $n \in BnBw_\alpha B$. So $n = b_1nb_2w_\alpha b_3$ for some $b_i \in B$ and $w_\alpha = b_2^{-1}n^{-1}b_1^{-1}nb_3^{-1}$. □

Lemma 6.4.12 [**b nbn**] *Let $n \in N$. Put $l = l(n)$, pick $\alpha_1, a_2 \ldots a_\lambda \in \Pi$ with $n \in w_{\alpha_1}w_{\alpha_2}\ldots w_{\alpha_l}T$ and put $\Sigma = \{\alpha_i \mid 1 \leq i \leq l\}$. Then*

$$\langle B, n^{-1}Bn\rangle = \langle B, n\rangle = BN_\Sigma B = P_\Sigma$$

Moreover, the set $\Sigma_n := \Sigma$ only depends on n and not on the choice of the $\alpha_1, \ldots, \alpha_n$.

Proof: Cleary $< B, n^{-1}Bn\rangle \leq \langle B, n\rangle \leq BN_\Sigma B$. Put $m = nw_{\alpha_l}^-1$. Then $l(m) = l(n) - 1$ and so by induction on l

$$(*) \quad w_{a_i} \in \langle B, m^{-1}Bm\rangle \text{ for all } 1 \leq i < l$$

Since $l(m) < l(n)$ we have $n(\alpha_l) \in \Phi^-$ and so by 6.4.11

$$(**)w_{\alpha_l} \in Bn^{-1}BnB \leq \langle B, n^{-1}Bn\rangle$$

Hence $\langle B, m^{-1}Bm\rangle = \langle B, w_\alpha^{-1}n^{-1}Bnw_\alpha\rangle \leq B, n^{-1}Bn, w_{\alpha_l}\rangle = \langle B, n^{-1}B\rangle$

So from (*) and (**), $P_\Sigma = BN_\Sigma B \leq \langle n, n^{-1}Bn\rangle$ and the the first statement of the lemma holds. The independence of Σ follows from 6.4.10(c) □

Corollary 6.4.13 [**b cap n**] *$B \cap N = T$.*

If $n \in B \cap T$, then $P_\emptyset = B = \langle B, n\rangle$ and so $\Sigma_n = \emptyset$, $l(n) = 0$ and $n \in T$. □

Lemma 6.4.14 [**overgroups of b**] *Let $B \leq P \leq G$. Then $P = P_\Sigma$ for a unique $\Sigma \subseteq \Pi$.*

Proof: Since $G = BNB$ we have $P = B(N \cap P)B = \langle B, N \cap P\rangle$. Let $\Sigma = \bigcup_{n \in P_\Sigma}$. Since $\langle B, n\rangle = P_{\{}Sigma_n\}$ we have

$$P = \langle B, N \cap P\rangle = \langle\, \langle B, n\rangle \mid n \in P \cap N\rangle = \langle P_{\Sigma_n} \mid n \in P \cap N\rangle = P_\Sigma$$

The uniqueness of Σ follows from 6.4.10(c). □

Lemma 6.4.15 [**conjugates**] *Let $\Sigma, \tilde{\Sigma} \subseteq \Pi$.*

(a) [**a**] *Let $g \in G$ with $g^{-1}P_\Sigma g \leq P_{\tilde{\Sigma}}$. Then $g \in P_{\tilde{\Sigma}}$ and $\Sigma \subseteq \tilde{\Sigma}$.*

(b) [**b**] *If P_Σ and $P_{\tilde{\Sigma}}$ are conjugate in G then $\Sigma = \tilde{\Sigma}$.*

(c) [**c**] $N_G(P_\Sigma) = P_\Sigma$.

(d) [**d**] $N_G(U_\Sigma) = P_\Sigma$.

Proof: Let $g = bn\tilde{b}$ with $b, \tilde{b} \in B$ and $n \in N$. Then $\tilde{b}^{-1}n^{-1}P_\Sigma n\tilde{b} = g^{-1}P_\Sigma g \leq P_{\tilde{\Sigma}}$ and conjuagtion with $\tilde{b}$ gives

$$n^{-1}P_\Sigma n \leq P_{\tilde{\Sigma}}$$

So by 6.4.12

$$n \in P_{\Sigma_n} = \langle B, n^{-1}Bn\rangle \leq \langle B, n^{-1}Bn\rangle \leq P_{\tilde{\Sigma}}$$

So also $g \in bn\tilde{b} \in P_{\tilde{\Sigma}}$. Since $g^{-1}P_\Sigma g \leq P_{\tilde{\Sigma}}$, conjugation with g gives $P_\Sigma \leq P_{\tilde{\Sigma}}$ and by 6.4.10(b) gives $\Sigma \leq \tilde{\Sigma}$

(b) Let $g \in G$ with $g^{-1}P_\Sigma g = P_{\tilde{\Sigma}}$. By (a) $\Sigma \subseteq \tilde{\Sigma}$ and by symmetry, $\tilde{\Sigma} \subseteq \Sigma$.

(c) Follows from (a) applied with $\Sigma = \tilde{\Sigma}$.

(d) Since $B \leq P_\Sigma \leq N_G(U_\Sigma)$ we have $N_G(\Sigma) = P_{\tilde{\Sigma}}$ for some $\Sigma \subseteq \tilde{\Sigma} \subseteq \Pi$. From 6.4.10(a), $\tilde{\Sigma} \subseteq \Phi^+ \cup \Phi_\Sigma$ and so $\tilde{\Sigma} = \Sigma$. □

For $w \in W$ pick $n_w \in N$ with $f(w) = n_w T$. For later convience choose $n_1 = 1$.

Lemma 6.4.16 [**w to bnwn**] *Let $n, \tilde{n} \in N$ with $BnB = B\tilde{n}B$. Then $nT = \tilde{n}T$ and so the map $W \to B\backslash G/B, w \to Bn_wB$ is a bijection.*

Proof: If $l(n) = 0$, then $n \in T \leq B$ and so $B\tilde{n}B = BnB = N$ and $m \in N \cap B = T$. By 6.4.13, $\tilde{n} \in T$ and so $nT = T = \tilde{n}T$.

Suppose now that $l(n) > 0$ and let $\widetilde{\in}\Pi$ and $l(nw_\alpha) < l(n)$. Put $m = nw_\alpha$. Then by 6.4.5(c) and b

$$BnB \cdot Bw_\alpha B = Bnw_\alpha B \cup BnB = BmB \cup BnB$$

and

$$B\tilde{n}B \cdot Bw_\alpha B \subseteq B\tilde{n}w_\alpha B \cup B\tilde{n}B$$

Since $BnB = B\tilde{n}B$, this implies

$$(*) \quad BmB \cup BnB \subseteq B\tilde{n}w_\alpha B \cup BnB$$

Since $mT \neq nT$ and $l(m) < l(n)$, induction on $l(n)$ gives $BmB \neq BnB$. Since double cosets are either equal or disjoint (*) implies $BmB = B\tilde{n}w_\alpha B$. Again by induction on $l(n)$, $nw_aT = mT = \tilde{n}w_\alpha T$ and so $nT = \tilde{n}T$. □

Proposition 6.4.17 [**unique factorization**] *Let $g \in G$.*

(a) [**a**] *There there exists unique $u \in U, h \in T, w \in W, v \in U_w^-$ with $g = uhn_w v$.*

(b) [**b**] *There exists unique $t_\alpha, \alpha \in \Phi^+$ and $s_\beta, b \in \Phi_w^+(-)$ with $u = \prod_{\alpha \in \Phi^+} \chi_\alpha(t_\alpha)$ and $v = \prod_{\beta \in \Phi_w^+(-)} \chi_\beta(s_\beta)$.*

Proof: (a) We first show the existence. Let $n \in N$. By 6.4.9 $nU^+n^{-1} \leq U$, $U = U_n^+ U_n^-$ and $U^+n \cap U^-n = 1$. Thus

$$BnB = BnTU^+U^- = B \cdot nTn^{-1} \cdot nU^+n^{-1} \cdot nU^- = UTnU_n^-$$

As $G = BNB$, this implies the exists of u, h, w and v. Since $B = UT$ and $U \cap T = 1$, any b in B can be uniquely written as ut with $u \in U$ and $t \in T$. Now let $b, \tilde{b} \in B, w, \tilde{w} \in W$, $v \in U_w^-$ and $\tilde{v} \in U_{\tilde{w}}^-$ with

$$bn_w v = \tilde{b} n_{\tilde{w}} \tilde{v}$$

Then $Bn_w B = Bn_{\tilde{w}} B$ and so by 6.4.16, $w = \tilde{w}$. Put $a = \tilde{b}^{-1}$ and $x = \tilde{v} v^{-1}$. Then $a \in B$, $x \in U_w^- \leq U$ and

$$n_w^{-1} a n_w = x$$

So by 6.4.9(c), $x \in U \cap n_w^{-1} U n_w \leq U_w^+$.

Thus $x \in U_w^+ \cap U_w^- = 1$. Hence also $a = 1$, $v = \tilde{v}$ and $b = \tilde{b}$, proving (a).

(b) follows from 6.4.9(e). □

6.5 Normal subgroups of Steinberg groups

Lemma 6.5.1 [**gprime and zg**] *G/G' is degenerate, while $G/Z(G)$ is non-degenerate.*

Proof: Let $\alpha \in \Phi$. $\mathcal{X}_\alpha$ and $\mathcal{X}_{-\alpha}$ are conjugate in G and so $\mathcal{X}_\alpha G' = \mathcal{X}_{-\alpha} G'$ and α is degenerate with respect to G/G'.

Since α is non-degerate in G, 6.1.7 implies that $\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha} \rangle$ is not nilpotent. Then also $\langle \mathcal{X}_\alpha, \mathcal{X}_{-\alpha} \rangle Z(G)/Z(G)$ is not nilpotent and so α is non-degenerate for $G/Z(G)$.

Lemma 6.5.2 [**b core**] *$\bigcap B^G \leq Z(G) \cap T$.*

Proof: By 6.4.9(d), $M := \bigcap B^G \leq B \cap n_0^{-1} B n_0 \leq T$. Let $\alpha \in \Phi^+$. Using 6.4.9(e),

$$[\mathcal{X}_\alpha, M] \mathcal{X}_\alpha \cap T \leq U \cap T = 1$$

Since $w_\alpha M w_a^{-1} = M$ we also have $[\mathcal{X}_{-\alpha}, M] = 1$ and so $M \leq Z(G)$. □

Proposition 6.5.3 [**normal subgroups**] *Let $M \lhd G$. Then one of the following holds:*

(a) [**a**] *G/M is degenerate. Also if in addition Φ is connected, $T \leq M = G'$, $|G/M| = |\mathbb{E}|$ and either $|\mathbb{E}| = 3$ and $\Phi \cong A_1$ or $|\mathbb{E}| = 2$ and $\Phi \cong A_1, B_2$ or G_2.*

(b) [**b**] *G/M is non-degenerate and $M \leq Z(G) = \bigcap B^G \leq T$*

Proof: Suppose first that G/M is degenerate. Then 6.2.1 implies (a), except that we still need to show $G' = M$. But $G' \leq M$, G/G' is degenerate and so $|G/G'| \leq |\mathbb{E}| = |G/M|$. Thus $G' = M$.

Suppose next that $\overline{G} := G/M$ is non degenerate and let $m \in M$. Then by 6.4.17 there exists $t_\alpha \in \mathbb{E}, \alpha \in \Phi^+$, $h \in T$, $w \in W$ and $s_\beta, \beta \in \Phi^+_w(-)$ with

$$m = \prod_{\alpha \in \Phi^+} \chi_\alpha(t) \cdot h \cdot n_w \cdot \prod_{\beta \in \Phi^-_w(-)} \chi_\beta(s_\beta)$$

Hence

$$1 = \overline{m} = \prod_{\alpha \in \Phi^+} \overline{\chi}_\alpha(t) \cdot \overline{h} \cdot \overline{n}_w \cdot \prod_{\beta \in \Phi^-_w(-)} \overline{\chi}_\beta(s_\beta)$$

Since $n_1 = 1$ we also have

$$1 = \prod_{\alpha \in \Phi^+} \overline{\chi}_\alpha(0) \cdot 1 \cdot \overline{n}_1 \cdot \prod_{\beta \in \Phi^-_w(-)} \overline{\chi}_\beta(0)$$

So the uniqueness assertions in 6.4.17 applied the the non-degenerate Steinberg group $\overline{G}$ gives $t_\alpha = 0$ for all $a \in \Phi$, $\overline{h} = 1$, $w = 1$ and $s_\beta = 0$ for all $\beta \in \Phi^+_w(-)$. Thus $m = h \in T \leq B$. So using 6.5.2 $M \leq \bigcap B^G \leq Z(G) \cap T$.

By 6.5.1, $G/Z(G)$ is non-degenerate and from what we just proved $Z(G) \leq B^G$. Thus $Z(G) = \bigcap B^G$ and all parts of (b) are proved. □

Corollary 6.5.4 [**g/zg**] *Let $\overline{G} = G/Z(G)$. Then $Z(\overline{G}) = \bigcap \overline{B}^{\overline{G}} = 1$. Moroever, if $\overline{G} = \overline{G}'$, then $\overline{G}$ is simple.*

Proof: By 6.5.3 applied to $\overline{G}$, $Z(\overline{G}) = \bigcap \overline{B}^{\overline{G}}$. Also $Z(G) = \bigcap B^G$ and thus $1 = \overline{\bigcap B^G} = \bigcap \overline{B}^{\overline{G}}$. If $\overline{G} = \overline{G}'$, 6.5.3 applied to $\overline{G}$ shows that $\overline{G}$ is simple. □

We group $G/Z(G)$ is called the adjoint Steinberg groups of type Φ over the field $\mathbb{E}$. Note that any non-degenerate Steinberg groups is a quotient of the universal groups and has the adjoint groups as a quotient.

6.6 The structure of the Cartan subgroup

In this section G continues to be a non-degenerate Steinberg group of type Φ over the field $\mathbb{E}$. We will investigate the structure of T.

Lemma 6.6.1 [**t abelian**] *T is abelian.*

Proof: Let $\alpha, \beta \in \Phi$ and $k, t \in \mathbb{E}^\sharp$. From 6.1.1(3) and 8 we have

$$h_\alpha(k)\omega_\beta(t)h_\alpha(k)^{-1} = \omega_\beta(k^{(\beta,\check{\alpha})}t)$$

Thus (freely using the Steinberg relations)

$$\begin{aligned} h_\alpha(k)h_\beta(t)\,\mathrm{ht}_\alpha(k)^{-1} &= h_\alpha(k)\omega_\alpha(1)^{-1}\,\mathrm{ht}_\alpha(k)^{-1} \\ &= \omega_\beta(k^{(\beta,\check{\alpha})}t)\omega_\beta(k^{(\beta,\check{\alpha})})^{-1} \\ &= h_\beta(k^{(\beta,\check{\alpha})}t)w_\alpha(h_\beta(k^{(\beta,\check{\alpha})})w_\alpha)^{-1} \\ &= h_\beta(k^{(\beta,\check{\alpha})}t)h_\beta(k^{(\beta,\check{\alpha})})^{-1} \\ &= h_\beta(k^{(\beta,\check{\alpha})}tk^{-(\beta,\check{\alpha})}) \\ &= h_\beta(t) \end{aligned}$$

So any two of the $h_\alpha(k)$ commute and since T is generated by the $h_\alpha(k)$, T is abelian.□

Definition 6.6.2 [**character**] *Let A and F be abelian groups. An F-*character *ξ for A is homomorphism from A to F. $\xi(A, F)$ denotes the set of F-characters.*

In our main applications of characters, A will be subgroup of $\mathbb{Q}\Phi$ and F the multiplicative group $\mathbb{E}^\sharp$. For this reason we will usually use additive notation for A and multiplicative notation for F.

Note that $\xi(A, F)$ is an abelian group via $(\xi_1 + \xi_2)(a) = \xi_1(a)\xi_2(a)$.

Lemma 6.6.3 [**xi and tensor**] *Let A and F be abelian groups and put $A^* = \xi(A, \mathbb{Z})$. Then there exists a unique homomorphism $\zeta : F \otimes_{\mathbb{Z}} A^* \to \xi(A, F)$ with $k \otimes b \to \xi_b(k)$ where $\xi_b(k)$ is defined by $\xi_b(k)(a) = k^{b(a)}$.*

Proof: Each $\xi_\beta(k)$ clearly is an F-character for A. Also the map $(k, b) \to x_b(k)$ is $\mathbb{Z}$-bilinear and so ζ exists and is unique. □

Lemma 6.6.4 [**xi for free abelian**]

(a) [**a**] *Let A be a free abelian group with basis $\mathcal{A}$. Then map $\rho : F^{\mathcal{A}} \to \xi(A, F)$ define by*

$$\rho((k_a)_{a\in A})(\sum_{b\in\mathcal{A}} n_b b) = \prod_{a\in A} k_a^{n_a}$$

is an isomomorphism.

(b) [**b**] *Let A be a free abelian group of finite rank, then ζ as defined in 6.6.3 is an isomorphism.*

Proof: (a) follows immediately from the definition of a basis via a universal property.

(b) Note that A^* is free abelian with $\mathbb{Z}$ basis the dual basis of $\mathcal{A}$. Thus every element in $F \otimes A^*$ can be uniquely written as $\sum k_\alpha \otimes a^*$ for some $k_a \in F$, $a \in \mathcal{A}$. Now

$$\zeta(\sum_a k_\alpha \otimes a^*)(\sum_b n_b b) = \prod_a \xi_{a^*}(k_a)(\sum n_b b) = \prod_\alpha k_\alpha^{a^*(\sum n_b b)} = \prod_a k_\alpha^{n_a}$$

Thus $\zeta(\sum_a k_a \otimes a^*) = \rho((k_a)_a)$ and so (b) follows from (a). □

Lemma 6.6.5 [**xi for zchphi**]

(a) [**a**] *There exists a isomorphism $g : \mathbb{Z}\check{\Phi} \to \xi(\Lambda, \mathbb{Z})$ with $g(d)(\lambda) = (\lambda, d)$ for all $d \in \mathbb{Z}\check{\Phi}, \lambda \in \Lambda$.*

(b) [**b**] *For $\alpha \in \Phi$ and $k \in \mathbb{E}^\sharp$ define $\xi_\alpha(k) \in \xi(\Lambda, \mathbb{E}^\sharp$ by $\xi_\alpha(k)(\lambda) = k^{(\lambda,\check{\alpha})}$ for all $\lambda \in \Lambda$. There exists an isomorphism $\zeta : \mathbb{E}^\sharp \otimes_\mathbb{Z} \mathbb{Z}\check{\Phi} \to \xi(\Lambda, \mathbb{E}^\sharp)$ with $\zeta(k \otimes \check{\alpha}) = \xi_\alpha(k)$ for all $k \in \mathbb{E}^\sharp$, $\alpha \in \Phi$ and $\lambda \in \Lambda$.*

Proof: (a) Note that $\check{\Pi}$ is a $\mathbb{Z}$-basis for $\check{\phi}$ and $(\check{a})^*, \alpha \in \Pi$ is a $\mathbb{Z}$-basis for Λ. Also $g(\check{\alpha}) = (\check{\alpha})^*$ and so (a) holds.

(b) follows from (a) and 6.6.4. □

Lemma 6.6.6 [**restriction of characters**] *Let A, F be an abelian group and $B \leq A$. Let* Res *: $\xi(A, F)) \to \xi(B, F), \xi \to \xi_{|}B$, be the restriction map. Denote by $\xi^A(B, F)$ the image of* Res *and by $\xi_B(A, F)$ the kernel of* Res*. Then $\xi^B(A, F) \cong \xi(A, F)/\xi_B(A, F)$ and $\xi_B(A, F) \cong \xi(A/B, F)$.*

Proof: obvious. □

Lemma 6.6.7 [**closure of pi**] *If Ψ is closed then $\Phi^+ \subseteq \Psi$ and if Ψ is closed and $\Psi = -\Psi$, then $\Psi = \Phi$.*

Proof: Suppose first that Ψ is closed and let $\alpha \in \Phi^+$. If $\operatorname{ht}\alpha = 1$, then $\alpha \in \Pi \subseteq \Psi$. If $\operatorname{ht}\alpha > 1$, then by 3.3.2(e) there exists $\beta \in \Pi$ and $\gamma \in \Phi^+$ with $\alpha = \beta + \gamma$ and $\operatorname{ht}\gamma < \operatorname{ht} a$. By induction on $\operatorname{ht}\alpha$, both β and γ are in Ψ. Since Ψ is closed, also $\alpha \in \Psi$. Thus $\Phi^+ \subseteq \Psi$.

Suppose in addition that $\Psi = -\Psi$, then $\Phi^- = -\Phi^+ \subseteq \Psi$ and $\Psi = \Phi$. □

Lemma 6.6.8 [**linear extension**] *Let A be an abelian group and $g : \Phi \to A$ and function such that $g(-\alpha) = -g(\alpha)$ for all $\alpha \in \Phi$ and $g(\alpha + \beta) = g(\alpha) + g(\beta)$ whenever $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \phi$. Then g can be uniquely extented to a $\mathbb{Z}$-linear map $\mathbb{Z}\Phi \to A$.*

Proof: Define $\tilde{g} : \mathbb{Z}\Phi \to A, \sum_{\alpha\in\Pi} n_\alpha \alpha \to \sum_{\alpha\in\Pi} n_\alpha f(\alpha)$. Let $\Psi = \{b \in \Phi \mid g(\beta) = \tilde{g}(\beta)$. Then Ψ is closed, $\Psi = -\Psi$ and $\Pi \subseteq \Psi$. Hence by 6.6.7, $\Psi = \Phi$. $\square$

Lemma 6.6.9 [**t as characters**]

(a) [**a**] *There exists a unique homomorphism $\tau : \xi(\Lambda, \mathbb{E}) \to T$ with $\tau(\xi_\alpha(k)) = h_\alpha(k)$ for all $\alpha \in \Phi$ and $k \in \mathbb{E}^\sharp$.*

(b) [**b**] *τ is onto.*

(c) [**c**] *$\tau(\xi)\chi_\beta(t)\tau(\xi)^{-1} = \chi_\beta(\xi(\beta)t)$ for all $\beta \in \Phi$, $\xi \in \xi(\Lambda, \mathbb{E})$, $t \in \mathbb{E}$.*

(d) [**d**] *$\tau(\xi) \in Z(G)$ iff $\xi \mid_\Phi = 0$. So $Z(G) = \tau(\xi_{\mathbb{Z}\Phi}(\Lambda, \mathbb{E}))$.*

Proof: By 6.6.1 T is abelian. Let $k \in \mathbb{E}^\sharp$. From 6.1.1(9) and 6.6.8 (applied to $\check{\Phi}$ in place of Φ) the map $\check{\alpha} \to h_\alpha(k)$ extends to a $\mathbb{Z}$-linear map $s_k : \mathbb{Z}\check{\Phi} \to T$. Define $s : \mathbb{E}^\sharp \times \mathbb{Z}\check{\Phi} \to T, (k, a) \to s_k(a)$.

Then clearly s is $\mathbb{Z}$-linear in the second coordinate. For $a \in \check{\Phi}$, 6.1.1(9:a) implies that s is also linear in the first coordinate. Since $\check{\Phi}$ generates $\mathbb{Z}phi$, s is $\mathbb{Z}$-bilinear. So there exists a homomorphism

$$\tilde{\tau} : \mathbb{E}^\sharp \otimes \mathbb{Z}\check{\Phi} \to T \text{ with } k \otimes \check{\alpha} \to h_\alpha(k) \,\forall k \in \mathbb{E}^\sharp, \alpha \in \Phi$$

let ζ be as in 6.6.5(b) and put $\tau = \tilde{\tau} \circ \zeta$. Then (a) holds.

(b) follows since T is generated by the $h_\alpha(k)$.

(c) Let Ξ be the set of $\xi \in \xi(\Lambda, \mathbb{E}^\sharp)$ which fulfill the equation in (c) for all $\beta \in \Phi$ and $t \in \mathbb{E}$. Then clearly Ξ is a subgroups of $\xi(\Lambda, \mathbb{E}^\sharp)$. Let $\alpha \in \Phi$ and $k \in \mathbb{E}^\sharp$. Then $\tau(\xi_\beta(k)) = h_\beta(k)$ and $\xi_\alpha(k)(\beta) = k^{(\beta,\check{\alpha})}$. So by 6.1.1(8), $\xi_\alpha(k) \in \Xi$. But the $\xi_\alpha(k)$ generate $\xi(\Lambda, \mathbb{E}^\sharp)$ and so (c) holds.

(d) Since G is generated by the $\chi_\beta(t)$, $\tau(\xi) \in Z(G)$ iff $\tau(\xi)\chi_\beta(t)\tau(\xi)^{-1} = \chi_\beta(t)$ for all $\beta \in \Phi$ and $t \in \mathbb{E}$. So (d) follows from (c).

Lemma 6.6.10 [**phicirc for faithful**] *Let V be a standard, faithful $L_\mathbb{K}$-module and put $\Phi^\circ = \Lambda_V(H_\mathbb{K})$. Then $\mathbb{Z}\Phi \leq \mathbb{Z}\Phi^\circ \leq \Lambda$.*

Proof: Let $\alpha \in \Phi$. Since V is faithful and $V = \bigoplus_{\mu\in\Phi^\circ}$, there exists $\mu \in \Phi^\circ$ with $L_a^\mathbb{K} V_\mu \neq 0$. Since $L_\alpha^\mathbb{K} V_\mu \leq V_{\alpha+\mu}$ we conclude that also $\mu + \alpha \in \Phi^\circ$. Hence $\alpha = (\mu + \alpha) - \mu \in \mathbb{Z}\Phi^\circ$. $\square$

Lemma 6.6.11 [**t for chevalley**] *Let M be an $\mathfrak{U}_\mathbb{Z}$ invarinat lattice in the standard, faithful $L^\mathbb{K}$-module $V^\mathbb{K}$. Put $\Phi^\circ = \Lambda_{V^\mathbb{K}}(H^\mathbb{K})$. Suppose that $G = G_\mathbb{E}(M)$.*

(a) [**a**] *Let $\mu \in \Phi^\circ$, $v \in V_\mu^\mathbb{E}$ and $\xi \in \xi(\Lambda, \mathbb{E}^\sharp)$. Then $\tau(\xi)(v) = \xi(\mu)v$.*

(b) [**b**] $\ker \tau = \xi_{\mathbb{Z}\Phi^\circ}(\Lambda, \mathbb{E}^\sharp)$.

(c) [**c**] $T \cong \xi^{\Lambda}(\mathbb{Z}\Phi^\circ, \mathbb{E}^\sharp)$

(d) [**d**] $Z(G) \cong \xi^{\Lambda/\mathbb{Z}\Phi}(\mathbb{Z}\Phi^\circ/\mathbb{Z}\Phi, \mathbb{E}^\sharp)$

Proof: (a) By the definition of $h_\alpha(k)$ for Chevalley groups, we have $h_\alpha(k)v = k^{(\mu,\check{\alpha})}v$. So the same argument as in the proof of 6.6.9(c) shows that (a) holds.

(b) By (a) $\tau(\xi) = 0$ iff $\xi(\mu) = 0$ for all $\mu \in \Phi^\circ$ and so iff $\xi(\lambda) = 0$ for all $\lambda \in \mathbb{Z}\Phi^\circ$. Thus (b) holds.

(c) By 6.6.10 we have $\mathbb{Z}\Phi \leq \mathbb{Z}\Phi^\circ$. So using 6.6.9(b), (b) and 6.6.6 we compute

$$\begin{array}{rcl} T \cong & \xi(\Lambda, \mathbb{E}^\sharp)/\ker\tau & = \xi(\Lambda, \mathbb{E}^\sharp)/\xi_{\mathbb{Z}\Phi^\circ}(\Lambda, \mathbb{E}^\sharp) \\ & = \cong \xi(\Lambda/\mathbb{Z}\Phi, \mathbb{E}^\sharp)/\xi_{\mathbb{Z}\Phi^\circ/\mathbb{Z}\Phi}(\Lambda/\mathbb{Z}\Phi, \mathbb{E}^\sharp) & \cong \xi^{\Lambda/\mathbb{Z}\Phi}(\mathbb{Z}\Phi^\circ/\mathbb{Z}\Phi, \mathbb{E}^\sharp) \end{array}$$

So (c) holds.

(d) Follows from (b), 6.6.9(d) and 6.6.6. $\square$

Lemma 6.6.12 [**characters for finite**] *Let A be a finite abelian group and $B \leq A$. Let $n \in \mathbb{Z}^+$ be minimal with $nA = 0$. Put $F = \{k \in \mathbb{E} \mid k^n = 1\}$ and $m = |F|$. Then*

(a) [**a**] $\xi(A, \mathbb{E}^\sharp) \cong \xi_{mA}(A, F) \cong A/mA$.

(b) [**b**] *Let $a \in A$. Then $\xi(a) = 1$ for all $\xi \in \xi(A, \mathbb{E}^\sharp)$ iff $a \in mA$.*

(c) [**c**] *Let $\Xi \leq \xi(A, \mathbb{E}^\sharp)$ and put $B = \{a \in A \mid \xi(a) = 1 \forall \xi \in \Xi\}$. Then $\Xi = \xi_B(A, \mathbb{E}^\sharp)$.*

Proof: Let $a \in A$ and $\xi \in \xi(A, \mathbb{E}^\sharp)$ then $\xi(a)^n = \xi(an) = \xi(0) = 1$ and so $\xi(a) \in F$. Hence $\xi(ma) = \xi(a)^m = 1$. Thus $\xi(A, \mathbb{E}^\sharp) \cong \xi_{mA}(A, F)$.

Replacing A by A/mA we may assume from now on that $m = n$. Since every finite abelian group is the direct sum of cylcic groups we can choose $\mathcal{A} \subseteq A$ with $A = \bigoplus_{\alpha \in A} \mathbb{Z}a$.

Let n_α be the order on a. Since finite subgroups of $\mathbb{E}^\sharp$ are cyclic, F is cylic of order $n = m$. In particular, there exists an element f_a of order n_α in F.

For $a \in \mathcal{A}$ define $\xi_a \in \xi(A, \mathbb{E}^\sharp)$ by $\xi_a(\sum_{b \in \mathcal{A}} m_b b) = f_\alpha^{m_a}$. If $\xi \in \xi(A, E^\sharp)$, then $\xi(a) = f_a^{m_\alpha}$ for some $m_\alpha \in \mathbb{Z}$. Then $\xi = \sum_{\alpha \in \mathcal{A}} m_\alpha xi_\alpha$. Also $\sum m_\alpha x_a = 0$ if and only if $f_a^{m_a} = 1$ and so $n_\alpha \mid m_a$ for all $a \in \mathcal{A}$. Thus

$$\xi(A, \mathbb{E}^\sharp) \cong \sum_{a \in A} \mathbb{Z}/n_a\mathbb{Z} \cong A$$

So (a) holds.

Let $b = \sum_{ain\mathcal{A}} m_a \in A$. Then $\xi(b) = 1$ for all $\xi \in \xi(A, \mathbb{E}^\sharp)$ iff $\xi_a(b) = 1$ for all $a \in \mathcal{A}$, iff $f_a^{m_a} = 1$, iff $n_a \mid m_a$ and iff $b = 0$. So (b) holds.

To prove (c) observe first that $\Xi \leq \xi_B(A, \mathbb{E}^\sharp) \cong \xi(A/B, \mathbb{E}^\sharp)$. By (a) $\xi(A/B, \mathbb{E}^\sharp)$ has order $|A/B|$. Define $\rho : A \to \xi(\Xi, \mathbb{E}^\sharp)$ by $\rho(a)(\xi) = \xi(a)$. Then $\ker\rho = B$. By (a) $\xi(\Xi, \mathbb{E}^\sharp)$ has order $|\Xi|$ and so $|A/B| \leq |\Xi|$. So all of our inequalities in this paragraph are equalities and (c) holds.

Lemma 6.6.13 [**intermediate lattices**]

(a) [**a**] *Let $\mathbb{Z}\Phi \leq \Lambda^\circ \leq \Lambda$. Then there exists a faithful $L^{\mathbb{K}}$ module V^K with $\Lambda^\circ = \mathbb{Z}\Lambda_{V^K}(H^K)$.*

(b) [**b**] *Let $\Xi \leq \xi_{\mathbb{Z}\Phi}(\Lambda, \mathbb{E})$. Then there exists $\mathbb{Z}\Phi \leq \Lambda^\circ \leq \Lambda$ with $\Xi = \xi_{\Lambda^\circ}(\Lambda, \mathbb{E}^\sharp)$*

Proof: (a) By 3.7.22(a), $\Lambda/\mathbb{Z}\Phi$ finite. So we can choose a finite subset $\mathcal{L}$ of Λ such that $\Lambda = \mathbb{Z}\langle\mathcal{L}\rangle + \mathbb{Z}\Phi$. (We remark for no particular reason that by 3.7.20(c) we can choose all $\lambda \in \mathcal{L}$ to be minimal). Put

$$V^K = L^{\mathbb{K}} \oplus \sum_{\lambda \in \mathcal{L}} V^K(\lambda)$$

By 5.4.4(c), $\lambda \in \Lambda_{V^{\mathbb{K}}(\lambda)}(H^K) \subseteq \lambda + \mathbb{Z}\Phi$. Thus (a) holds.

(b) Put $\Lambda^\circ = \{\lambda \in \Lambda \mid \xi(\lambda) = 0, \forall \xi \in \Xi\}$. Then $\mathbb{Z}\Phi \leq \Lambda^\circ \leq \Lambda$. Since $\xi_{\mathbb{Z}\Phi}(\Lambda, \mathbb{E}^\sharp) \cong \xi(\Lambda/\mathbb{Z}\Phi, \mathbb{E}^\sharp)$, (b) now follows from 6.6.12(c).

Lemma 6.6.14 [**z for universial**] *Let $G = G_\Phi(\mathbb{E})$.*

(a) [**a**] *Let $\rho : G \to G_{\mathbb{E}}(M)$ be the canonical epimorphism. Then* $\ker \rho \circ \tau = \xi_{\mathbb{Z}\Phi^\circ}(\Lambda, \mathbb{E})$

(b) [**b**] *τ is an isomorphism and so $T \cong \xi(\Lambda, \mathbb{E}^\sharp)$ and $Z(G) \cong \xi_{\mathbb{Z}\Phi}(\Lambda, \mathbb{E}^\sharp)$.*

(c) [**c**] *$G \cong G_{\mathbb{E}}(M)$ iff $\xi(\Lambda/\mathbb{Z}\Phi^\circ, \mathbb{E}^\sharp) = 1$ iff $\mathbb{E}^\sharp$ containes no non-trivial elements of order dividing $|\Lambda/\mathbb{Z}\Phi|$.*

(d) [**d**] *The isomorphism type of $G_{\mathbb{E}}(M)$ only dependes on $\mathbb{Z}\Phi^\circ$ and in particular is independent of the choice of the lattice M in V^K.*

Proof: (a) Since $G_{\mathbb{E}}(M)$ is non-degenerate, $G/\ker\rho$ is non-degenerate and so $\ker\rho \leq Z(G) \leq T$. (a) now follows from 6.6.11(b) applied to $G_{\mathbb{E}}(M)$.

(b) From 6.6.13(a), we can choose V^K such that $\mathbb{Z}\Phi^\circ = \Lambda$. So by (a), $\ker\rho\tau = 1$ and so also $\ker\tau = 1$.

(c) The first "iff" follows easily from (a) and (b). The second from 6.6.12(a).

(d) follows immediately from (a). □

Proposition 6.6.15 [**chevalley = steinberg**] *Every non-degenerate Steinberg group is isomorphic to a Chevalley group of the same type and over the same field.*

Proof: By 6.5.3 any non degenerate Steinberg groups is isomorphic to G/Z, where $G = G_\Phi(\mathbb{E})$ and Z is some subgroup of $Z(G)$. Let $\Xi = \tau^{-1}(Z) \leq \xi(\Lambda, \mathbb{E}^\sharp)$. By 6.6.13(b) there exists $\mathbb{Z}\Phi \leq \Lambda^\circ \leq \Lambda$ with $\Xi = \xi_{\Lambda^\circ}(\Lambda, \mathbb{E}^\sharp)$. Then by 6.6.13(a), we may choose $V^{\mathbb{K}}$ such that $\Lambda^\circ = \mathbb{Z}\Phi^\circ$. 6.6.14(a) and (b) now show that $\ker\rho = Z$ and so $G/Z \cong G_{\mathbb{E}}(M)$. □

6.7 Minimal weights modules

Lemma 6.7.1 [**minimal weight modules**] *Let $\lambda \neq 0$ be a minimal weight for $\check{\Phi}$, M a $\mathfrak{U}_{\mathbb{Z}}$ invariant lattice in $V^{\mathbb{K}}(\Lambda)$, $V = \mathbb{E} \otimes_{\mathbb{Z}} V(\mathbb{K}\lambda)$, $\Phi^\circ = \Phi_{V^{\mathbb{K}}}(H^{\mathbb{K}})$ and $G = G_{\mathbb{E}}(M)$.*

(a) [**a**] *$W(\Phi)$ acts transively on Φ°.*

(b) [**b**] *Let $n \in \mathbb{Z}^+$, $\alpha \in \Phi$ and $\mu \in \Phi^\circ$ with $\frac{x_\alpha^n}{n!} V_\mu \neq 0$. Then $n = 1$ and $(\mu, \check{\alpha}\,) = -1$.*

(c) [**c**] *Let $\alpha \in \Phi$. Then $\frac{x_\alpha^n}{n!} = 0$ for all $n \leq 2$ and $\chi_\alpha(t) = 1 + t x_\alpha$.*

(d) [**d**] *Let $a \in \alpha \in \Phi$, $\mu \in \Phi^\circ$ with $(\mu, \check{\alpha}\,) = -1$ and $v \in V_\mu$. Then $x_\alpha v = w_\alpha(v)$ and $\chi_\alpha(t) v = v + w_\alpha v$.*

(e) [**e**] *V is a simple $\mathbb{Z}G$-module and a simple $L^{\mathbb{E}}$-module.*

(f) [**f**] $\dim_{\mathbb{E}} V = |\Phi^\circ| = |W(\Phi)/W_{\Pi \cap \lambda}|$.

Proof: (a) Let $\mu \in \Phi^\circ$. Then by 3.5.5(a), there exists $w \in W(\Phi)$ such that $w(\mu)$ is dominant. By 5.4.10, $w(\mu) \in \Phi^\circ$. By 5.4.4, $w(\mu) \prec \lambda$ and by 3.7.20(b), λ is $\prec$-mimimal in Λ^+. Thus $w(\mu) = \lambda$.

(b) Since $\frac{x_\alpha^n}{n!} V_\mu = V_{\mu + n\alpha}$. Thus both μ and $\mu + n\alpha$ are in Φ° and so by (a), both μ and $\mu + n\alpha$ are minimal weights on $\check{\Phi}$. Thus

$$1 \geq (\mu + n\alpha, \check{\alpha}\,) = (\mu, \check{\alpha}\,) + 2n \geq 2n - 1 \geq 1$$

hence equality most hold for each of the preceeding inequalities. Therefore $n = 1$ and $(\mu, \check{\alpha}\,) = -1$.

(c) Follows immediately from (a) and the definition of $\chi_\alpha(t)$.

(d) By definition of w_α and (c) we have

$$(*) \quad w_\alpha = \chi_\alpha(1)\chi_{-\alpha}(-1)\chi_\alpha(1) = (1 + x_\alpha)(1 - x_{-\alpha})(1 + x_\alpha)$$

From $[x_\alpha, x_a] = h_\alpha$ we have $-x_{-\alpha} x_\alpha = -x_\alpha x_{-\alpha} + h_\alpha$ and so

$$(**) \quad (1 - x_{-a})(1 + x_\alpha) = 1 + x_\alpha - x_{-\alpha} - x_\alpha x_\alpha + h_\alpha$$

From (a) $x_a v = 0$ and since $v \in V_\mu$, $h_\alpha v = (\mu, \check{\alpha}\,) v = -v$. Thus from (**)

$$(***) \quad (1 - x_{-a})(1 + x_\alpha) v = v + x_\alpha v - v = x_\alpha v$$

From (b), $x_\alpha x_\alpha v = 2(\frac{x_\alpha^2}{2!} v = 0$ and so by (***)

$$(1 + x_\alpha)(1 - x_{-a})(1 + x_\alpha) v = x_\alpha v$$

So by (*), $w_\alpha c = \chi_\alpha(v)$. So the first statement in (d) holds. The second follows from the first and (c).

(e) Let $0 \neq I$ be either a $\mathbb{Z}G$-submodule of V or an $L^{\mathbb{E}}$-submodule of V. For $v \in V$ define $v_\mu \in V_\mu$ be $v = \sum_{\mu \in \Phi^\circ} v_\mu$. If $v \neq 0$ define $\operatorname{ht} v = \min\{\operatorname{ht} mu \mid \mu \in \Phi\circ\}$. Choose $0 \neq i \in I$ with $\operatorname{ht} i$ maximal. We claim that $i \in V_\lambda$. For this choose $\rho \in \Phi^\circ$ with $v_\rho \neq 0$ and $\operatorname{ht} \rho = \operatorname{ht} i$. Suppose that ρ is not dominant. Then there exists $\alpha \in \Pi$ with $(\rho, \check{\alpha}) < 0$. Since ρ is minimal, $(\rho, \check{\alpha}) = -1$. Thus by (d),

$$(\chi_\alpha(1) - 1)i_\rho == x_\alpha v_\rho = w_\alpha(i_\rho) \neq 0$$

On the otherhand by (c)

$$(\chi_\alpha(1) - 1)i = x_\alpha i = \sum_{\mu \in \Phi^\circ} x_\alpha i_\mu$$

Since $x_\alpha i_\mu \in V_{\mu+\alpha}$ and $x_\alpha i_\rho \neq 0$ we get $\operatorname{ht} x_\alpha v = \operatorname{ht}(\rho + \alpha) = \operatorname{ht} \rho + \operatorname{ht} \alpha > \operatorname{ht} \rho$, a contradiction to the maximality of $\operatorname{ht} i$.

Thus ρ is dominant and so by (a) and 3.5.5(a), $\rho = \lambda$. By definition, $i_\mu = 0$ for all μ of height less then $\operatorname{ht} i = \operatorname{ht} \rho$ and so by 3.4.2, $i = i_\rho = i_\lambda \in V_\lambda$.

Let $v \in V_\lambda$. Since V_λ is 1-dimensional there exists $k \in \mathbb{E}$ with $v = ki$. Since $\lambda \neq 0$ there exists $\alpha \in \Pi$ with $(\lambda, \check{\alpha}) \neq 1$ and since λ is minimal, $(\lambda, \check{\alpha}) = 1$. Hence $h_\alpha(k)i = k^{(\lambda, \check{\alpha})}i = ki = v$ and so $V_\lambda \leq I$. Let $\mu \in \Phi^\circ$. By (a) there exists $n \in N$ with $n(\lambda) = \mu$. Hence $V_\mu = V_{n(\lambda)} = nV_\lambda \leq I$. We conclude that $V = \sum_{\mu \in \Phi^\circ} V_\mu \leq I$. Thus (e) holds.

(f) From (a) and $\dim V_\lambda = 1$ we have $\dim V = |\Phi^\circ| = |W/C_W(\lambda)$. By 3.4.5(a) we have $C_W(\lambda) = W(\Pi \cap \lambda^\circ)|$ and so (f) is proved. □

Lemma 6.7.2 [**orbits on perp**] *Let α be a dominant root in Φ and $\lambda \in \overline{\mathfrak{C}}$. and $\Delta = W(\Phi) \cdot \lambda$ be the the orbit of λ under $W(\Phi)$. Then there is a 1-1 correspondence between the orbits of $W(\Pi \cap \alpha^\perp)$ on $\Delta \cap \alpha^\perp$ and the connected components of $\Pi \cap \lambda^\perp$ containing roots conjugate to α under $W(\Phi)$.*

Proof: Let $W = W(\Phi)$, $\Psi = W \cdot \alpha$ and $\Sigma = \{(\delta, \beta) \mid d \in \Delta, \beta \in \Psi, d \perp \beta\}$. Since W is transitive on Ψ and Δ there 1-1 correspondence between the orbits of $C_W(\alpha)$ on $\Delta \cap \alpha^\perp$, the orbits of W on Σ and the orbits of $C_W(\lambda)$ on $\Psi \cap \lambda^\perp$. By 3.4.5(a) $C_W(\alpha) = W(\Pi \cap \alpha^\perp)$ and $C_W(\lambda) = W(\Pi \cap \lambda^\perp)$. Also by 3.5.4(c) and 3.3.2(h) there is a 1-1 correspondence between the orbits of $W(\Pi \cap \lambda^\perp)$ on $\Psi \cap \lambda^\perp$ and the connected components of $\Pi \cap \lambda^\perp$ containing an element of Ψ. □

Corollary 6.7.3 [**orbits on minimal weights**] *Let α be a dominant root in Φ and λ a dominant minimal weight. Let $\Delta = W(\Phi) \cdot \lambda$ be the the orbit of λ under $W(\Phi)$. For $i \in \{-1, 0, 1\}$ let $\Delta_i = \{\delta \in \Delta \mid (\delta, \alpha) = i$. Then*

(a) [**a**] $\Delta = \Delta_{-1} \cup \Delta_0 \cup \Delta_1$.

(b) [**b**] $\omega_\alpha(Delta_1) = \Delta_{-1}$ *and* ω_α *fixes every member of* Δ_0.

(c) [**c**] *$W(\Pi\cap\alpha^{\perp})$ acts transitively on Δ_1 and Δ_{-1}.*

(d) [**d**] *There here is a 1-1 correspondence between the orbits of $W(\Pi\cap\alpha^{\perp})$ on Δ_0 and the connected components of $\Pi\cap\lambda^{\perp}$ containing roots conjugate to α under $W(\Phi)$.*

Proof: (a) Each $\delta\in\Delta$ is a minimal weight and so $(\delta,\alpha)\in\{-1,0,1\}$ and so (a) holds.

(b) $(\omega_\alpha(\delta),\alpha)=(\delta,\omega_\alpha(a))=(\delta,-\alpha)=-(\delta,\alpha)$ and $\Delta_0=\Delta\cap\alpha^{\perp}$. So (b) holds.

(c) We may assume that $\Delta_1\neq\emptyset$. Then by 3.5.5(c), $(\lambda,\alpha)=1$ and so by 3.5.5(d) applied with $e=\alpha$ and $d=\lambda$, $W(\Pi\cap\alpha^{\perp})$ is transitive on Δ_1. As $\omega_\alpha(\Delta_1)=\Delta_{-1}$ and ω_α commutes with $W(\Pi\cap\alpha^{\perp}$, $W(\Pi\cap\alpha^{\perp}$ is also tranistive on Δ_{-1}.

(d) Follows from 6.7.2. □

6.8 Steinberg Groups of type A_n

In this section we have a closer look at the Steinberg groups of type A_n. Let E_0 be the $n+1$ dimensional Euklidean space over $\mathbb{Q}$ with orthonormal basis $e_0,e_1,\dots e_n$. Let $\Phi=\{e_i-e_j\mid 0\leq i\neq j\leq n\}$ and $E=\mathbb{Q}E_0=\{\sum_{i=0}^n k_ie_i\mid k_i\in\mathbb{Q},\sum_{i=0}^n k_i=0\}$. Put $\alpha_k=e_{k-1}-e_k$. Th Then Φ is a roots system of type A_n in $\mathbb{E}$ with base $\Pi=\{\alpha_1,\alpha_2\dots,\alpha_k\}$.

Let V be a finite dimensional vector space over $\mathbb{K}$ with basis $v_0,v_1,\dots v_n$. Let $L=\mathfrak{sl}(V)$, the lie algebra of trace zero endomorphisms of V. Define $E_{ij}\in\mathrm{End}(V)$ by $E_{ij}v_k=\delta_{jk}v_i$. Define $x_{e_i-e_j}=E_{ij}$ and $h_{e_i-e_j}=E_{ii}-_{jj}$. Slightly abusing notations we write x_{i-j} for $x_{e_i-e_j}$ and h_{i-j} for $h_{e_i-e_j}$. Then x_{i-j} and h_{i-j} are in L.

Let $H=\mathbb{K}\langle h_\alpha\mid\alpha\in\Phi=\{\sum_{i=0}^n k_iE_{ii}\mid k_i\in\mathbb{K},\sum_{i=0}^n k_i=0$. Then H is an abelian subalgebra of L. For $e=\sum_{i=0^n}q_ie_i\in E$ define $\eta(e)\in H^*$ by $\tilde{e}(\sum k_iE_{ii})=\sum q_ik_i$. Then $\eta e=0$ if and only if $q_i=q_j$ for all i. Inparticular, $\eta\mid_{E_0}$ is $1-1$ and we identify E_0 with its image in H^*. Let $h=\sum k_iE_{ii}\in H$. It is easy to compute that $[h,x_{i-j}]=(k_i-k_j)x_{i-j}$ and so x_{i-j} is weight vector with weight e_i-e_j for H in V.

Also $[x_{i-j},x_{j-i}]=E_{ij}Eji-E_{ji}E_{ij}=E_{ii}-E_{jj}=h_{i-j}$ and
if $i\neq j\neq k\neq 1$, $[x_{i-j},x_{j-k}]=E_{ij}E_{jk}-E_{jk}E_{ij}=E_{ik}=x_{ik}$ and so
$(x_{i-j},\alpha_k,\mid 0\leq i\neq j\leq n.1\leq k\leq n\}$ is a Chevalley basis for L.

From $x_{i-j}v_k=\delta_{jk}v_i$ and $x_{i-j}^m=0$ for all $m\in\mathbb{N},\geq 2$ we see that $M=\mathbb{Z}\langle v_0,\dots v_n\rangle$ is a $\mathfrak{U}_{\mathbb{Z}}$ invariant lattice in V. Let $1\leq m\leq n$ and put $V_m=\bigwedge_{\mathbb{K}}^m V$ and $M_m=\bigwedge_{\mathbb{Z}}^m M$, with M_m viewed as a $\mathbb{Z}$-submodule of V_i. For $i\in\mathbb{N}$ let $\mathbb{N}_i=\{\{0,1,\dots i\}$ and for $I\subseteq\mathbb{N}_n$ with $|I|=m$ put $v_I=v_{i_1}\wedge v_{i_2}\wedge\dots v_{i_m}$ where $I=\{i_j\mid 1\leq j\leq m\}$ with $i_1<i_2\dots i_m$. Then $(v_I\mid I\subseteq\mathbb{N}_n,|I|=m\}$ is a $\mathbb{K}$ basis for V_i and a $\mathbb{Z}$-basis for M_i. It follows that M_i is an $\mathfrak{U}_{\mathbb{Z}}$ invarinant lattice in V_i.

Define $\delta_{Ij}=1$ if $j\in I$ and $\delta_{Ij}=0$ if $j\notin I$. A straighforward computation shows that

$$(*)\quad h_{i-j}v_I=(\delta_{Ii}-\delta_{Ij})v_I$$

Thus v_I is a weight vector for H on V_m of weight say $\lambda(I)$. Suppose that $I\neq\mathbb{N}_{m-1}$. Then there exists $j\leq 1$ with $j\in I$ but $j-1\notin I$. Then by (*) $h_{\alpha_j}v_I=-v_I$ and so $\lambda(I)$ is not dominant. If $I = N_{m-1}$, then $h_{a_j} v_I = \delta_{jm} v_I$ and so $\lambda(\mathbb{M}_{m-1}) = \alpha_m^*$. Thus α_m^* is the unique dominat weight for H on V_m and 5.4.11(b), $V_m \cong V(\alpha_m^*)$.

It now follows from (*) and or equally well from 3.8.1, α_m^* is minimal weight. Hence by 6.7.1 $V_{\mathbb{E}}(\alpha_m^*)$ is a simple $L^{\mathbb{E}}$ and a simple $G_{\mathbb{E}}(M_m)$-module. Note also that

$$V_{\mathbb{E}}(\alpha_m^*) \cong \mathbb{E} \otimes M_m \cong \bigwedge^m (E \otimes M) \cong \bigwedge^m V_{\mathbb{E}}(\alpha_1^*)$$

.

and that the image of L^E in $V_E := \mathbb{E} \otimes M$ is $\mathfrak{sl}(V_E)$.

To round up this section we will now show that $G_{\mathbb{E}}(M) = SL(V_{\mathbb{E}})$. Let $u \in \mathfrak{U}_{\mathbb{Z}}$. Abusing notations we will denote the element $1 \otimes u$ of $\operatorname{End}(V_{\mathbb{E}})$ also by u. Similarly we write v_i for $1 \otimes v_i$, V for $V_{\mathbb{E}}$ and $\chi_{i-j}(t) = \sum k = o^{\infty} t^i \frac{x_{i-j}^k}{k!} \in GL(V_{\mathbb{E}})$. Then $G := G_{\mathbb{E}}(M)$ is the subgroup of $GL(V)$ generated by all the $\chi_{i-j}(t)$. Since $\frac{x_{i-j}^k}{k!} = 0$ for all $k \leq 2$ we have $\chi_{i-j}(t) = 1 + t x_{i-j}$ and $\chi_{i-j}(t) \in Sl(V)$ and so

1° [**chiij**] $\quad \chi_{ij} v_k = v_k + t\delta_{jk} v_i$.

In particular, $\chi_{ij} \in SL(V)$ and so

2° [**g in sl**] $\quad G \leq Sl(V)$

Next we show:

3° [**op point**] *Let $i \in \mathbb{N}_n$ and for $i \neq j \in \mathbb{N}_n$ let $k_i \in \mathbb{E}$. Then there exists $g \in G$ with $gv_i = v_i$ and $gv_j = v_j + k_j v_i$ for all $i \neq j \in \mathbb{N}_n$.*

Just choose $g = \prod_{j \in \mathbb{N}_n \setminus \{i\}} \chi_{i-j}(k_j)$.

4° [**g transitive**] *Let $0 \neq v \in V$ and $i \in mbN$. Then there exists $k \in \mathbb{E}$ and $g \in G$ such that $kgv_i = v$ and if $n \geq 1$, then $k = 1$.*

Without loss $i = n$. Let $v = sv_n + \tilde{v}$ with $s \in \mathbb{E}$ and $\tilde{v} \in \mathbb{E}\langle v_0, \dots v_{n-1}\rangle$. If $\tilde{v} = 0$, then (4°) holds with $g = 1$ and $k = s$. So we may assume that $\tilde{v} \neq 0$. In particular, $n \geq 1$. By induction on n, there exists $\tilde{g} \in \langle \chi_{i-j}(t) \mid 0 \leq i \neq j \leq n-1. t \in \mathbb{E}\rangle$ and $k \in \mathbb{E}$ with $k\tilde{g}v_0 = \tilde{v}$. Note that $\tilde{g}v_n = v_n$ and so $\tilde{g}(sv_n + kv_0) = v$. Put $t = k^{-1}(s-1)$. Then $s = kt + 1$. By (3°) there exists $g_1 \in G$ with $g_1 v_n = kv_0 + v_n$ and $g_2 \in G$ with $g_2 v_0 = v_0 + tv_n$ and $g_2 v_n = v_n$. Then

$$\begin{aligned}
\tilde{g} g_2 g_1 v_n &= \tilde{g} g_2 (kv_0 + v_n) \\
&= \tilde{g} k g_2 (v_0 + k^{-1} v_n) \\
&= \tilde{g} k (v_0 + tv_n + k^{-1} v_n \\
&= \tilde{g} k v_0 + (kt+1) v_n \\
&= \tilde{g} k v_0 + s v_n \\
&= v
\end{aligned}$$

So (4°) holds.

5° [**g multi transitive**] *Let $0 \leq m \leq n$ and $w_0, w_1 \dots w_m$ be linearly independent in V. Then there exists $g \in G$ and $k \in \mathbb{E}$ such that $kgv_m = w_m$ and $gv_j = w_j$ for all $0 \leq j < m$ and if $m \neq n$, then $k = 1$.*

By (4°) there exists $g_1 \in G$ and $s \in \mathbb{E}$ with $sg_1v_0 = w_0$. If $n = 0$ we are done. So suppose that $n > 0$. Then we can choose $s = 1$. For $1 \leq j \leq m$ let $g_1^{-1}w_i = l_iv_0 + \tilde{w}_i$ with $l_i \in \mathbb{E}$ and $\tilde{w}_i \in \mathbb{E}\langle v_1, \dots v_n\rangle$. By induction on n there exists $\tilde{g} \in \langle \chi_{i-j} \mid 1 \leq h$ and $0 \neq k_i \in E$ with $k_i\tilde{g}v_i = \tilde{w}_i$, with $k_i = 1$ for $1 \leq i < m$ and if $m < n$, $k_m = 1$. Note that $\tilde{g}v_0 = v_0$. By (3°) there exists $g_2 \in G$ with $g_2v_0 = v_0$ and $g_2v_i = k_i^{-1}l_i + v_i$ for all $1 \leq i \leq m$. Put $g = g_1\tilde{g}g_2$. Then $gv_0 = g_1v_o = w_0$ and for all $1 \leq i \leq m$:

$$\begin{aligned}
g_1\tilde{g}g_2k_iv_i &= g_1\tilde{g}(k_i(k_i^{-1}l_iv_0 + v_i) \\
&= g_1\tilde{g}l_iv_0 + k_iv_i) \\
&= g_1l_iv_0 + \tilde{w}_i \\
&= g_1(g_1^{-1}(w_i)) \\
&= w_i
\end{aligned}$$

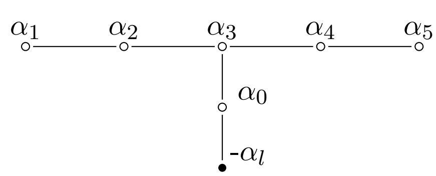
and so (5°) holds.

We are now able to show that $G = SL(V)$. Let $h \in SL(V)$. Then $(hv_i \mid 0 \leq i \leq n)$ is linearly independent so by (5°) there exist $g \in G$ and $k \in \mathbb{E}$ with $kgv_m = hv_m$ and $hv_i = hv_i$ for all $0 \leq i < n$. Thus $g^{-1}hv_m = kv_m$ and $g^{-1}hv_i = v_i$ for all $1 \leq i \leq n$. Thus $\det(g^{-1}h) = k$. On the otherhand $\det g = \det h = 1$ and so $k = 1$ and $g = h$. So $h \in G$ and $G = SL(V)$.

6.9 Steinberg groups of type E_6

In this section we explicitly determine the roots system of type E_6 and show if λ is a non-zero minimal weight for $\check{E}_6$, then $V(\lambda)$ has dimension 27 and that Chevalley group $G_{\mathbb{E}}(M(\lambda))$ is the universal Steinberg group of type E_6 over $\mathbb{E}$, where $M(\lambda)$ is a $\mathfrak{U}_{\mathbb{Z}}$ invariant lattice ein $V(\lambda)$.

According to 3.8.1 the affine diagram for $\Phi = E_6$ is



Here $\Pi = \{\alpha_i \mid 0 \le i \le 5\}$ and α_l is the highest root in Φ. Let $\Sigma = \{\alpha_i \mid 1 \le i \le 5\}$. So $\langle \Sigma \rangle$ is a root sytem of type A_5. We now will define explicit embedding of Φ into an eight dimensional euclidean space with orthormal basis $e_i \mid 0 \le i \le 7$. For $1 \le i \le 5$ we choose $\alpha_i = e_i - e_{i+1}$. Also let $\alpha_l = -e_0 + e_7$. Note that this implies $(\alpha, \alpha) = 2$ and $\alpha = \check{\alpha}$ for all $\alpha \in \Phi$. Let E be the 6 dimensional space spanned by Σ and α_l. So

$$E = \{\sum_{i=0}^{7} k_i e_i \mid k_0 + k_7 = 0, \sum_{i=1}^{6} k_i = 0\}$$

Let $\alpha_0 = \sum_{i=0}^{7} k_i e_i$. From $(\alpha_0, -\check{\alpha_l}) = -1$ we have $-k_0 + k_7 = 1$. Also $k_0 = -k_7$ and so $k_7 = \frac{1}{2} = -k_0$. From $(\alpha_0, \check{\alpha_i}) = 0$ for $i = 1, 2, 4, 5$ we have $k_1 = k_2 = k_3$ and $k_4 = k_5 = k_6$. From $(\alpha_0, \check{\alpha_3}) = -1$ we have $k_3 - k_4 = -1$. So $k_4 = k_3 + 1$. Also $0 = \sum_{i=1}^{6} k_i = 6k_1 + 3$ and so $k_1 = k_2 = k_3 = -\frac{1}{2}$ and $k_4 = k_5 = k_6 = \frac{1}{2}$. For subset I of $\mathbb{N}_7$ let $e_I = \sum_{i \in I} e_i$. Also write e_{ij} for $e_{\{ij\}}$ and so on. Then

$$\alpha_0 = \frac{1}{2}(-e_{0123} + e_{4567})$$

Since $\omega_{e_i - e_j}$ fixed each e_k for $k \neq i, j$ and interchanges e_i and e_j. It follows that the orbit of α_0 under $W(\Sigma)$ is

$$W(\Sigma) \cdot \alpha_0 = \{\frac{1}{2}(-e_{0ijk} + e_{lmn7} \mid \{1, 2, 3, 4, 5, 6\} = \{i, j, k, l, m, n\}.$$

To show that we found all positive roots by now we prove the following general lemma.

Lemma 6.9.1 **[orbits on long roots]** *Let Φ be a connected root system and put $\Sigma = \Pi \cap \lambda^{\perp}$. Let $\Delta \subseteq \Sigma$ such that Δ contains exactly one long root from each connected component of $\Gamma(\Sigma)$ which contains a long root. Then each orbit of $W(\Sigma)$ on Φ_l contains exactly one element of*

$$\Delta \cup \Pi_l \setminus \Sigma \cup -(\Pi_l \setminus \Sigma) \cup \{\alpha_l, -\alpha_l\}.$$

Proof: Note that by 3.5.5(d), $\Phi \cap \alpha_l = \langle \Sigma \rangle$. Let $\alpha \in \Phi_l$. Then $-2 \le (\alpha, \check{\alpha_l}) \le 2$. If $(\alpha, \check{\alpha_l}) = \pm 2$. Then $\alpha = \pm \alpha_l$.

Suppose that $(\alpha, \check{\alpha_l}) = 1$ and let δ be the unique conjugate of δ under $W(\Sigma)$ such that $-\delta$ is dominant on $\Pi \cap \lambda^{\perp}$. We need to show that $\delta \in \Pi$. Suppose not. Then by 3.3.2(e) there exist $\beta \in \Pi$ with $(\beta, \check{\delta}) > 0$ and $\gamma = \omega_\beta(\delta) \in \Phi^+$. Then $\beta \notin \Sigma$ and so $(\beta, \check{\alpha_l}) > 0$. Thus

$$0 \le (\gamma, \check{\alpha_l}) = (\beta, \check{\alpha_l}) - (\delta, \check{\beta})(\beta, \check{\alpha_l}))$$

It follows that $(\delta, \check{\beta}) = 1 = (\beta, \check{\alpha_l})$, β is long, $\gamma = \delta - \beta$, $(\delta, \check{\alpha_l}) = 0$ and $(\gamma, \check{\delta}) = 2 - 1 = 1$. So $\gamma \in \Phi \cap \alpha_l = \langle \Sigma \rangle$. Since $\gamma \in \Phi$ and $-\delta$ is dominant on Σ, $(\delta, \check{\gamma}) \le 0$, a contradiction.

If $(\alpha, \check{\alpha_l}) = -1$, then by the preceeding case $-\alpha$ is conjugate unde $W(\Sigma)$ to an unique element of $\Pi_l \setminus \Sigma$.

If $(\alpha, \check{\alpha}_l) = 0$, then $\alpha_l \in \Phi \cap \alpha_l^\perp = \langle \Sigma \rangle$ and so by 3.5.4(e) (applied to each connected component of$\langle \Sigma \rangle$, α is conjugate under $W(\Sigma)$ to an unique element of Δ. $\square$.

Back to $\Phi = E_6$. The lemma shows that

$$\begin{array}{rcl} \Phi^+ = \{ & e_i - e_j, & 1 \leq i < j \leq 6 \\ & \frac{1}{2}(-e_{0ijk} + e_{lmn7}, & \{1,2,3,4,5,6\} = \{i,j,k,l,m,n\} \\ & -e_0 + e_7 & \} \end{array}$$

Thus $|\Phi^+| = \binom{6}{2} + \binom{6}{3} + 1 = 15 + 20 + 1 = 36$ and $|\Phi| = 72$.

Let $\lambda = \alpha_6^*$, so $(\lambda, \alpha)_i = \delta_{6i}$. By 3.8.1 λ is a minimal weight. Let $\lambda = \sum_{i=0}^{7} k_i e_i$. Since $\lambda \perp \alpha_i$ for all $1 \leq i \leq 5$ we have $k_1 = k_2 = k_3 = k_4 = k_5$. Since $(\lambda, \alpha_6) = 1$, $k_5 - k_6 = 1$ and so $k_6 = k_5 + 1$. Thus $0 = \sum_{i=1^6} k_i = 6k_5 + 1$. Thus $k_i = -\frac{1}{6}$ for all $1 \leq i \leq 5$ and $k_6 = 1 - \frac{1}{6}$. Since $\lambda \perp \alpha_0$ we have $k_0 + k_1 + k_2 + k_3 = k_4 + k_5 + k_6 + k_7$. Also $k_7 = -k_0$ and so $2k_0 = 1$. So $k_0 = \frac{1}{2} = -k_7$. Thus

$$\lambda = \frac{1}{2}(e_0 - e_7) + e_6 + \frac{1}{6} e_{123456}$$

Let Φ° be the orbit of λ under $W(\Phi)$. Let $i \in \{-1, 0, 1\}$ let $\Phi_i^\circ = \{\mu \in \Phi^\circ \mid (\lambda, \check{\alpha}_l) = i$. By 6.7.3, $\Phi^\circ = \Phi_1^\circ \cup \Phi_0^\circ \cup \Phi_{-1}^\circ$ and $W(\Pi \cap \alpha_l^\perp)$ acts tranistively on Φ_{-1}°, Φ_0° and Φ_a°. So

$$\Phi_1^\circ = \{\mu_i := \frac{1}{2}(e_0 - e_7) + e_i + \frac{1}{6} e_{123456} \mid 1 \leq i \leq 6\}$$

Also $\omega_{\alpha_l}(\Phi_1^\circ) = \Phi_{-1}^\circ$ and so

$$\Phi_{-1}^\circ = \{\mu_{-i} := \frac{1}{2}(-e_0 + e_7) + e_i + \frac{1}{6} e_{123456} \mid 1 \leq i \leq 6\}.$$

Alos $(\mu_3, \alpha_0) = \frac{1}{2}(\frac{1}{2}(-1-1) - 1) = -1$ and so

$$\omega_{\alpha_0}(\mu_1) = \mu_3 + \alpha_0 = e_1 - \frac{1}{3} e_{123} + \frac{2}{3} e_{456} = -\frac{1}{3} e_{12} + \frac{2}{3} e_{3456}$$

Thus

$$\Phi_0^\circ := \{\mu_{ij} := -\frac{1}{3} e_{ij} + \frac{2}{3} e_{klmn} \mid \{1,2,3,4,5,6\} = \{i,j,k,l,m,m\}\}$$

Hence $|\Phi^\circ| = 6 + \binom{6}{2} + 6 = 27$.

By 3.8.1 $|\det \Pi| = 3$. Thus $|\Lambda / \mathbb{Z}\Phi| = 3$ and so $\Lambda = \mathbb{Z}\Phi + \mathbb{Z}\lambda = \mathbb{Z}\Phi^\circ$. So 6.6.14(c) implies that $G_{\mathbb{E}}(M(\Lambda)$ is a univeral Steinberg group. Thus the universal Steinberg group of type E_6 has a faithful simple module of dimension 27.

6.10 Automorphism of Chevalley groups

In this section we determine some automorphism Chevalley groups.

Lemma 6.10.1 [**base for orbit**] *Suppose that Φ is the disjoint union of the W invariant subsets Φ_1 and Φ_2. For $\Psi \subseteq \Phi$ and $i \in \{1,2\}$ put $\Psi_i = \Psi \cap \Phi_i$. Also put $\Sigma_1 = \bigcup \Pi_1^{W(\Pi_2)}$.*

(a) [**a**] *$W(\Phi_1)$ is normal in W and $W = W(\Phi_1)W(\Pi_2) = W(\Phi_1)W(\Phi_2)$.*

(b) [**b**] *Φ_1^+ is invarinant under $W(\Pi_2)$, Σ_1 is a base for Φ_1 and $\Phi_1^+ = \Phi_1 \cap \mathbb{N}\Sigma_1$.*

Proof: (a) Since Φ is W invariant, $W(\Phi_1)$ is normal in W. L Also $W = W(\Pi) = \langle W(\Pi_1), W(\Phi_2)\rangle = W(\Phi_1)W(\Pi_2) \leq W(\Phi_1)W(\Phi_2) \leq W$ and (a) holds.

(b) Let $\alpha \in \Pi_2$ and $\beta \in \Phi_1^+$. By 3.3.2(d), $\omega_\alpha(\beta)$ is positive. Thus Φ_1^+ is $W(\Pi_2)$ invariant.

Let $e \in \mathfrak{C}$. Then $(\Phi_1)_e^+ = \Phi_1^+$ and so by 3.2.6 $\Delta := \Phi_1^+ \setminus (\Phi_1^+ + \Phi_1^+)$ is a base for Φ_1. Since Φ_1^+ is $W(\Pi_2)$ invariant, so is Δ. Also $\Pi_1 \subseteq \Delta$ and so $\Sigma_1 \subseteq \Delta$.

Now let $\delta \in \Delta$. We will show by induction on $\operatorname{ht}\delta$ that $\delta \in \Sigma_1$. If $\operatorname{ht}\delta = 1$, then $\delta \in \Pi \cap Phi_1 = \Pi_1 \subseteq \Sigma_1$. Suppose now that $\operatorname{ht}\delta > 1$, then $\delta \notin \Pi$ and so by 3.3.2(e) there exists $\alpha \in \Pi$ with $(\delta, \alpha) > 0$, $\omega_\alpha(\delta) \in \Phi^+$ and $\operatorname{ht}\omega_\alpha(\delta) < \operatorname{ht}\delta$. Since Δ is obtuse, $\alpha \notin \Pi_1$. Thus $\alpha \in \Pi_2$ and so $\omega_\alpha(\delta) \in \Delta$ and by induction $\omega_\alpha(\delta) \in \Sigma_1$. Since Σ_1 is $W(\Pi_2)$ invariant, also $\delta \in \Sigma$.

Thus $\Sigma_1 = \Delta$ and (b) is proved. □

Lemma 6.10.2 [**auto for zphi**] *Let Φ be a connected root systems with two root lengths. Let $p = p_\Phi = \frac{(\alpha_l,\alpha_l)}{(\alpha_s,\alpha_s)}$. Define*

$$\overline{a} = a + p\mathbb{Z}\Phi, \quad \forall a \in \mathbb{Z}\Phi$$

and

$$\widetilde{b} = b + \mathbb{Z}(\Phi_s)\check{} + p\mathbb{Z}\check{\Phi}, \quad \forall b \in \mathbb{Z}\check{\Phi}.$$

Then

(a) [**a**] *$W(\Phi_s)$ acts trivially on $\overline{\mathbb{Z}\Phi_l}$ and $\widetilde{\mathbb{Z}\check{\Phi}}$.*

(b) [**b**] *$(\overline{\alpha} \mid \alpha \in \Pi_l)$ is an $\mathbb{F}_p$ basis for $\overline{\mathbb{Z}\Phi_l}$.*

(c) [**c**] *$(\widetilde{\check{\alpha}} \mid \alpha \in \Pi_l)$ is an $\mathbb{F}_p$ basis for $\widetilde{\mathbb{Z}\check{\Phi}}$.*

(d) [**d**] *There exists a unique $\mathbb{F}_p$-linear isomorphism $\rho : \overline{\mathbb{Z}\Phi_l} \to \widetilde{\mathbb{Z}\check{\Phi}}$ with $\rho\overline{\alpha} = \widetilde{\check{\alpha}}$ for all $\alpha \in \Phi_l$.*

Proof: (a) Let $\alpha \in \Phi_l$ and $\beta \in \Phi_s$. Then

$$(\alpha, \check{\beta}) = \frac{(\beta, \beta)}{(\alpha, \alpha)}(\beta, \check{\alpha}) = p(\alpha, \check{\alpha}) \in p\mathbb{Z}$$

Thus $\omega_\beta(\alpha) = \alpha + (\alpha, \check{\beta})\beta \in \alpha + p\mathbb{Z}\Phi$ and so $\overline{\omega_\beta(\alpha)} = \overline{\alpha}$. Thus ω_β acts trivially on $\overline{\mathbb{Z}\Phi_l}$. Since $W(\Phi_s) = \langle \omega_\beta \mid \beta \in \Phi_s \rangle$, also $W(\Phi_2 s)$ acts trivially on $\overline{\mathbb{Z}\Phi_l}$.

Next let $\alpha \in \Phi$ and $\beta \in \Phi_s$. Then

$$\omega_\beta(\check{a}) = \check{a} + (\check{,}\check{\alpha})\beta\beta = \check{\alpha} + (\check{\alpha}, \beta)\check{\beta} \in \check{a} + \mathbb{Z}(\Phi_s)^{\check{}} + p\mathbb{Z}\check{\Phi}$$

and so $\widetilde{\omega_\beta(\check{\alpha})} = \tilde{\check{\alpha}}$. Thus (a) holds.

(b) By 6.10.1(b), $\{w(\alpha) \mid w \in W(\Pi_s), \alpha \in \Phi_l\}$ is a base for Φ_l. Thus $\overline{w(\alpha)} \mid w \in W(\Pi_l), \alpha \in \Pi_l\}$ spans $\overline{\mathbb{Z}\Phi_l}$. By $\overline{w(\alpha)} = \overline{\alpha}$ and so $(\overline{\alpha} \mid \alpha \in \Pi_l)$ spans $\overline{\mathbb{Z}\Phi_l}$. Since Π is a $\mathbb{Z}$ basis for $\mathbb{Z}\Phi$, $\overline{\Pi}$ is an $\mathbb{F}_p$ basis for $\overline{\mathbb{Z}\Phi}$. Thus $\overline{\Pi}_l$ is linearly independent and (b) holds.

(c) To apply (b) to $\check{\Phi}$ we now put $\overline{b} = b + p\mathbb{Z}^\Phi$ for $b \in \mathbb{Z}$
$ch\Phi$. Then by (a) $\overline{(\Pi_s)^{\check{}}} = \overline{(\check{\Pi})_l}$ is an $\mathbb{F}_p$ basis for $\overline{(\check{\Phi})_l} = \overline{(\Phi_s)^{\check{}}}$. Also $\overline{\check{\Pi}}$ is an $\mathbb{F}_p$ basis for $\overline{\mathbb{Z}\check{\phi}}$. Thus $(\overline{\check{a}} + \overline{\mathbb{Z}(\Phi_s)^{\check{}}} \mid \alpha \in \Pi_l)$ is a $\mathbb{F}_p$-basis for $\overline{\mathbb{Z}\check{\Phi}}/\overline{\mathbb{Z}(\Phi_s)^{\check{}}}$. Thus (c) holds.

(d) By (b) and (c), there exists a unique $\mathbb{F}_p$-linear isomorphism $\rho : \overline{\mathbb{Z}\Phi_l} \to \widetilde{\mathbb{Z}\check{\Phi}}$ with $\rho(\overline{\alpha}) = \tilde{\check{\alpha}}$ for all $\alpha \in \Pi_l$. We need to show that $\rho(\overline{\alpha}) = \tilde{\check{\alpha}}$ for all $\alpha \in \Phi_l$. So let $\alpha\ in\Phi_l$ and pick $w \in W(\Phi)$ and $\beta \in \Pi$ with $\alpha = w(\beta)$. Then $\beta \in \Pi_l$. By 6.10.1(a) there exists $w_s \in W(\Phi_s)$ and $w_l \in W(\Pi_l)$ with $w = w_s w_l$. Put $\delta = w_l(\beta)$. Then $\delta \in \langle \Pi_l \rangle \subseteq \mathbb{Z}\Pi_l$ and so $\delta = \sum_{\alpha \in \Pi_l} n_\alpha \alpha$ for some $n_\alpha \in \mathbb{Z}^+$. Since δ is long we have $(\delta, \delta) = (\alpha, \alpha)$ for all $\alpha \in \Pi_l$ and so

$$\check{\delta} = \frac{2}{(\delta, \delta)} \sum_{\alpha \in \Pi_l} n_\alpha \alpha = \sum_{\alpha \in \Pi_l} n_\alpha \check{\alpha}.$$

Hence

$$\rho(\overline{\delta}) = \sum_{\alpha \in \Pi_l} n_\alpha \rho(\overline{a}) = \sum_{\alpha \in \Pi_l} n_\alpha \tilde{\check{a}} = \widetilde{\sum_{\alpha \in \Pi_l} n_\alpha} \check{\alpha} = \tilde{\check{d}}$$

By (a) we $\overline{w_s(\delta)} = \overline{\delta}$ and $\widetilde{w_s(\check{\delta})} = \tilde{\check{\delta}}$. Also $\alpha = w_s(\delta)$ and so

$$\rho(\overline{\alpha}) = \rho(\overline{w_s(\delta)}) = \rho(\overline{\delta}) = \tilde{\check{\delta}} = \widetilde{w_s(\check{\delta})} = \tilde{\check{\alpha}}$$

and so also (d) is proved.

Lemma 6.10.3 [**sum of roots**] *Let Φ be a connected root system with two root lengths. Let $\alpha, \beta \in \Phi$ with $\alpha + \beta \in \Phi$.*

(a) [**a**] *If α and β are long then $\alpha + \beta$ is long and $(\alpha + \beta)^{\check{}} = \check{\alpha} + \check{\beta} \in \check{\Phi}$.*

(b) [**b**] *If α is long and β is short, then $\alpha + \beta$ is short, $(\alpha + \beta)^{\check{}} = p_\Phi \check{\alpha} + \check{\beta}$ and $\check{\alpha} + \check{\beta} \in \check{\Phi}$.*

(c) [**c**] *If α and β are short and $\alpha+\beta$ is long, then $k_{\alpha\beta} = \pm p_\Phi$, $p_\phi(\alpha+\beta)\check{} = \check{\alpha} + \check{b}) \notin \Phi$.*

Proof: Note first that $(\alpha+\check{\beta})\check{} = \frac{2}{(\alpha+\beta,\alpha+\beta)}(\alpha+\beta)$ and so

$$(*) \quad (\alpha+\beta)\check{} = \frac{(\alpha,\alpha)}{(\alpha+\beta,\alpha+\beta)}\alpha + \frac{(\beta,\beta)}{(\alpha+\beta,\alpha+\beta)}\beta$$

(a) By 3.5.4(d), Φ_l is $\mathbb{Z}$-closed. So $\alpha+\beta$ is long. By (*) $(\alpha+\beta)\check{} = \check{\alpha}+\check{\beta}$ and (a) holds.

(b) Suppose that $\alpha+\beta$ is long. Then by (a) $\beta = (-\alpha)+(\alpha+\beta)$ is long, a contradiction. Thus $\alpha+\beta$ is short and by (*), $(\alpha+\beta)\check{} = p_\phi\check{\alpha}+\check{\beta}$. In particular, $s_{\check{\alpha}\check{\beta}} \neq 0$ and $\check{\alpha}+\check{\beta}$ is a root in $\check{\Phi}$.

(c) By 5.1.2(c) $r_{\alpha\beta}+1 = s_{\alpha\beta}\frac{(\alpha+\beta,\alpha+\beta)}{(\beta,\beta)} = s_{\alpha\beta}p_\phi$. By 5.1.2(d), $r_{\alpha\beta} \le p_\Phi$ and since $\alpha+\beta$ is a root $s_{\alpha\beta} \ge 1$. Thus $r_{\alpha\beta}+1 = p_\Phi$. By 5.1.8(c:c), $k_{\alpha\beta} = \pm(r_{\alpha\beta}+1 = p_\Phi$. By (*) $(\alpha+\beta)\check{} = \frac{1}{p_\Phi}(\check{a}+\check{\beta})$. Since $(\alpha+\beta)\check{} \in \check{\Phi}$, and $\check{\Phi}$ is a root system, 3.1.1(iv) implies that $\check{\alpha}+\check{\beta} \notin \Phi$. □

Lemma 6.10.4 [**matching kab**] *Let Φ be a connected root system and $\mathbb{K}$ a standard field. Then there exist Chevalley basis $(x_\alpha, h_\beta \mid \alpha\in\Phi, \beta\in\Pi)$ and $(x_{\check{\alpha}}, h_{\check{b}} \mid \alpha\in\Phi,\beta\in\Pi)$ for $L_{\mathbb{K}}(\Phi)$ and $L_{\mathbb{K}}(\check{P}hi)$, respectively, such that $k_{\alpha\beta} = k_{\check{\alpha}\check{\beta}}$ for all $\alpha\beta \in \Phi_l$.*

Proof: Let $L = L_{\mathbb{K}}(\Phi)$ and $L_l = \langle L_\alpha \mid \alpha\in\Phi_l\rangle_{Lie}$. Also let $\check{L} = L_{\mathbb{K}}(\check{P}hi)$ and $\check{L}_s = \langle L_\alpha \mid \alpha \in (\check{P}i)_s\rangle$. Let Σ be base for Φ_l. Note that $\check{} : \Phi_l \to \check{\Phi}_s, \alpha\to\check{\alpha}$ is an isomorphism of root systems.

Let $(x_{\check{\alpha}}, h_{\check{b}} \mid \alpha\in\Phi,\beta\in\Pi)$ and $(y_{\check{a}}, h_{\check{b}} \mid \alpha\in\Phi,\beta\in\Pi)$ be any Chevalley bases for L and $\check{L}$, respectively. Then $(x_\alpha, h_\beta \mid\in \Phi_l,\Sigma)$ is a Chevalley basis for L_l and $(y_{\check{a}}, h_{\check{b}} \mid\in \Phi_l,\Sigma)$ is a Chevalley basis for $\check{L}_s$. By 5.1.7 there exists Lie isomorphism $\sigma : L_l \to \check{L}_s$ and $\epsilon_\alpha \in \{\pm a\}, \alpha\in\Phi_l$ with with $\epsilon_\alpha = \epsilon_{-\alpha}$ and $\sigma(x_\alpha) = \epsilon_\alpha y_{\check{a}}$ for all $\alpha \in \pm\Sigma$. For $\alpha in \Phi_s$ pick $\epsilon_\alpha \in \Phi_s$ arbitrarily subject to $\epsilon_\alpha = \epsilon_{-\alpha}$. For each $a\in\Phi$ define $x_{\check{a}} = e_\alpha y_{\check{\alpha}}$. Then by 5.1.7 (applied with $\rho = \mathrm{id}_\Phi$), $(x_{\check{a}}, h_{\check{b}} \mid \alpha\in\Phi,\beta\in\Pi)$ is a Chevalley basis for $\check{L}$. Let $\alpha,\beta\in\Phi_l$ with $\alpha+\beta\in\Phi$. Then by 6.10.3 we have $\alpha+\beta\in\Phi_l$ and so $(\alpha+\beta)\check{} = \check{\alpha}+\check{b}$. We compute

$$\sigma([x_\alpha, x_\beta]) = \sigma(k_{\alpha\beta}x_{\alpha+\beta}) = k_{\alpha\beta}x_{(\alpha+\beta)\check{}} = k_{\alpha\beta}x_{\check{a}+\check{b}}$$

$$[\sigma(x_\alpha), \sigma(x_\beta)] = [x_{\check{\alpha}}, x_{\check{b}} = k_{\check{a}\check{b}}x_{\check{\alpha}+\check{b}}$$

Since σ is a Lie homomorphism, this implies $k_{\alpha\beta} = k_{\check{\alpha}\check{\beta}}$. □

Lemma 6.10.5 [**chi for adjoint**] *Let Φ be a roots system, $L = L_{\mathbb{E}}(\Phi)$ and $G = G_{\mathbb{E}}(L\mathbb{Z})$. Then*

(a) [**a**] *If α is long then $\frac{x_\alpha^n}{n!} * L = 0$ for all $n \ge 3$ and $\chi_\alpha(t) = 1 + tx_\alpha + t^2\frac{x_\alpha^2}{2!}$.*

(b) [**b**] *If α is not long then* $\frac{x_\alpha^n}{n!} * L = 0$ *for all* $n > p_\alpha$ *and* $\chi_\alpha(t) = \sum_{i=0}^{p_{\Phi\alpha}} t^i \frac{x_\alpha^i}{i!}$.

Proof: Let $\alpha\,\beta \in \Phi$ and $n > 2$. Then none of $(n+1)\alpha, n\alpha$ and $(n-1)\alpha$ is in Φ and so $\frac{x_\alpha^n}{n!}$ annihilates x_α, h_β and $x_{-\alpha}$. If$\beta + n\alpha$ is a root, then by 5.1.2(d), α is not long and $n \leq p_{\phi\alpha}$. $\square$

Lemma 6.10.6 [**action on l and wl**] *Let Φ be a connected root system with two root lengths. Let $\mathbb{E}$ be a field with $p := \operatorname{char}\mathbb{E} = p_\Phi$. Let $(x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$ be a Chevalley basis for $\mathbb{E}$. Put $L_s = \mathbb{E}\langle x_\alpha, h_\alpha \mid a \in \Phi_s\rangle$.*

(a) [**a**] *Let $\beta \in \Phi_s$.*

- *(a)* [**a**] *Let $\alpha \in \Phi_l$.*
  - *(a)* [**a**] $x_\alpha * h_\beta = 0.$
  - *(b)* [**b**] $x_\alpha * x_b = \begin{cases} k_{\alpha\beta}x_{\alpha+\beta} & \textit{if } \alpha+\beta \in \Phi_s \\ 0 & \textit{otherwise} \end{cases}$
  - *(c)* [**c**] $\frac{x_\alpha^p}{p!} * L_s = 0.$
- *(b)* [**b**] *Let $\alpha \in \Phi_s$. Then*
  - *(a)* [**a**] $x_\alpha * h_\beta = -(\alpha, \check{\beta}\,)x_\alpha$
  - *(b)* [**b**] $x_\alpha * x_\beta = \begin{cases} h_\beta & \textit{if } \alpha = -\beta \\ k_{\alpha\beta}x_{\alpha+\beta} & \textit{if } \alpha+\beta \in \Phi_s \\ 0 & \textit{otherwise} \end{cases}$
  - *(c)* [**c**] $\frac{x_\alpha^p}{p!} * h_\beta = 0$
  - *(d)* [**d**] $\frac{x_\alpha^p}{p!} * x_\beta = \begin{cases} x_\alpha & \textit{if } \alpha = -\beta \textit{ and } p = 2 \\ 0 & \textit{otherwise} \end{cases}$

(b) [**b**] *L_s is invariant under $\mathfrak{U}_\mathbb{Z}$, L_s is am ideal in L, $L_s = \langle L_\alpha \mid \alpha \in \Phi_s\rangle_{Lie}$, and $\widetilde{L} := L/L_s$ is a module for $\mathfrak{U}_\mathbb{Z}$*

(c) [**c**] *Let $\beta \in \Phi_l$. Then*

- *(a)* [**a**] *Let $\alpha \in \Phi_l$. Then*
  - *(a)* [**a**] $x_\alpha * \widetilde{h_\beta} = -(\alpha, \check{\beta}\,)\widetilde{x_\alpha}$
  - *(b)* [**b**] $x_\alpha * \widetilde{x_\beta} = \begin{cases} h_\beta & \textit{if } \alpha = -\beta \\ k_{\alpha\beta}\widetilde{x_{\alpha+\beta}} & \textit{if } \alpha+\beta \in \Phi_l \\ 0 & \textit{otherwise} \end{cases}$
  - *(c)* [**c**] $\frac{x_\alpha^p}{p!} * \widetilde{h_\beta} = 0$

(d) [**d**] $\frac{x_\alpha^p}{p!} * \widetilde{x_\beta} = \begin{cases} \widetilde{x_\alpha} & \text{if } \alpha = -\beta \text{ and } p{=}2 \\ 0 & \text{otherwise} \end{cases}$

(b) [**b**] *Let* $\alpha \in \Phi_s$. *Then*

(a) [**a**] $x_\alpha * \widetilde{L} = 0.$

(b) [**b**] $\frac{x_\alpha^p}{p!} * \widetilde{h_\beta} = 0$

(c) [**c**] $\frac{x_\alpha^p}{p!} * \widetilde{x_\beta} = \begin{cases} m_{\alpha\beta p}\, \widetilde{x_{p\alpha+\beta}} & \text{if } p\alpha + \beta \in \Phi_l \\ 0 & \text{otherwise} \end{cases}$

(d) [**d**] *Define* $f : \mathfrak{U}_{\mathbb{E}}(L_{\mathbb{Z}})) \to \mathrm{End}_{\mathbb{E}}(\widetilde{L})$ *by* $f(u)\widetilde{l} = \widetilde{ul}$ *for all* $u \in U_{\mathbb{E}}(L_{\mathbb{Z}}))T$ *and* $l \in L$. *Then*

(a) [**a**] *Let* $a \in \Phi_l$. *Then* $f(\chi_\alpha(t)) = 1 + tf(x_\alpha) + t^2 f(\frac{x_\alpha^2}{2!}$

(b) [**b**] *Let* $\alpha \in \Phi_s$. *Then* $f(\chi_\alpha(t)) = 1 + t^p f(\frac{x_\alpha^p}{p!}$.

(c) [**c**] $G := G_{\mathbb{E}}(L_{\mathbb{Z}})$ *acts faithfully on* $\widetilde{L}$, *that is* $f\,|_G$ *is 1-1.*

(e) [**e**] *Define* $g : \mathfrak{U}_{\mathbb{E}}(L_{\mathbb{Z}})) \to \mathrm{End}_{\mathbb{E}}(L_s)$ *by* $g(u)ll = ul$ *for all* $u \in U_{\mathbb{E}}(L_{\mathbb{Z}}))T$ *and* $l \in L_s$. *Then*

(a) [**a**] *Let* $a \in \Phi_l$. *Then* $g(\chi_\alpha(t)) = 1 + tg(x_\alpha)$.

(b) [**b**] *Let* $\alpha \in \Phi_s$. *Then* $f(\chi_\alpha(t)) = 1 + tg(x_\alpha) + t^2 g(\frac{x_\alpha^2}{2!}$

(c) [**c**] *G acts faithfully on* L_s, *that is* $g\,|_G$ *is 1-1.*

Proof: We will use 5.2.2 without further reference.

(a:a) $(\beta, \check{\alpha}) = p_\phi(\alpha, \check{\beta})$ and since $\mathrm{char}\,\mathbb{E} = p_\Phi$, $[x_\alpha, h_\beta] = -(\alpha, \check{\beta})x_\alpha = 0$. So (a:a:a) holds. α and β have different lenghts and $a \neq \pm\beta$. If $\alpha+\beta \in \Phi$ then by 6.10.3(b), $\alpha+\beta \in \Phi_s$ and so $[x_\alpha, x_\beta] = k_{\alpha\beta}x_{\alpha\beta}$. If $0 \neq \alpha + \beta \notin \Phi$ then $[x_\alpha, x_\beta] = 0$. Thus (a:a:b) holds. Since α is long, and β is short, $p\alpha + \beta$ is not a root and so $\frac{x^{x_\alpha}}{x_\alpha!}{}_\beta = 0$. $p\alpha$ is not a root and so $\frac{x_\alpha^p}{p!}h_\beta = 0$. Thus (a:a:c) holds.

(a:b) (a:b:a) is obvious. If $\alpha+\beta \in \Phi_l$ then by 6.10.3(c), we have $k_{\alpha\beta} = \pm p_\Phi$ and so since $\mathrm{char}\,\mathbb{E} = p_\Phi$, $[x_\alpha, x_\beta] = k_{\alpha\beta}x_{\alpha\beta} = 0$. Thus (a:b:b) holds. $p\alpha$ is not a root and so (a:b:c) holds. Since both α and β are short, $p\alpha + \beta$ is not a root unless $\beta = a$ and $p = 2$. In the latter case $\frac{2}{x_\alpha} * x_{-\alpha} = -x_\alpha = x_\alpha$ and (a:b:d) holds.

(b) Let $\alpha, \beta \in \Phi$. Indeed, by (a), L_s is invariant under x_α and $\frac{x_\alpha^p}{p!}$. If $n > p$, the by 6.10.5 $\frac{x_\alpha^n}{n!} * L_s = 0$. Suppose that $1 \leq n < p$. $n!$ is invertible in $\mathbb{E}$. Since L_s is an $\mathbb{E}$-subspace invariant under x_α^n and since $(n!)\frac{x_\alpha^n}{n!} = x_\alpha^n$ we conclude that L_s is invariant under $\frac{x_\alpha^n}{n!}$.

So L_s is invariant under all $\frac{x^n}{n!}{}_\alpha$, $n \in \mathbb{N}, \alpha \in \Phi$ and since this elements generate $\mathfrak{U}_{\mathbb{Z}}$ as a ring, L_s is invaraint under $\mathfrak{U}_{\mathbb{Z}}$. In particular $\widetilde{L} = L/L_s$ is a module for $\mathfrak{U}_{\mathbb{Z}}$. Since $L_{\mathbb{Z}} \leq \mathfrak{U}_{\mathbb{Z}}$ and L_s is a $\mathbb{E}$-subspace, L_s is invariant under $L = \mathbb{E} \otimes L_{\mathbb{Z}}$ and so L_s is an ideal in L. In particular, L_s is a Lie subalgebra and $\langle L_\alpha \mid \alpha \in \Phi_s\rangle \leq L_s$. But $h_\alpha = [x_\alpha, x_{-\alpha}] \in \langle L_\alpha \mid \alpha \in \Phi_s\rangle$ for all $\alpha \in \Phi_s$ and $\langle L_\alpha \mid \alpha \in \Phi_s\rangle = L_s$. Thus (b) holds.

(c:a) (c:a:a) is obvious. If $\alpha+\beta \in \Phi_s$, then $\widetilde{x_{\alpha+\beta}} = 0$. So (c:a:b) holds. (c:a:c) and (c:a:d) are proved justed as (a:b:c) and (a:b:d).

(c:b) By (c) $[x_\alpha, L] \leq [L_s, L] \leq L_s$ and (c:b:a) holds. (c:b:b) is obvious. α and β have different lenghts and so $\alpha \neq \pm\beta$. Also if $p\alpha+\beta$ is a roots then $p\alpha+\beta$ is long and (c:b:c) holds.

The first two statement of (a) and of (e) follow easily from (a)-(c) and 6.10.5. It remains to show that G acts faithfully on $\widetilde{L}$ and L_2. Note that $\mathbb{Z}\Lambda_{L^{\mathbb{K}}}(H^{\mathbb{K}}) = \mathbb{Z}\Phi$ and so by 6.6.11(d), $Z(G) = 1$. Let $\alpha \in \Phi$. It follows easily from the above that $f(\not{x}_\alpha) \neq f(\not{x}_{-\alpha})$ and $g(\not{x}_\alpha) \neq g(\not{x}_{-\alpha}$. Hence by 6.1.7 both $f(G)$ and $g(G)$ are non-degenerate. So by 6.5.3 $\ker f\mid_G = 1 = \ker g|mid_G$. □

Lemma 6.10.7 [**he**] *Let Φ be a root system, $\mathbb{E}$ a field, $p = \operatorname{char}\mathbb{E}$ and $\mathbb{F}_p = \mathbb{Q}$ if $p = 0$ and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ if $p \neq 0$. We view $\mathbb{F}_p$ as a subfield of $\mathbb{E}$.*

(a) [**a**] *There exists a $\mathbb{Z}$-linear map $f : \mathbb{E} \otimes_{\mathbb{Z}} \mathbb{Z}\check{\Phi} \to H_{\mathbb{E}}$ with $f(k \otimes \check{\alpha}) = kh_\alpha$ for all $k \in \mathbb{E}$, $\alpha \in \Phi$. Moreover, f is an $\mathbb{E}$-linear isomorphism.*

(b) [**b**] *There exists an isomorphism $g : \mathbb{E} \otimes_{\mathbb{Z}} \mathbb{Z}\check{\Phi} \to (\mathbb{E} \otimes_{\mathbb{F}_p} (\mathbb{F}_p \otimes_{\mathbb{Z}} Z\check{\Phi})$ with $g(k \otimes l \otimes d) = kl \otimes d$.*

(c) [**c**] *If $p = 0$ there exists an isomorphism $\tilde{f} : F_p \otimes_{\mathbb{Z}} Z\check{\Phi}) \to \mathbb{Q}\check{\Phi}$ with $\tilde{f}(k \otimes d) = kd$.*

(d) [**d**] *Suppose $p \neq 0$ and let $\overline{d} = d + p\mathbb{Z}\check{\Phi}$ for $d \in \mathbb{Z}\check{\Phi}$. The there exists an isomorphism $\tilde{f} : F_p \otimes_{\mathbb{Z}} Z\check{\Phi}) \to \overline{\mathbb{Z}\check{\Phi}}$ with $\tilde{f}(k \otimes d) = k\overline{d}$..*

Proof: (a) Since $H_{\mathbb{E}} = \mathbb{E} \otimes H_{\mathbb{Z}}$, (a) follows from 5.2.1(a).

(b) The maps $\mathbb{F}_p \otimes \mathbb{E}$ with $k \otimes e \to ke$ is an isomorphism. So (b) follows from the associtive property of tensor products.

(c) Clearly there exists a $\mathbb{F}_p$ linear map $\tilde{f} : F_p \otimes_{\mathbb{Z}} Z\check{\Phi}) \to \mathbb{Q}\check{\Phi}$ with $\tilde{f}(k \otimes d) = kd$. Also $\tilde{f}$ sends the $\mathbb{F}_p$-basis $1 \otimes \check{\alpha}, \alpha \in \Pi$ to the $\mathbb{F}_p$ basis $\check{\alpha}, \alpha \in \Pi$ of $\mathbb{Q}\check{\Phi}$.

(d) For $n \in \mathbb{Z}$ let $\overline{n} = n + p\mathbb{Z}$. Then it is easy to see that $\mathbb{F}_p \otimes \mathbb{Z} \to \mathbb{F}_p, \overline{a} \otimes n \to \overline{an}$ is an $\mathbb{Z}$-isomorphism. Since $\mathbb{Z}\check{\Phi}$ is free abelian (with basis $|\check{\Pi}|$) we see that (d) hold. □

Theorem 6.10.8 [**lie-hom from l to chl**] *Let Φ be a connected root system with two root lengths. Let $\mathbb{E}$ be a field with $\operatorname{char}\mathbb{E} = p_\Phi$. Let $(x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$ and $(x_{\check{\alpha}}, h_{\check{b}} \mid \alpha \in \Phi, \beta \in \Pi)$ be Chevalley basis for $L := L_{\mathbb{E}}(\Phi)$ and $\check{L} := L_{\mathbb{E}}(\check{P}hi)$, respectively with $k_{\alpha\beta} = k_{\check{\alpha}\check{\beta}}$ for all $\alpha\beta \in \Phi_l$. Then there exists a unique Lie-homomorphism $\rho : L \to \check{L}$ with $\rho(x_\alpha) = x_{\check{\alpha}}$ for all $\alpha \in \Phi_l$ and $\rho(x_\alpha) = 0$ for all $\alpha \in \Phi_s$. Moreover, $\operatorname{Im}\rho = \check{L}_s := \langle \check{L}_\alpha \mid \alpha \in (\check{\Phi})_s\rangle_{Lie}$ and $\ker\rho = L_s := \langle L_\alpha \mid \alpha \in \Phi_s\rangle_{Lie}$.*

Proof: Let $H = H_{\mathbb{E}}(\Phi)$, $\check{H} = H_E(\check{P}hi)$, $H_s = \mathbb{E}\langle h_\alpha \mid a \in \Phi_s\rangle$ and $\check{H}_s = \mathbb{E}\langle h_\alpha h_\alpha \mid a \in (\check{\Phi})_s\rangle$. Let $p = \operatorname{char}\mathbb{E} = p_\Phi$.

From 6.10.6(b) we have

1° [**1**] $L_s = H_s \oplus \bigoplus_{\alpha\in\Phi_s} L_\alpha$ *and* $\check{L}_s = \check{H}_s \oplus \bigoplus_{\alpha\in(\check{\Phi})_l} \check{L}_{\check{\alpha}}$

Next we show.

2° [**2**] *There exists a $\mathbb{E}$-linear map $\sigma : H \to \check{H}$ such that $\sigma(h_\alpha) = h_{\check{a}}$ for all $\alpha \in \Phi_l$, $\sigma(h_\alpha) = 0$ for all $\alpha \in \Phi_s$, $\ker\sigma = H_s$ and $\operatorname{Im}\sigma = \check{H}_s$.*

For $a \in \mathbb{Z}\Phi$ let $\overline{a} = a + p\mathbb{Z}\Phi$. For $a \in \mathbb{Z}\check{\Phi}$ let $\overline{a} = a + p\mathbb{Z}\check{\Phi}$ and $\widetilde{a} = \overline{\alpha} + \overline{\mathbb{Z}(\Phi_l)}\check{}$. By 6.10.7 there exists an isomorphism $g : H \to \mathbb{E}\otimes_{\mathbb{F}_p} \overline{\mathbb{Z}\check{\Phi}}$ with $g(h_\alpha) = 1 \otimes \overline{\check{a}}$ for all $\alpha \in \Phi$. Then $g(H_s) = \mathbb{E}\otimes_{\mathbb{F}_p} \overline{\mathbb{Z}(\Phi_l)}\check{}$ and so there exists an $\mathbb{E}$- isomorphism

$$g_1 : H/H_s \to \mathbb{E}\otimes_{\mathbb{F}_p} \widetilde{\mathbb{Z}\check{\Phi}} \text{ with } g_1(h_\alpha) = 1 \otimes \widetilde{\check{a}}\,\forall \alpha \in \Phi$$

By 6.10.2(d) there exists an $\mathbb{E}$-isomorphism

$$g_2 : \mathbb{E}\otimes_{\mathbb{F}_p} \widetilde{\mathbb{Z}\check{\Phi}} \to \mathbb{E}\otimes_{\mathbb{F}_p} \overline{\mathbb{Z}\Phi_l} \text{ with } g_2(1\otimes\widetilde{\check{\alpha}}) = 1\otimes\overline{\alpha}\,\forall\alpha\in\Phi_l$$

By 6.10.7 applied to $\check{\Phi}$ there exists an isomorphism

$$g_3 : \tilde{\mathbb{Z}\check{\Phi}} \to \mathbb{E}\otimes_{\mathbb{F}_p} \overline{\mathbb{Z}\Phi_l} \to \check{H} \text{ with } g_3(1\otimes\overline{\check{a}}) = h_{\check{\alpha}}\,\forall\alpha\in\Phi$$

For $h \in H$ define $\sigma(h) = g_3g_2g_1(h + H_s)$. Then clearly (2°) holds.

Let ρ be the unique $\mathbb{E}$-linear map with $\rho(x_\alpha) = x_{\check{\alpha}}$ for $\alpha \in \Phi_l$, $\rho(\alpha) = 0$ for $\alpha \in \Phi_l$ and $\rho(h) = \sigma(h)$ for $h \in H$. Note that

3° [**3**] $\ker\rho = L_s$ *and* $\operatorname{Im}\rho = \check{L}_s$.

Next we will show that ρ is a Lie-homomorphism, that is $\rho([l_1, l_2] = [\rho(l_1), \rho(l_2)]$ for all $l_1, l_2 \in L$.

4° [**3.5**] *If $l_1 \in L_s$, then $[rho(l_1), \rho(l_2)] = \rho([l_1, l_2])$.*

By 6.10.6(b) L_s is an ideal in L and so $[l_1, l_2] \in L_s$. Hence $\rho(l_1) = 0$ and $\rho([l_1, l_2]) = 0$. Thus (4°) holds.

Since $L = L_s + \mathbb{E}\langle x_\alpha, h_\alpha \mid \alpha \in \Phi_l\rangle$ and since ρ is $\mathbb{E}$-linear it remains to consider l_1 and l_2 of the form x_α or h_α for $\alpha \in \Phi_l$. For this let $\alpha, \beta \in \Phi_l$.

5° [**4**] $\rho([h_\alpha, h_\beta]) = 0 = [\rho(h_\alpha), \rho(h_b)]$

Clear since H and $\check{H}$ are abelian.

6° [**5**] $\rho([h_\alpha, x_\beta]) = (\alpha, \check{\beta})x_{\check{\beta}} = [\rho(h_\alpha), \rho(x_\beta)]$

$$[\rho(h_\alpha), \rho(x_\beta)] = [h_{\check{a}}, x_{\check{b}}] = (\check{\alpha}, \check{\check{\beta}})x_{\check{b}} = (\alpha, \check{\beta})x_{\check{\alpha}}.$$

7° [**7**] *If α is long, then $\rho([x_\alpha, x_{-\alpha}]) = h_{\check{\alpha}} = [\rho(x_\alpha), \rho(x_{-\alpha})]$.*

Obvious.

8° [**9**] *If $\alpha + \beta \in \Phi$, then $\alpha + \beta$ is long and $\rho([x_\alpha, x_\beta]) = k_{\alpha\beta} x_{\check{\alpha}+\check{b}} = [\rho(x_\alpha), \rho(x_\beta)]$.*

By 6.10.3(a), $\alpha + \beta$ is long and $(\alpha+\beta)\check{} = \check{\alpha} + \check{\beta}$. From the assumptions of the lemma $k_{\alpha\beta} = k_{\check{\alpha}\check{\beta}}$ and so (8°) holds.

9° [**11**] *If $0 \neq \alpha + \beta \notin \Phi$, then $\rho([x_\alpha, x_\beta]) = k_{\alpha\beta} x_{\check{\alpha}+\check{\beta}} = [\rho(x_\alpha), \rho(x_\beta)]$.*

We have $[x_\alpha, x_\beta] = 0$. If $\check{\alpha} + \check{\beta} \notin \check{\Phi}$ we conclude that (9°) holds. So suppose that $\check{\alpha} + \check{\beta} \in \check{\Phi}$. Since $\alpha + \beta$ is not a root we conlude from 6.10.3(applied to $\check{\Phi}$) that $\check{\alpha} + \check{\beta}$ is long in $\check{\Phi}$ and $k_{\check{\alpha}\check{b}} = p_\Phi$. Hence $[x_{\check{a}}, x_{\check{\beta}}] = 0$ and again (9°) holds.

From (4°) to (9°), ρ is a Lie-homomorphism and the Theorem is proved. □

Lemma 6.10.9 [**nice choice of kab**] *Let Φ be a connected root system, $\mathbb{K}$ a standard field and put $p = p_\Phi$. Then there exists Chevalley basis $(x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$ and $(x_{\check{\alpha}}, h_{\check{b}} \mid \alpha \in \Phi, \beta \in \Pi)$ for $L_{\mathbb{K}}(\Phi)$ and $L_{\mathbb{K}}(\check{P}hi)$, respectively, such that $k_{\alpha\beta} = k_{\check{\alpha}\check{\beta}}$ for all $\alpha\beta \in \Phi_l$ with $\alpha + \beta \in \Phi$ and $m_{\alpha\beta p} \equiv k_{\check{a}\check{b}} \mod p$ for all $\alpha \in \Phi_s, \beta \in \Phi_l$ with $p\alpha + \beta \in \Phi$.*

Proof: By 6.10.4 there exist Chevalley basis $(x_\alpha, h_\beta \mid \alpha \in \Phi, \beta \in \Pi)$ and $(y_{\check{\alpha}}, h_{\check{b}} \mid \alpha \in \Phi, \beta \in \Pi)$ for $L_{\mathbb{K}}(\Phi)$ and $L_{\mathbb{K}}(\check{P}hi)$, respectively, such that $k_{\alpha\beta} = k_{\check{\alpha}\check{\beta}}$ for all $\alpha\beta \in \Phi_l$ with $\alpha + \beta \in \Phi$. Let α be long and β be short with $\alpha + \beta \in \Phi$. Then $s_{\alpha\beta} \geq p$ and so by 5.1.2(d), $r_{\alpha\beta} = 0$ and so by 5.2.2(b), $m_{\alpha\beta p} = \pm\binom{p}{p} = \pm 1$. Also by 6.10.3(b), $\check{\alpha} + \check{\beta} \in \check{\Phi}$. Since $\check{\alpha}$ is long in $\check{\Phi}$, 5.1.2(d) implies $r_{\check{\alpha}\check{\beta}} = 0$ and so $k_{\check{\alpha}\check{\beta}} = \pm 1$.

If $p = 2$ we conclude $m_{\alpha\beta p} \equiv 1 \equiv k_{\check{\alpha}\check{b}} \mod 2$ and so the lemma holds in this case. So suppose that $p = 3$ and so Φ is of type G_2. For short root α pick long root $\beta = \beta(\alpha)$ with $p\alpha + \beta \in \Phi$. Define $\epsilon_\alpha = m_{\alpha\beta 3}\, \kappa_{\check{\alpha}\check{\beta}} \in \{\pm 1\}$. We will show that ϵ_α is independent of the choice of β and that $\epsilon_\alpha = e_{-\alpha}$. From Figure 3.1, there exists exactly one root $\delta \neq \beta$ with $\phi\alpha + \delta \in \Phi$. Let

The α sting through β is

$$\beta_0 = \beta, \beta_1 + \alpha, \beta_2 = \beta + 2\alpha, \beta_3 = b + 3\alpha$$

So

$$-\beta_3 = -\beta - 3\alpha, -\beta_2 = -\beta - 2\alpha, -\beta_1 = -\beta - \alpha, -\beta_0 = -\beta$$

is an α string through α and so $\delta = -\beta_3 = -\beta - 3\alpha$.

Since $m_{\alpha\beta 3} = \pm 1$, $\eta_{\alpha\beta} = \mathrm{sgn} k_{\alpha\beta}$ we have

$$m_{\alpha\beta 3} = \frac{1}{3!}\prod_{i=0}^{2} k_{\alpha\beta_i} = \prod_{i=0}^{2} \eta_{\alpha\beta_i}$$

and

$$m_{\alpha\delta 3} = \prod_{i=0}^{2} \eta_{\alpha-\beta_{3-i}} = \prod_{i=0}^{2} \eta_{\alpha\beta_{i+1}}.$$

Since $\alpha + \beta_i = \beta_{i+1}$, 5.1.9 implies $\eta_{\alpha b_i} = \eta_{-\beta i+1,\alpha} = -\eta_{\alpha,-\beta i+1}$. Thus

$$m_{\alpha\beta 3} = (-1)m_{\alpha\delta 3} = -m_{\alpha\delta 3}$$

On the otherhand by $-\check{\delta} = (\beta + 3\alpha)\check{} = \check{a} + \check{\beta}$ and by 5.1.9, $\epsilon_{\check{a}\check{\beta}} = -\epsilon_{\check{a}\check{\delta}}$. Thus

$$e_\alpha = m_{\alpha\beta 3}\ \kappa_{\check{\alpha}\check{\beta}} == m_{\alpha\delta 3}\, \kappa_{\check{\alpha}\check{\delta}}$$

is indeed independent of β.

Using 5.1.9

$$m_{-\alpha-\beta 3} = \prod_{i=0}^{2} \eta_{-\alpha-\beta_i} = -\prod_{i=0}^{2} \eta_{\alpha\beta_i} = m_{\alpha\beta 3}$$

and $k_{-\check{\alpha}-\check{b}} = -k_{\check{\alpha}\check{\beta}}$ and thus $\epsilon_{-\alpha} = e_\alpha$.

Let $x_{\check{\alpha}} = y_{\check{\alpha}}$ for $\alpha \in \Phi_l$ and $x_{\check{\alpha}} = \epsilon_\alpha y_{\check{\alpha}}$ for $\alpha \in \Phi_l$. By 5.1.7 $(x_{\check{\alpha}}, h_{\check{\beta}} \mid \alpha \in \Phi, \beta in \Pi)$ is Chevalley basis for $\check{L}$. Note that the change of basis did not effect the $k_{\check{\alpha}\check{\beta}}$ for $\alpha, \beta \in \Phi_l$ with $\alpha + \beta \in \Phi$. On the otherhand if $\alpha \in \Phi_s$ and $\beta \in \Phi_l$ with $p\alpha + \beta \in \Phi_l$, then $\check{\alpha} + \check{\beta} = p\alpha\check{+}\beta$ is long and so $k_{\check{\alpha}\check{\beta}}$ is changed by ϵ_α. Thus for the new base $m_{\alpha\beta p} = k_{\check{\alpha}\check{\beta}}$ and the lemma is proved.

Theorem 6.10.10 [**monomorphism**] *Let Φ be a connected rootsystem with two root lengths and $\mathbb{E}$ a field with $p := \mathrm{char}\,\mathbb{E} = p_\Phi$. Choose a Chevalley basis $(x_\alpha, h_\beta, \alpha \in \Phi, \beta \in \Pi)$ for $L := L_{\mathbb{E}}(\Phi)$ and $(x_{\check{\alpha}}, h_{\check{\beta}}, \alpha \in \Phi, \beta \in \Pi$ for $\check{L} := \mathbb{L}_{\mathbb{E}}(\check{\Phi})$ as in 6.10.9. Let $\widetilde{L} = L/L_s$. Let $\rho : L \to \check{L}$ be as in 6.10.8 and let $\widetilde{\rho} : \widetilde{L} \to \check{L}_s, \widetilde{l} \to \rho(l)$ be the isomorphism induced by ρ. By 6.10.6 there exists a homomorphisms $f : \mathfrak{U}_{\mathbb{E}}(L_{\mathbb{Z}})) \to \mathrm{End}_{\mathbb{E}}(\widetilde{L})$ and $\check{g} : \mathfrak{U}_{\mathbb{E}}(\check{L}_{\mathbb{Z}}) \to \mathrm{End}_{\mathbb{E}}(\check{L}_s)$. Define $\sigma : \mathrm{End}_{\mathbb{E}}(\widetilde{L}) \to \mathrm{End}_{\mathbb{E}}(\check{L}_s), r \to \widetilde{\rho} \circ r \circ \widetilde{\rho}^{-1}$.*

(a) [**a**] $\sigma(f(\frac{x_\alpha^n}{n!})) = g(\frac{x_{\check{a}}^n}{n!})$ *for $n = 1, 2$ and all $\alpha \in \Phi_l$.*

(b) [**b**] $\sigma(f(\frac{x_\alpha^n}{n!})) = 0 = g(\frac{x_{\check{a}}^n}{n!})$ *for all $n \geq 3$ and all $\alpha \in \Phi$*

(c) [**c**] $\sigma(f(\frac{x_\alpha^p}{p!})) = g(x_{\check{\alpha}})$ *for all $\alpha \in \Phi_s$.*

(d) [**d**] $\sigma(f(\frac{x_\alpha^n}{n!})) = 0 = g(\frac{x_{\check{\alpha}}^m}{m!})$ *for all* $n, m \in \mathbb{Z}^+$, $nn \neq p$, $m \neq 1$.

(e) [**e**] $\sigma f(\chi_\alpha(t^{p_\alpha}) = g(\chi_{\check{\alpha}})(t)$ *for all* $\alpha \in \Phi$, $t \in \mathbb{E}$.

(f) [**f**] *There exists a monomorphism*

$$\nu : G \to \check{G} \text{ with } \chi_\alpha(t)) \to \chi_{\check{\alpha}}(t^{p_\alpha})$$

for all $\alpha \in \Phi$, $t \in \mathbb{E}$.

(g) [**g**] *If* $\mathbb{E}$ *is perfect, that is* $\mathbb{E} = \mathbb{E}^p := \{k^p \mid k \in \mathbb{E}\}$, *then* ν *is an isomorphism.*

Proof: Observe that if $p \neq 2$ and $\sigma(f(x_\alpha)) = \sigma(g(x_\alpha)$, then also $\sigma(f(x_\alpha))^2 = \sigma(g(x_\alpha)^2$ and $\sigma(f(\frac{x_\alpha^2}{2!})) = \sigma(g(\frac{x_\alpha^2}{2!})$. (a)=(d) now follow easily from 6.10.6.

(e). Suppose first that α is long. Then $p_\alpha = 1$ an by (a) and (b)

$$f(\chi_\alpha(t)) = \sum_{i=0}^{\infty} t^i f(\frac{x_\alpha^i}{i!}) = \sum_{i=0}^{\infty} t^i g(\frac{x_{\check{\alpha}}^i}{i!} = g(\chi_{\check{\alpha}}(t))$$

Supppose next that α is short. Then $p_\alpha = p_\Phi = p$ and by (c) and (d)

$$f(\chi_\alpha(t^p)) = 1 + t^p f\frac{x_\alpha^p}{p!} = 1 + t^p g(x_{\check{\alpha}}) = g(\chi_{\check{\alpha}}(t^p))$$

So (e) holds.

Since both $f\mid_G$ and $g\mid_G$ are 1-1, (f) follows from (e).

(g) follows from (a).

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