Chapter 1

General Representation Theory

1.1 Basic Definitions

With ring we always mean a ring with 1 and all ring homomorphisms send 1 to 1.

**Definition 1.1.1.** Let $R$ be a ring and $M$ an abelian group. An $R$-module structure on $M$ is a function

$$
\cdot : \ R \times M \to M, \ (r, m) \to r \cdot m
$$

such that

(a) $r(m + \tilde{m}) = rm + r\tilde{m}$ for all $r \in R$ and $m, \tilde{m} \in M$.

(b) $(r + \tilde{r})m = rm + \tilde{r}m$ for all $r, \tilde{r} \in R$ and $m \in M$.

(c) $(r\tilde{r})m = r(\tilde{r}m)$ for all $r, \tilde{r} \in R$ and $m, \tilde{m} \in M$.

(d) $1m = m$.

An $R$-module is a pair $(M, \cdot)$, where $M$ is an abelian group and $\cdot$ is an $R$-module structure on $M$.

**Example 1.1.2.** Let $M$ be an abelian group.

(1) There exists a unique $\mathbb{Z}$-module structure on $M$. Indeed $1m = m$, $2m = (1 + 1)m = m + m$ and so inductively

$$
nm = m + \cdots + m
$$

for all $n \in \mathbb{Z}^+$ and $m \in M$. Also $0m = 0$ and $(-n)m = -(nm)$. So there exists at most one $\mathbb{Z}$-module structure on $M$. Conversely, it is easy to see that the above actually defines a $\mathbb{Z}$-module structure on $M$.

(2) $\text{End}(M)$ denotes the endomorphism ring on $M$. So, as a set, $\text{End}(M)$ consists of all group homomorphisms from $M$ to $M$. For $\alpha, \beta \in \text{End}(M)$, $\alpha + \beta$ and $\alpha \beta$ are defined by

$$(\alpha + \beta)(m) = \alpha(m) + \beta(m) \quad (\alpha \beta)(m) = \alpha(\beta(m)).$$

Then $M$ is an $\text{End}(M)$ module via $\alpha m = \alpha(m)$ for all $\alpha \in \text{End}(M), m \in M$. 

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(3) Let \( \mathbb{K} \) is a field. Then the \( \mathbb{K} \)-modules are exactly the \( \mathbb{K} \)-spaces, that is the vector spaces over \( \mathbb{K} \).

**Definition 1.1.3.** Let \( R \) be a ring and let \( M, N \) be \( R \)-modules.

(a) A (group) homomorphism \( \alpha : M \to N \) is called \( R \)-linear provided that \( \alpha(rm) = ra(m) \) for all \( r \in R \) and \( m \in M \).

(b) \( \text{Hom}_R(M, N) \) denotes the set of all \( R \)-linear homomorphisms from \( M \) to \( N \).

(c) \( \text{End}_R(M) \) denotes the set of \( R \)-linear endomorphisms of \( M \). So \( \text{End}_R(M) = \text{Hom}_R(M, M) \).

(d) \( \text{GL}_R(M) \) consists of all \( R \)-linear isomorphisms of \( M \).

(e) \( M \) and \( N \) are called isomorphic \( R \)-modules provided that there exists an \( R \)-linear isomorphism from \( M \) to \( N \).

Note that \( \text{End}_R(M) \) is a subring of \( \text{End}(M) \) and \( \text{GL}_R(M) \) is a subgroup of \( \text{Aut}(M) \).

**Remark 1.1.4.** Let \( R \) be a ring and let \( M, N \) be \( R \)-modules. Then

(a) \( \text{Hom}_R(M, N) \) is an subgroup of the abelian group \( \text{Hom}(M, N) \).

(b) \( \text{End}_R(M) \) is a subring of the ring \( \text{End}(M) \).

(c) \( \text{GL}_R(M) \) is a subgroup of group \( \text{Aut}(M) \).

**Lemma 1.1.5.** Let \( R \) be a ring and \( M \) an abelian group. Then there exists a natural 1-1 correspondence between set \( \text{Hom}_{\text{ring}}(R, \text{End}(M)) \) of ring homomorphism from \( R \) to \( \text{End}(M) \) and the set of \( R \)-module structure on \( M \).

**Proof.** Let \( \phi : R \to \text{End}(M) \) be a ring homomorphism. Define

\[
\cdot_{\phi} : \quad R \times M \to M, \quad (r, m) \mapsto \phi(r)(m).
\]

Then it is readily verified that \( \cdot_{\phi} \) is an \( R \)-module structure on \( M \).

Conversely, suppose that \( \cdot : R \times M \to M \) is an \( R \)-module structure. For \( r \in R \) define

\[
\phi_r : \quad M \to M, \quad m \mapsto rm.
\]

Then it is easy to verify that \( \phi_r \in \text{End}(M) \) and

\[
\phi : \quad R \to \text{End}(M), \quad r \to \phi_r
\]

is a ring homomorphism.

Note also that these two functions are inverse to each other. Indeed

\[
\phi \cdot_{\phi} (r)(m) = r \cdot_{\phi} m = \phi(r)(m)
\]

and so \( \phi \cdot_{\phi} = \cdot \). Also

\[
\cdot_{\phi} \cdot m = \phi_r(m) = rm = r \cdot m
\]

and so \( \cdot_{\phi} = \cdot. \)
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Observe, that for any ring $R$ there exists a unique ring homomorphism $\mathbb{Z} \to R$, namely $n \mapsto n1_R$. In particular, for any abelian group $M$ there exists a unique ring homomorphism $\mathbb{Z} \to \text{End}(M)$. Together with the preceding lemma gives a second proof that there exists a unique $\mathbb{Z}$-module structure on a given abelian group $M$.

Definition 1.1.6. Let $R$ be a ring, $M$ an $R$-module and $G$ a group. Then an $RG$-module structure on $M$ is a binary operation

$$\cdot : \quad G \times M \to M, \quad (g, m) \mapsto g \cdot m$$

such that

(i) $(g \cdot g)m = g(gm)$ for all $g, \tilde{g} \in G$ and $m \in M$.

(ii) $1_Gm = m$ for all $m \in M$.

(iii) $g(rm) = r(gm)$ for all $g \in G, r \in R$ and $m \in M$.

(iv) $g(m + \tilde{m}) = gm + g\tilde{m}$ for all $g \in G$ and $m, \tilde{m} \in M$.

An $RG$-module is a pair $(M, \cdot)$ where $M$ is an $R$-module and $\cdot$ is $RG$-module structure on $M$.

Note that (i) and (ii) in the preceding definition just say that $G$ acts on the set $M$, while (iii) and (iv) say for each $g \in G$ the function $m \mapsto gm$ is a $R$-linear homomorphism.

Definition 1.1.7. Let $R$ be a field, $M$ an $R$-module and $G$ a group. A representation of $G$ on $M$ over $R$ is a group homomorphism $\rho : G \to \text{GL}_R(M)$.

Lemma 1.1.8. Let $R$ be a ring, $M$ an $R$-module and $G$ a group. Then there exists a natural 1-1-correspondence between representations of $G$ on $M$ over $R$ and $RG$-module structures on $M$.

Proof. If $\rho : G \to \text{GL}_R(M)$ is a homomorphism, then $G \times M \to M, (g, m) \mapsto \rho(g)(m)$ is an $R$-module structure. Conversely if $G \times M \to M, (g, m) \mapsto gm$ is an $R$-module structure, define $\rho(g) \in \text{GL}_R(M)$ by $\rho(g)(m) = gm$. Then $\rho : G \to \text{GL}_R(M)$ is a representation of $G$ on $M$ over $R$. □

Definition 1.1.9. (a) Let $(A_i)_{i \in I}$ be a family of sets. Then

$$\times_{i \in I} A_i := \left\{ f : I \to \bigcup_{i \in I} A_i \bigg| f(i) \in A_i, \forall i \in I \right\}.$$  
We denote $f \in \times_{i \in I} A_i$ by $(f(i))_{i \in I}$. The set $\times_{i \in I} A_i$ is called the direct product of $(A_i)_{i \in I}$.

(b) Let $(A_i)_{i \in I}$ be family of monoids, then

$$\bigoplus_{i \in I} A_i := \left\{ (a_i)_{i \in I} \in \times_{i \in I} A_i \bigg| \{i \in I \mid a_i \neq 1_{A_i} \text{ is finite} \} \right\}.$$  
The set $\bigoplus_{i \in I} A_i$ is called the direct sum of $(A_i)_{i \in I}$.

(c) Let $A$ and $I$ be sets. Then $A^I = \times_{i \in I} A$.

(d) Let $A$ be a monoid and $I$ a set. Then $A^I = \bigoplus_{i \in I} A$.

Remark 1.1.10. (a) We often write $(a_i)$ for $(a_i)_{i \in I}$.
Remark 1.1.12. If \((A_i)_{i \in I}\) is a family of monoids (groups), then both \(\times_{i \in I} A_i\) and \(\bigoplus_{i \in I} A_i\) are monoids (groups) via \((a_i)(b_i) = (a_ib_i)\).

(c) If \((A_i)_{i \in I}\) is a family of rings, then \(\times_{i \in I} A_i\) is a ring. If \(I\) is finite, then also \(\bigoplus_{i \in I} A_i\) is a ring, but if \(I\) is infinite, \(\bigoplus_{i \in I} A_i\) might not have a multiplicative identity.

(d) If \((M_i)_{i \in I}\) is a family of \(R\)-modules then both \(\times_I M_i\) and \(\bigoplus_{i \in I} M_i\) are \(R\)-modules via \(r \cdot (m_i) = (rm_i)\).

Definition 1.1.11. Let \(R\) be a ring and \(G\) a monoid. Then the monoid ring \(R[G]\) for \(G\) over \(R\) is the ring with \(R[G] = R_G\) as an abelian group and multiplication defined by

\[
(r_g)_{g \in G} \cdot (s_h)_{h \in G} := \left( \sum_{g \in G, h \in H, r_g \neq 0, s_h \neq 0} r_g s_h \right)_{k \in G}.
\]

For \(g, h \in G\) define

\[
\delta_{gh} := \begin{cases} 1_R & \text{if } g = h \\ 0_R & \text{if } g \neq h \end{cases}
\]

We identify \(r \in R\) with \((\delta_{1r})_r\) in \(R[G]\) and \(h \in G\) with \((\delta_{gh})_g\) in \(R[G]\). Then \(R\) is a subring of \(R[G]\) and \(G\) as a submonoid of the multiplicative monoid \((R[G], \cdot)\).

Remark 1.1.12. Let \(R\) be a ring and \(G\) a monoid.

(a) \(R[G]\) is a ring with identity \(1_R 1_G\).

(b) Let \(r \in R\) and \(g \in G\). Then \(rg = gr = (\delta_{gh})_h\).

(c) For each \(a \in R[G]\) there exists a unique \((r_g)_{g \in G} \in R_G\) with \(a = \sum_{g \in G} r_g g\), namely \((r_g)_{g \in G} = a\).

(d) \(R[G]\) is generated by \(R\) and \(G\) as a ring.

Proof. (a) and (b) are readily verified. From (b) we conclude that \(\sum_{g \in G} r_g g = (r_g)_{g \in G}\) and so (c) holds. Now (d) follows from (c).

Example 1.1.13. Compute \((1 + (12) + (13)) \cdot ((123) + (23))\) in \(\mathbb{Z}_2[\text{Sym}(3)]\).

\[
(1 + (12) + (13)) \cdot ((123) + (23)) = (123) + (23) + (12)(123) + (12)(23) + (13)(123) + (13)(23) \\
= (123) + (23) + (23) + (123) + (12) + (132) \\
= (12) + (132)
\]

Lemma 1.1.14. Let \(R\) and \(S\) be rings and \(G\) a monoid. Let \(\alpha : R \to S\) be a ring homomorphism and \(\beta : G \to (S, \cdot)\) be a monoid homomorphism. Suppose that \(\alpha(r)\beta(g) = \beta(g)\alpha(r)\) for all \(r \in R\) and \(g \in G\). Then

\[
\gamma : R[G] \to S, \quad \sum_{g \in G} r_g g \mapsto \sum_{g \in G} \alpha(r_g)\beta(g)
\]

is the unique ring homomorphism \(\gamma : R[G] \to S\) with \(\gamma(r) = \alpha(r)\) and \(\gamma(g) = \beta(g)\) for all \(r \in R\), \(g \in G\).
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Proof. Define \( \gamma : R[G] \to S \) by \( \gamma(\sum_{g \in G} r_g s g) := \sum \alpha(r_g) \beta(g) \). Then

\[
\begin{align*}
\gamma(1_{R[G]}) &= \gamma(1_R \cdot 1_G) = \alpha(1_R) \cdot \beta(1_G) = 1_S \cdot 1_S = 1_S, \\
\gamma(g) &= \gamma(1_{R[G]}) = \alpha(1_R) \beta(g) = 1_S \beta(g) = \beta(g), \\
\gamma(r) &= \gamma(r1_G) = \alpha(r) \beta(1_G) = \alpha(r) 1_S = \alpha(g)
\end{align*}
\]

for all \( r \in R, g \in G \). Since \( \alpha \) is an additive homomorphism, \( \gamma \) is an additive homomorphism as well. To check

that \( \gamma \) is a multiplicative homomorphism we compute

\[
\gamma \left( \sum_{g \in G} r_g s g \cdot \sum_{h \in G} s_h h \right) = \gamma \left( \sum_k \left( \sum_{g, h \in G, gh = k} r_g s_h \right) k \right) = \sum_k \alpha \left( \sum_{g, h \in G, gh = k} r_g s_h \right) \beta(k)
\]

\[
= \sum_k \sum_{g, h \in G, gh = k} \alpha(r_g) \alpha(s_h) \beta(k) = \sum_k \sum_{g, h \in G, gh = k} \alpha(r_g) \alpha(s_h) \beta(gh)
\]

\[
= \sum_{g \in G} \alpha(r_g) \alpha(s_g) \beta(g) \beta(h) = \sum_{g \in G} \alpha(r_g) \beta(g) \alpha(s_g) \beta(h)
\]

\[
= \left( \sum_{g \in G} \alpha(r_g) \beta(g) \right) \cdot \left( \sum_{h \in G} \alpha(s_h) \beta(h) \right) = \gamma \left( \sum_{g \in G} r_g s g \right) \cdot \gamma \left( \sum_{h \in G} s_h h \right)
\]

Thus \( \gamma \) is a ring homomorphism.

Now suppose that \( \gamma : RG \to S \) is any ring homomorphism with \( \gamma(r) = \alpha(r) \) and \( \gamma(g) = \beta(g) \). Then

\[
\gamma(\sum_{g \in G} r_g s g) = \sum_{g \in G} \gamma(r_g) \gamma(s_g) = \sum_{g \in G} \alpha(r_g) \beta(g)
\]

and so \( \gamma \) is unique. \( \square \)

Lemma 1.1.15. Let \( R \) be a ring and \( G \) a group. Then there exists a natural 1-1 correspondence between \( R \)-modules and \( \text{End}(R)[G] \)-modules.

Proof. By 1.1.8 there exists a natural 1-1 correspondence between \( RG \)-modules and group homomorphisms \( \beta : G \to \text{GL}_g(M), M \) an \( R \)-module.

By 1.1.5 there exists a natural 1-1 correspondence between \( \text{End}(M) \)-modules and ring homomorphism \( \gamma : \text{End}(R)[G] \to \text{End}(M), M \) an abelian group.

Let \( M \) be an \( RG \)-module and \( \beta : G \to \text{GL}_g(M) \) the corresponding groups homomorphism. As \( \text{GL}_g(M) \subseteq \text{End}(M) \), \( \beta \) is also a monoid homomorphism \( \beta : G \to (\text{End}(M), \cdot) \). Since \( M \) is an \( R \)-module, the function \( \alpha : R \to \text{End}(M), r \mapsto (m \mapsto rm) \) is a homomorphism of rings. Let \( r \in R, g \in G \) and \( m \in M \). Since \( \beta(g) \in \text{GL}_g(M) \) we have \( r(\beta(g)m) = \beta(g)(rm) \).

We compute

\[
(\alpha(r)\beta(g))(m) = \alpha(r)(\beta(g)(m)) = r(\beta(g)(m)) = \beta(g)(rm) = \beta(g)(\alpha(r)(m)) = (\beta(g)\alpha(r))(m).
\]

Hence \( \alpha(r)\beta(g) = \beta(g)\alpha(r) \). Thus we can apply 1.1.14 and conclude that there exists a (unique) ring homomorphism \( \gamma : \text{End}(R)[G] \to \text{End}(M) \) with \( \gamma(r) = \alpha(r) \) and \( \gamma(g) = \beta(g) \). So \( M \) is an \( \text{End}(R)[G] \)-module.
Conversely, suppose $\cdot : R[G] \times M \to Ma \to am$ is an $R[G]$-module structure on the abelian group $M$. Recall that we view $R$ as a subring of $R[G]$. So $\cdot : R \times M \to M, (r, m) \to rm$ is an $R$-module structure. Also $G$ is a submonoid of $(R, \cdot)$ and so $\cdot : G \times M \to M, (g, m) \to M$ is an action of $G$ on $M$. Since $rg = gr$ in $R[g]$ we have

$$r(gm) = (rg)m = (gr)m = g(rm)$$

for all $r, g \in G$ and $m \in M$ and so $G \times M$ is an $RG$-module structure for $G$ on the $R$-module $M$. □

**Definition 1.1.16.** Let $R$ be a ring and $M$ an $R$-module.

(a) An $R$-submodule of $M$ is a subgroup $N$ of $M$ with $rn \in N$ for all $r \in R, n \in N$.

(b) $M$ is a simple $R$-module $M \neq 0$ and if $0$ and $M$ are the only $R$-submodules of $M$.

**Example 1.1.17.** Let $R$ be a ring, $I$ a set and $G$ a group action on $I$. Note that both $R^I$ and $R_I$ are $R[G]$-modules via

$$r \cdot (r_i)_{i \in I} = (rr_i)_{i \in I} \quad \text{and} \quad g \cdot (r_i)_{i \in I} = (r_{g^{-1}i})_{i \in I}$$

for all $r \in R, g \in G, (r_i)_{i \in I} \in R^I$.

(a) Let $J$ be a $G$-invariant subset of $I$. Identify, $(r_j)_{j \in J}$ with $(s_i)_{i \in I}$, where $s_i = r_i$ if $i \in J$ and $s_i = 0$ if $i \notin J$. Then $R^I$ is an $R[G]$-submodule of $R^I$ and $R_J$ an $R[G]$-submodule of $R_I$.

(b) Let $K$ be an ideal in $R$. Then $K^I$ is an $R[G]$-submodule of $R^I$ and $K_I$ is an $R[G]$-submodule of $R_I$.

(c) Define $W := \{(r_i)_I \in R_I \mid \sum_{i \in I} r_i = 0\}$. Then $W$ is an $R[G]$-submodule of $R_I$ and $R^I$.

(d) Define $Z := \{(r)_I \mid r \in R\}$. Then $Z$ is an $R[G]$-submodule of $R^I$.

(e) Put $n = |I|$. If $n = \infty$ then $W \cap Z = 0$. If $n < \infty$, then $W \cap Z = \{(r)_{i \in I} \mid r \in R, nr = 0\}$. If $R$ is a field, $n$ is finite and $\text{char } R \nmid n$, then $W \cap Z = 0$. If $R$ is a field, $n$ is finite and $\text{char } R \mid n$, then $Z \subseteq W$.

(f) Suppose $R$ is a field and $|I| \geq 2$. If $|I| = 2$ suppose also that $\text{char } p \neq 2$. Then $W + Z/Z$ is a simple $R[G]$-module.

**Definition 1.1.18.** Let $R$ be a ring and $M$ an $R$-module.

(a) $N \leq_R M$ means that $N$ is an $R$-submodule of $M$.

(b) For $I \subseteq M$ define $\langle I \rangle_R = \bigcap\{N \mid I \subseteq N \leq_R M\}$. $\langle I \rangle_R$ is called the $R$-submodule of $R$ generated by $I$. Observe that $R$ is the smallest $R$-submodule of $M$ containing $R$.

(c) For a family $(N_i)_{i \in I}$ of $R$-submodules in $M$, let

$$\sum_{i \in I} N_i := \left\{(n_i)_{i \in I} \in \bigoplus_{i \in I} N_i \right\}.$$ 

Observe that $\sum_{i \in I} N_i = \langle \bigcup_{i \in I} N_i \rangle_R$.

(d) We say that $M$ is the internal direct sum of the family $(N_i, i \in I)$ of $R$-submodules in $M$ if for each $m \in M$ there exists a unique $(n_i) \in \bigoplus_{i \in I} N_i$ with $m = \sum_{i \in I} n_i$. 
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(e) We say that a family \((N_i, i \in I)\) of \(R\)-submodules in \(M\) is linearly independent if \(N_i \neq 0\) for all \(i \in I\) and if \((n_i)_{i \in I} \in \bigoplus_{i \in I} N_i\) with \(\sum_{i \in I} n_i = 0\) implies \(n_i = 0\) for all \(i \in I\).

Lemma 1.1.19. Let \((N_i)_{i \in I}\) be a family of \(R\)-submodules of the \(R\)-module \(M\). Then \((N_i)_{i \in I}\) is linearly independent iff \((N_i)_{i \in J}\) is linearly independent for all finite subsets \(J\) of \(I\).

Proof. Obvious. \(\square\)

Lemma 1.1.20. Let \(M\) be an \(R\)-module \((M_i, i \in I)\) a family of non-zero \(R\)-submodules of \(M\). Let \(W = \sum_{i \in I} M_i\). Then the following are equivalent.

(a) \(W\) is the internal direct sum of \((M_i)_{i \in I}\).

(b) \((M_i)_{i \in I}\) is linearly independent.

(c) The function \(\phi : \bigoplus_{i \in I} M_i \to W, (m_i) \to \sum_{i \in I} m_i\) is an \(R\)-linear isomorphism.

(d) For each \(k \in I\), \(M_k \cap \bigoplus_{j \neq k} M_j = 0\).

Proof. Observe that \(\phi\) is \(R\)-linear and onto. The definition of an internal direct sum implies that \(\phi\) is a bijection if and only if \(W\) is the internal direct sum of the \((M_i, i \in I)\). Also \((M_i)_{i \in I}\) is linearly independent iff \(\ker \phi = 0\). So (a), (b) and (c) are equivalent.

Suppose (d) holds and let \(m \in M_k \cap \bigoplus_{j \neq k} M_j\). Then there exists \((m_j) \in \bigoplus_{j \neq k} M_j\) with \(\sum m_j = m\). Put \(m_k = -m\). Then \(\phi((m_i)) = 0\). Thus \(m_i = 0\) for all \(i\) and so \(m = -m_k = 0\). Thus (d) holds.

Suppose that (d) holds and let \((m_i) \in \bigoplus_{i \in I} M_i\) with \(\phi((m_i)) = 0\). Let \(k \in I\). Then

\[-m_k = \sum_{j \neq k} m_j \in M_k \cap \bigoplus_{j \neq k} M_j = 0.\]

Thus \(m_k = 0\), \((m_i) = 0\) and \(\phi\) is one to one. So (d) holds. \(\square\)

By the previous lemma, if \(\sum M_i\) is the internal direct sum of \((M_i)_{i \in I}\), then \(\sum M_i \cong \bigoplus_{i \in I} M_i\). In this case we usually identify \(\sum_{i \in I} M_i\) with \(\bigoplus_{i \in I} M_i\). In particular, we will write \(\sum M_i = \bigoplus M_i\) to indicate that \(\sum M_i\) is the internal direct sum of \((M_i, i \in I)\). We will also write just “direct sum” instead of “internal direct sum”.

Definition 1.1.21. Let \(R\) a ring, \(M\) an \(R\)-module, \(I\) a set and \(b = (b_i)_{i \in I} \in M^I\).

(a) \(M\) is a free \(R\)-module with respect to \(b\) if for all \(R\)-modules \(N\) and all \(c \in N^I\) there exists a unique \(R\)-linear functions \(f : M \to N\) with \(c_i = f(b_i)\) for all \(i \in I\) (that is with \(c = f \circ b\)).

(b) \(b\) is called an \(R\)-basis for \(M\) if for all \(m \in M\) there exists a unique \((r_i) \in \bigoplus_i R\) with \(m = \sum_{i \in I} r_i m_i\).

Example 1.1.22. (1) Let \(R\) be a ring and \(I\) a set. Put \(e_i := (\delta_{ij})_{j \in I}\). Then \((e_i)_{i \in I}\) is an \(R\)-basis for \(R_I\).

(2) Let \(R\) be a ring and \(G\) a monoid. Then \(G\) is an \(R\)-basis for \(R[G]\), or to be more precise \((g)_{g \in G}\) is an \(R\)-basis for \(R[G]\).

Lemma 1.1.23. Let \(R\) be a ring, \(I\) a set and let \(M\) and \(N\) free \(R\)-modules. Suppose that \(M\) is free with respect to \(b \in M^I\) and \(N\) is free with respect to \(c \in N^I\). Then there exists a unique \(R\)-isomorphism \(f : M \to N\) with \(f(b_i) = c_i\) for all \(i \in I\).

Proof. By definition of a free \(R\)-module there exist \(R\)-linear function \(f : M \to N\) and \(g : N \to M\) with \(c = f \circ b\) and \(b = g \circ c\). Then \((g \circ f) \circ b = g \circ (f \circ b) = g \circ c = b = \text{id}_M \circ b\). The uniqueness assertion in the definition of free \(R\)-modules now shows that \(g \circ f = \text{id}_M\). By symmetry \(f \circ g = \text{id}_N\). \(\square\)
Lemma 1.1.24. Let $I$ be a set, $M$ an $R$-module and $b = (b_i)_{i \in I} \in M^I$. Then the following are equivalent.

(a) The function $\alpha : R_I \to M, (r) \to \sum r_i b_i$ is an isomorphism.

(b) $b$ is an $R$-basis for $M$.

(c) $M$ is a free $R$-module with respect to $b$.

Proof. Clearly (a) and (b) are equivalent.

Suppose (b) holds, that is $b$ is an $R$-basis for $M$. Let $N$ be an $R$-module and $c \in N^I$, function. If $f : M \to N$ is linear with $f(b_i) = c_i$, then

\[ f(\sum_{i \in I} r_i b_i) = \sum_{i \in I} r_i c_i \]

for all $r \in R_I$. Hence $f$ is unique. Conversely, since $b$ is a basis for $M$, (*) defines an $R$-linear function $M \to N$ with $f(b_i) = c_i$. Thus $M$ is a free $R$-module with respect to $b$.

Suppose now that (c) holds, that is $M$ is a free $R$-module with respect to $b$.

By Example 1.1.22(a) $e := (e_i)_{i \in I}$ is a $R$-basis for $R_I$. As (b) implies (c) this shows that $M$ is a free $R$-module with respect to $e$. Hence 1.1.23 shows that there exists a $R$-isomorphism $\alpha : R_I \to M$ with $\alpha(e_i) = b_i$ for all $i \in I$. This gives (a). □

Definition 1.1.25. Let $R$ be a ring and $M$ an $R$-module.

(a) We say that $M$ is semisimple if $M$ is the direct sum of simple $R$-submodules.

(b) $M$ is directly indecomposable if $M$ is not the direct sum of two proper $R$-submodules.

(c) Let $N$ be an $R$-submodule of $M$. Then we say that $N$ is a direct summand of $M$ (or that $M$ splits over $N$) as an $R$-module if there exists an $R$-submodule $K$ of $M$ with $M = N \oplus K$.

Lemma 1.1.26. Let $S$ a set of simple $R$-submodules of the $R$-module $M$. Also let $N$ be a $R$-submodule of $M$ and suppose that $M = \sum S$.

(a) There exists a linearly independent subset $M$ of $S$ with $M = N \oplus \oplus M$.

(b) $N$ is a direct summand of $M$.

(c) $M = \oplus T$ for some linearly independent subset $T$ of $S$.

(d) $M/N \cong \oplus M$ for some linearly independent subset $M$ of $S$.

(e) $M/N$ is semisimple.

(f) $N \cong \oplus N$ for some linearly independent subset $N$ of $S$.

(g) $N$ is semisimple.

(h) If $N$ is simple then $N \cong S$ for some $S \in S$. 

1.1. BASIC DEFINITIONS

**Proof.** (a): In the following regular capital letter like \( M, N, W \) will denote submodules of \( M \). Calligraphic letters like \( \mathcal{D}, \mathcal{M} \) denotes subsets of \( S \) and so are sets of submodules of \( M \). Bourbaki letters like \( \mathfrak{B} \) and \( \mathfrak{D} \) will denote sets of subsets of \( S \). Let \( \mathfrak{B} \) consists of all the linearly independent subsets \( \mathcal{T} \) of \( S \) with \( N \cap \bigcap \mathcal{T} = 0 \). Since \( \emptyset \in \mathfrak{B} \) we see that \( \mathfrak{B} \neq \emptyset \). Order \( \mathfrak{B} \) by inclusion and let \( \mathcal{D} := \bigcup \mathfrak{B} \).

Let \( \mathcal{F} = \{F_1, \ldots, F_n\} \) be a finite subset of \( \mathcal{D} \). Then for each \( 1 \leq i \leq n \) there exists \( D_i \in \mathfrak{D} \) with \( F_i \in \mathcal{D}_i \). As \( \mathfrak{D} \) is a chain we may assume that \( D_1 \subseteq D_2 \subseteq \ldots \subseteq D_n \). Thus \( \mathcal{F} \subseteq D_n \). As \( D_n \) is linearly independent we conclude that also \( \mathcal{F} \) is linearly independent. Hence 1.1.19 shows that \( \mathcal{D} \) is a linearly independent subset of \( S \).

Let \( m \in M \cap \sum \mathcal{D} \). Then there exists \( D_i \in \mathcal{D}, 1 \leq i \leq n \) and \( d_i \in D_i \) with \( m = \sum_{i=1}^{n} d_i \). For each \( D_i \) there exists \( D_j \in \mathfrak{D} \) with \( D_i \subseteq D_j \). As above we may assume that \( D_1 \subseteq D_2 \subseteq \ldots \subseteq D_n \) Then \( D_i \in D_n \) for all \( 1 \leq i \leq n \) and so \( m \in N \cap \sum \mathcal{D}_n = 0 \).

Therefore \( N \cap \sum \mathcal{D} = 0 \) and so \( \mathcal{D} \in \mathfrak{B} \). So we can apply Zorn’s lemma to obtain a maximal element \( M \) in \( \mathfrak{B} \). Put \( W := \sum \mathcal{M} \). As \( \mathcal{M} \in \mathfrak{B} \) we know that \( \mathcal{M} \) is linearly independent and \( N \cap W = 0 \). In particular, \( W = \bigoplus \mathcal{M} \). If \( M = N + W \) we conclude that \( M = N \oplus W = N \oplus \bigoplus \mathcal{M} \) and (a) holds.

Suppose we may assume for a contradiction that \( M \neq N + W \). Since \( \sum \mathcal{S} = M \) there exists \( S \in \mathcal{S} \) with \( S \subseteq N + W \). Then \( S \subsetneq W \) and so also \( S \notin \mathcal{M} \). Put \( \mathcal{M}^* = \mathcal{M} \cup \{S\} \). Then \( \mathcal{M} \subseteq \mathcal{M}^* \) and we will obtain a contradiction by showing that \( \mathcal{M}^* \in \mathfrak{B} \). Note that \( \sum \mathcal{M}^* = W + S \). So we need to show that \( N \cap (W + S) = 0 \) and \( \mathcal{M}^* \) is linearly independent.

From \( S \subsetneq N + W \) we get \( S \neq (N + W) \cap S \) and since \( S \) is simple:

\[(N + W) \cap S = 0.
\]

Note that

\[W \subseteq (N + W) \cap (W + S) \subseteq W + S
\]

and so the modular law shows that

\[(N + W) \cap (S + W) = W + ((N + W) \cap (W + S) \cap S) = W + ((N + W) \cap S) = W + 0 = W.
\]

It follows that

\[N \cap \sum \mathcal{M}^* = N \cap N \cap (S + W) \subseteq N \cap (N \cap W) \alpha' (S + W) \subseteq N \cap W = 0.
\]

For \( (N \cap W) \cap S = 0 \) we have \( W \cap S = 0 \) and so

\[\sum \mathcal{M}^* = W + S = W \oplus S = (\bigoplus \mathcal{M}) \oplus S = \bigoplus \mathcal{M}^*.
\]

Thus \( \mathcal{M}^* \) is linearly independent and so \( \mathcal{M}^* \in \mathfrak{B} \), a contradiction to the maximality of \( \mathcal{M} \).

(a) follows from (a).

(b) follows from (a) applied with \( N = 0 \).

(c) follows from (a).

(d) follows from (a).

(e) follows from (a).

(f) Note that \( N \cong M/W \). So (f) follows from (d) applied to \( W \) in place of \( N \).

(g) follows from (f).

(h): Suppose \( N \) is simple. Then the set \( \mathfrak{N} \) from (c) only contains one element, say \( S \). So \( N \cong S \) and (h) is proved. \[\square\]
Lemma 1.1.27. Let $R$ be a ring and $M$ an $R$-module. Then the following are equivalent

(a) $M$ is a sum of simple $R$-modules.

(b) Every $R$-submodule of $M$ is semisimple.

(c) $M$ is a semisimple $R$-module.

(d) Every non-zero $R$-submodule of $M$ is a direct summand of $M$ and contains a simple $R$-submodule.

(e) If $N$ is an $R$-submodule of $M$ with $N \neq M$, then there exists simple $R$-submodule $S$ with $S \subseteq N$.

Proof. By 1.1.26(g), (a) implies (b). Clearly (b) implies (c).

Suppose that (c) holds and let $N$ be a non-zero $R$-submodule of $M$. By 1.1.26(b) $N$ is a direct summand of $M$ and by 1.1.26(g) $N$ is semisimple. So since $N \neq 0$, $N$ has a simple submodule.

Suppose now that (d) holds and let $N \neq M$ be an $R$-submodule of $M$. By (d) $M = N \oplus W$ for some $R$-submodule $W$. Since $N \neq M$, $W \neq 0$. Thus (d) implies $W$ has a simple $R$-submodule $S$. Since $N \cap W = 0$, $S \subseteq N$ and (e) holds.

Suppose (e) holds. Let $N$ be the sum of all the simple $R$-submodules in $M$. If $N \neq M$, then (e) implies the existence of a simple $R$-submodule $S$ of $M$ with $S \subseteq N$. But this contradicts the definition of $N$. So $N = M$ and (a) holds.

□

Corollary 1.1.28. Let $M$ semisimple $R$-module. Then all $R$-sections of $M$ are semisimple.

Proof. Let $A \leq B \leq M$ be $R$-submodules of $M$. Then 1.1.26(g) implies that $B$ is semisimple. Then 1.1.26(c) applied to $(A, B)$ in place of $(N, M)$ shows that $B/A$ is semisimple.

□

Definition 1.1.29. Let $R$ be a ring.

(a) $\mathcal{S}(R)$ is set of all isomorphism classes of simple $R$-modules.

(b) For an $R$-module $M$ and $S \in \mathcal{S}(R)$ define $M_S := \sum \{ S \in S \mid S \leq_R M \}$.

Lemma 1.1.30. Let $R$ be a ring and $M$ a semisimple $R$-module. Then

$$M = \bigoplus_{S \in \mathcal{S}(R)} M_S.$$ 

Proof. Let $S$ be a simple $R$ module and $S$ the class of $R$-modules isomorphic to $S$. Then $S \in S \in \mathcal{S}(R)$. It follows that $S \leq S_S \leq \sum S \in \mathcal{S}(R) M_S$. As $M$ is a semisimple $R$-module we know that $M$ is sum of it simple $R$-modules and we conclude that

$$M = \sum_{S \in \mathcal{S}(R)} M_S$$

Let $S \in \mathcal{S}(R)$ and define $W := \sum_{T \in \mathcal{S}(R), S} M_T$. Suppose for a contradiction that $M_S \cap W \neq 0$. Since $M$ is semisimple 1.1.27 shows that also $M_S \cap W$ is a semisimple $R$-module and so there exists a simple $R$-submodule $U$ of $M_S \cap W$. Then

$$U \leq M_S = \sum \{ S \in S \mid S \leq R M \}$$

and so 1.1.26(h) shows that $U \cong_R S$ for some $S \in S$ with $S \leq R M$. Since $S$ is an isomorphism class of simple $R$-modules this implies that $U \in S$. Similarly,
\[ U \subseteq W = \sum \{ T \in \mathcal{T} \mid S \subseteq T \in \mathcal{S}(R), T \leq R \} M \]

and \([1.1.26(\text{h})]\) shows that \( U \cong T \) for some \( T \in \mathcal{T} \) with \( T \leq R \) and \( S \subseteq \mathcal{S}(R) \) with \( \mathcal{T} \neq S \). Since \( \mathcal{T} \) is isomorphism class of simple \( R \)-modules this implies that \( U \in \mathcal{T} \). Hence \( U \in \mathcal{T} \cap \mathcal{S} \) and so \( \mathcal{T} = \mathcal{S} \), a contradiction.

This contradiction shows that \( M_S \cap W = 0 \) and so \( M \) is the internal direct sum of the family \( (M_S)_{S \in \mathcal{S}(R)} \).

\[ \square \]

### 1.2 Krull-Schmidt Theorem

**Definition 1.2.1.** A ring with a largest proper left ideal is called a local ring.

**Lemma 1.2.2.** Let \( G \) be a monoid and \( a, r, l \in G \). If \( la = ar = 1 \), then \( a \) is unit and \( r = l \) is the inverse of \( a \).

**Proof.** \( l = l1 = l(ar) = (la)r = 1r = 1 \). \[ \square \]

**Lemma 1.2.3.** Let \( \alpha : A \rightarrow B \) be homomorphism of \( R \)-modules and \( D \) an \( R \)-submodule of \( A \)

(a) If \( \alpha|_D \) is 1-1, then \( D \cap \ker \alpha = 0 \).

(b) If \( \alpha|_D \) is onto, then \( A = D + \ker \alpha \).

(c) If \( \alpha|_D \) is an isomorphism, \( A = D \oplus \ker \alpha \).

**Proof.** (a) is obvious.

(b): Let \( a \in A \) and pick \( d \in D \) with \( \alpha(a) = \alpha(d) \). Then \( a - d \in \ker \alpha \) and so \( a = d + (a - d) \in D + \ker \alpha \).

(c) follows from (a) and (b). \[ \square \]

**Lemma 1.2.4.** Let \( R \) be a non-zero ring. Let \( R^* \) be the set of units in \( R \) and put \( I := T \setminus R^* \).

(a) Suppose \( I \) is an ideal in \( R \). Then \( I \) is the largest proper left ideal of \( R \) and \( I \) is the largest proper right ideal in \( R \).

(b) \( R \) is local ring if and only if \( I \) is an ideal in \( R \).

**Proof.** (a) Suppose first that \( I \) is an ideal. Since \( 1 \notin I \) we see that \( I \) is a proper left ideal in \( R \). Let \( J \) be any proper left ideal in \( R \) and let \( j \in J \). Then \( Rj \subseteq J \neq R \) and so \( Rj \neq R \). Hence \( j \) is not a unit in \( R \) so \( j \notin I \). Thus \( J \subseteq I \) and so \( I \) is the largest proper left ideal of \( R \). By symmetry, \( I \) is also the largest proper right ideal in \( R \).

(b) If \( I \) is an ideal in \( R \), then (a) shows that \( I \) is a largest proper ideal of \( R \). Thus \( R \) is a local ring.

Suppose next that \( R \) is a local ring. Then \( R \) has largest proper ideal \( J \). We will first show that \( J \) is also a right ideal in \( R \). For this let \( r \in R \).

Suppose for a contradiction that \( Jr = R \). View \( R \) as a \( R \)-module by left multiplication. Then \( \alpha : R \rightarrow R, t \rightarrow tr \) is \( R \)-linear and so \( \ker \alpha \) is a left ideal in \( R \). Also \( \alpha|_J \) is onto and so by \([1.2.3(b)]\), \( R = J + \ker \alpha J \). It follows that \( \ker \alpha \not\subseteq J \). As \( \ker \alpha \) is left ideal and \( J \) is the largest proper left ideal this implies that \( R = \ker \alpha \). It follows that \( R = Jr = 0 \), contrary to the hypothesis of the lemma.

Thus \( Jr \neq R \). Hence \( Jr \) is a proper left ideal in \( R \) and since \( J \) is the largest proper left ideal if \( R \) we get \( Jr \subseteq J \). Thus \( J \) is a right ideal. As \( J \) is a left ideal this shows that \( J \) is an ideal in \( R \).
Next we will show $J = I$. As $I = R \setminus R^*$ we see that $J = I$ if and only if $R \setminus J = R^*$. If $r \in R^*$, then $Rr = R \not\subseteq J$. Thus $r \notin J$ and so $R^* \subseteq \setminus J$.

Now let $r \in R \setminus J$. Since $r \in Rr$, this gives $Rr \not\subseteq J$. As $Rr$ is left ideal and $J$ is the largest proper left ideal this shows that $Rr = R$. In particular, there exists $s \in R$ with $sr = 1$. If $s \in J$, then, since $J$ is a right ideal, also $1 = sr \in J$, a contradiction. Thus $s \notin J$. So $s$ fulfills the same hypothesis as $r$ and thus $ks = 1$ for some $k \in R$. We proved that $ks = 1$ and $sr = 1$. Thus by 1.2.4 the set $\langle s \rangle$ is an ideal in $R$ with inverse of $r$. Hence $r \in R^*$ and so $R \setminus J = R^*$. We proved that $R \setminus J = R^*$ and so $I = J$. As $J$ is an ideal of $R$ this shows that $I$ is an ideal of $iR$. Thus (b) is proved. \hfill \qedsymbol

**Lemma 1.2.5.** Let $R$ be a local ring and $r_1, \ldots, r_n \in R$ such that $r_1 + r_2 + \ldots + r_n$ is a unit. Then $r_i$ is a unit for some $1 \leq i \leq n$.

**Proof.** By 1.2.4 the set $I$ of non-units is an ideal in $R$. If $r_i \in I$ for all $1 \leq i \leq n$, then also $r_1 + r_2 + \ldots + r_n \in I$, a contradiction. \hfill \qedsymbol

**Lemma 1.2.6.** Let $\alpha : A \to B$ and $\beta : B \to C$ be $R$-linear functions such that $\beta \circ \alpha$ is invertible.

(a) $\alpha$ is 1-1 and $\alpha : A \to \text{Im} \, \alpha$ is an isomorphism.
(b) $\beta$ is onto and $\beta|_{\text{Im} \, \alpha}$ is an isomorphism.
(c) $B = \text{Im} \, \alpha \oplus \ker \beta$.
(d) If, in addition, $B$ is directly indecomposable and $A \neq 0$, then both $\alpha$ and $\beta$ are isomorphisms.

**Proof.** (a) and (b) are readily verified.

(a) From (b) we know that $\beta|_{\text{Im} \, \alpha}$ is an isomorphism. Thus 1.2.3((c)) shows that $B = \text{Im} \, \alpha \oplus \ker \beta$.

(b) Suppose now that $B$ is directly indecomposable. By (c) $B = \text{Im} \, \alpha \oplus \ker \beta$ and so either either $\text{Im} \, \alpha = 0$ and $\ker \beta = B$ or $\text{Im} \, \alpha = B$ and $\ker \beta = 0$. In the first case $\text{Im} \, \alpha = 0$, by (a) we know $A \cong \text{Im} \, \alpha$, so also shows that $A = 0$, a contradiction. In the second case $\text{Im} \, \alpha = B$ and (a) and (b) show that both $\alpha$ and $\beta$ are both isomorphisms. \hfill \qedsymbol

**Definition 1.2.7.** Let $M$ be an $R$-module.

(a) We say that $M$ fulfills the ascending chain condition (ACC) if each countable ascending chain
\[ M_1 \leq M_2 \leq \ldots \leq M_n \leq M_{n+1} \leq \ldots \]
terminates, that is there exists $m \in \mathbb{Z}^+$ with $M_k = M_m$ for all $k \geq m$.

(b) We say that $M$ fulfills the descending chain condition (DCC) if each countable descending chain
\[ M_1 \geq M_2 \geq \ldots \geq M_n \geq M_{n+1} \geq \ldots \]
 terminates.

**Lemma 1.2.8.** Let $M$ be an $R$-module. Then the following are equivalent.

(a) $M$ fulfills DCC.

(b) Every nonempty set of $R$-submodules of $M$ has a minimal element.

(c) If $M$ is a non-empty set of $R$-submodules, then there exists a finite subset $N$ of $M$ with $\bigcap M = \bigcap N$. 

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Proof. (a) $\implies$ (b): Let $M$ be a non-empty set of $R$-submodules of $M$ and suppose $M$ has no minimal element. Let $M_1 \in M$ and inductively assume we already found $M_1, M_2, \ldots, M_k \in M$ with

$$M_1 > M_2 > \ldots > M_k$$

Since $M_k$ is not a minimal element, there exists $M_{k+1} \in M$ with $M_k > M_{k+1}$. Hence $\{M_k \mid k \in \mathbb{Z}^+\}$ is a non-terminating descending chain of $R$-submodules, a contradiction to DCC.

(b) $\implies$ (c): Let $M$ be a set of $R$-submodules and define

$$\mathcal{F} := \left\{ N \mid N \text{ a finite subset of } M \right\}.$$ 

By (b), $\mathcal{F}$ has a minimal element $W$. Then $W = \bigcap N$ for some finite $N \subseteq M$. Let $N \in M$. Then $W \cap N = \bigcap (N \cup \{N\}) \in \mathcal{F}$ and so by minimality of $W$, $W = W \cap N \leq N$. Thus $W = \bigcap M$ and (b) holds.

(c) $\implies$ (a): Let $M_1 \geq M_2 \geq M_3 \ldots$ be an descending chain of $R$-submodules of $M$. By (c) applied to $M = \{M_i \mid 1 \leq i < \infty\}$ there exists a finite subset $N$ of $M$ with $\bigcap N = \bigcap M$. Since $N$ is finite and total ordered $\bigcap N = M$, where $M$ is the minimal element of $N$. It follows that $M_i \leq M_j$ for all $j$ and so $M_i = M_j$ for all $j \geq i$. \hfill $\square$

Lemma 1.2.9 (Fitting). Let $M$ be an $R$-module and $f \in \text{End}_R(M)$.

(a) If $f$ is onto, then $f^n$ is onto for all $n \in \mathbb{Z}^+$.

(b) If $f$ is onto and $M$ fulfills ACC, then $f$ is an isomorphism.

(c) If $M$ fulfills DCC, there exists $n \in \mathbb{Z}^+$ with $\text{Im } f^n = \text{Im } f^{n+1}$.

(d) If $M$ fulfills ACC and DCC then there exists $n \in \mathbb{Z}^+$ such that $M = \ker f^n \oplus \text{Im } f^n$.

(e) If $M$ is directly indecomposable and fulfills ACC and DCC, then $f$ is either invertible or nilpotent.

Proof. (a): Suppose inductively that $f^n$ is onto. Then $f^{n+1}(M) = f(f^n(M)) = f(M) = M$. So also $f^{n+1}$ is onto.

(b): By (a) we know that $f^n$ is onto and so we obtain on isomorphism:

$$\alpha : M/\ker f^n \to M, \quad m + \ker f^n \to f^n(a)$$

Note that $\alpha(m + \ker f^n) \leq \ker f$ if and only if $f(f^n(m)) = 0$ and so if only if $m \in \ker f^{n+1}$. Thus $\alpha$ induces an isomorphism from $\ker f^{n+1}/\ker f^n$ to $\ker f$. Hence

$$(*) \quad \ker f^{n+1}/\ker f^n \cong \ker f.$$ 

Now $0 \leq \ker f \leq \ker f^2 \leq \ldots$ is an ascending chain of $R$ modules and so by ACC there exists $n$ with $\ker f^{n+1} = \ker f^n$. Thus $(*)$ implies that $\ker f = 0$ and so $f$ is 1-1.

(c): Just observe that $M \geq \text{Im } f \geq \text{Im } f^2 \geq \text{Im } f^3 \geq \ldots$, is an descending chain of $R$-submodules in $M$.

(d): Choose $n$ as in (c). Then $f : \text{Im } f^n \to \text{Im } f^n$ is onto. Hence also $f^n : \text{Im } f^n \to \text{Im } f^n$ is onto. By (c) we conclude that $f^n : \text{Im } f^n \to \text{Im } f^n$ is an isomorphism. So we can apply 1.2.3(c) to $f^n : M \to \text{Im } f^n$ in place of $\alpha$ and $\text{Im } f^n$ in place of $D$. Therefore $M = \text{Im } f^n \oplus \ker f^n$.

(e): Since $M$ is directly indecomposable, (d) implies that either $\ker f^n = M$ and $\text{Im } f^n = 0$, or $\ker f^n = 0$ and $\text{Im } f^n = M$. In the first case $f^n = 0$ and so $f$ is nilpotent. In the second case $f^n$ is an isomorphism and so invertible. But then also $f$ is invertible. \hfill $\square$
Lemma 1.2.10. Let $R$ be a non-zero ring.

(a) Let $I$ be a maximal left ideal in $R$ and suppose that all elements of $I$ are nilpotent. Then $I$ is the largest proper left ideal of $R$. In particular, $R$ is a local ring.

(b) Suppose each element of $R$ is invertible or nilpotent. Then $R$ is a local ring.

Proof. (a): Let $r \in R \setminus I$. Then $R_r + I$ is left ideal of $R$ and the maximality of $R$ shows that $R_r + I = R$. Thus $1 = ar + i$ for some $a \in R$ and $i \in I$. Since $i \in I$ we know that $i$ is nilpotent. In particular, $1 + i$ is has an inverse $j$, namely $j = \sum_{k=0}^{n-1} i^k$, where $n \in \mathbb{Z}^+$ with $i^n = 0$. As $1 = ar + i$ we have $1 - i = ar$ and so $(jr)(ar) = j(1 - i) = 1$. It follows that $R_r = R$ and so $r$ is not contained in any proper left ideal of $R$. Hence every proper left ideal of $R$ is contained in $I$.

(b): Since $R$ is a non-zero ring with identity, $R$ has maximal left ideal $I$. Note that no element of $I$ is invertible, and so the hypothesis of (b) shows that all elements of $I$ are nilpotent. Hence (b) now follows from (a). □

Lemma 1.2.11. Let $M$ be an $R$-module.

(a) If $\text{End}_R(M)$ is a local ring, $M$ is directly indecomposable.

(b) If $M$ fulfills ACC and DCC, then $M$ is directly indecomposable if and only if $\text{End}_R(M)$ is a local ring.

Proof. Put $A := \text{End}_R(M)$.

(a): Suppose for a contradiction that $A$ is a local ring and $M$ is directly decomposable. Let $M = X \oplus Y$ for some proper $R$-submodules $X$ and $Y$. Let $\pi_X : M \to M$ be the projection map defined by, $\pi_X(x + y) = x$ for all $x \in X$ and $y \in Y$. Similarly define $\pi_Y$. Then $\pi_X$ and $\pi_Y$ are in $A$ and $\pi_X + \pi_Y = \text{id}_M$. Since $X = \ker \pi_Y$, $\pi_Y$ is not invertible. Similarly $\pi_X$ is not invertible. But this contradicts 1.2.5.

(b) By (a) the backward direction in (b) holds. For the forward direction suppose that $M$ fulfills ACC and DCC and $M$ is directly indecomposable. Then 1.2.9(c) shows that all each elements of $R$ is invertible or nilpotent. Now 1.2.10(b) implies that $A$ is a local ring. □

Proposition 1.2.12. Let $M$ be an $R$-module, $\mathcal{B}$ a finite set of directly indecomposable $R$-submodules of $M$ with $M = \bigoplus \mathcal{B}$. If $A$ is a direct summand of $M$ such that $\text{End}_R(A)$ is a local ring, then there exists $B \in \mathcal{B}$ such that

$$M = A \oplus E, \quad \text{where } E = \bigoplus_{B \neq D \in \mathcal{B}} D.$$ 

In particular,

$$A \cong_R M/E \cong_R B \quad \text{and} \quad M/A \cong_R E \cong_R M/B.$$ 

Proof. Let $X$ be an $R$-submodule of $M$ with $M = A \oplus X$. Let

$$\iota_A : A \to M, \ a \mapsto a \quad \text{and} \quad \pi_A : M \to A, \ a + x \mapsto a$$

be the associated inclusion and projection maps. For $D \in \mathcal{B}$, let

$$\iota_D : D \to M, \ d \mapsto d \quad \text{and} \quad \pi_D : M \to D, \ \sum_{B \in \mathcal{B}} m_B \mapsto m_D$$

be the inclusion and projection map associated to the direct sum decomposition $M = \bigoplus \mathcal{B}$. Then

$$\pi_A \iota_A = \text{id}_A \quad \text{and} \quad \sum_{B \in \mathcal{B}} \iota_B \pi_B = \text{id}_M.$$
1.3. MASCHKE’S THEOREM

Let \( h \) be a field and \( A \) a finite ring. \( (K_\text{rull}-S_\text{chmidt}) \)

Theorem 1.2.14

Let \( \mathcal{A} \) be sets of \( R \) modules. We say that \( \mathcal{A} \) and \( \mathcal{B} \) are \( R \)-isomorphic and write \( \mathcal{A} \cong \mathcal{B} \) or \( \mathcal{A} \cong _R \mathcal{B} \) if there exists a bijection \( \alpha : \mathcal{A} \to \mathcal{B}, \mathcal{A} \to \mathcal{A}' \) with \( \mathcal{A} \cong _R \mathcal{A}' \) for all \( \mathcal{A} \in \mathcal{A} \).

Theorem 1.2.14 (Krull-Schmidt). Let \( \mathcal{A} \) and \( \mathcal{B} \) be sets of directly indecomposable \( R \)-modules. Suppose that \( \mathcal{B} \) is finite and that for each \( A \in \mathcal{A}, \text{End}_R(A) \) is a local ring. If \( \bigoplus \mathcal{A} \cong _R \bigoplus \mathcal{B} \) then \( \mathcal{A} \cong _R \mathcal{B} \).

Proof. Note that the theorem holds if \( \mathcal{B} = \emptyset \). We proceed by induction on \( |\mathcal{B}| \). We may assume that \( M := \bigoplus \mathcal{A} = \bigoplus \mathcal{B} \). Let \( A \in \mathcal{A} \). Then by 1.2.12 there exists \( B \in \mathcal{B} \) such that \( A \cong _R B \) and \( M/A \cong _R M/B \).

Let \( \mathcal{A}^* = \mathcal{A} \setminus \{A\} \) and \( \mathcal{B}^* = \mathcal{B} \setminus \{B\} \). Then \( M/A \cong \bigoplus \mathcal{A}^* \) and \( M/B \cong \bigoplus \mathcal{B}^* \). Thus \( \bigoplus \mathcal{A}^* \cong _R \bigoplus \mathcal{B}^* \). By induction \( \mathcal{A}^* \cong _R \mathcal{B}^* \) and since \( A \cong _R B \), also \( \mathcal{A} \cong _R \mathcal{B} \).

1.3 Maschke’s Theorem

Theorem 1.3.1 (Maschke). Let \( K \) be a field and \( G \) a finite group such that \( \text{char } K \) does not divide \( |G| \).

(a) Let \( V \) be a \( K[G] \)-module. Then every \( K[G] \)-submodule of \( V \) is a direct summand of \( V \) as a \( K[G] \)-module.

(b) Every \( K[G] \)-module is semisimple.

Proof. Let \( V \) be a \( K[G] \) module.

(a) Let \( W \) a \( K[G] \) submodule of \( V \). Note that there exists a \( K \)-subspace \( Z \) of \( V \) with \( V = W \oplus Z \). Define \( \pi : V \to W \) by \( \pi(w + z) = w \) for all \( w \in W, z \in Z \). Let \( g \in G \) and \( v \in V \). Then \( \pi(gv) \in W \) and since \( W \) is a \( K[G] \)-module we have

\[ g^{-1}(\pi(gv)) \in W. \]

Since \( \text{char } K \) does not divide \( |G| \), \( \frac{1}{|G|} \) is a well defined element of \( K \) and we can define

\[ \rho : V \to W, \quad v \mapsto \frac{1}{|G|} \sum_{g \in G} g^{-1}(\pi(gv)). \]

We claim that \( \rho \) is \( K[G] \)-linear. Note that \( \rho \) is a sum of compositions of \( K \)-linear functions and so \( K \)-linear. Let \( h \in G \) and \( v \in V \). Then

\[ \rho(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1}(\pi(g(hv))) = \frac{1}{|G|} \sum_{g \in G} g^{-1}(\pi(g(hv))) = \frac{1}{|G|} \sum_{g \in G} g^{-1}(\pi(ghv)) = \frac{1}{|G|} \sum_{g \in G} g^{-1}(\pi(ghv)). \]
\[ \rho(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1}(\pi(g(hv))) \]
\[ = \frac{1}{|G|} \sum_{g \in G} (h(gh)^{-1}) \pi((gh)v)) \]
\[ = h \frac{1}{|G|} \sum_{g \in G} (gh)^{-1}(\pi((gh)v)) \]
\[ = h \frac{1}{|G|} \sum_{g \in G} g^{-1}(\pi(gv)) \]
\[ = h(\rho(v)) \]

Thus \( \rho \) is indeed \( \mathbb{K}G \)-linear. For \( w \in W \) we have \( gw \in W \). It follows that
\[ \pi(gw) = gw, \quad g^{-1}(\pi(gw))) = g^{-1}(gw) = w, \quad \text{and} \quad \rho(w) = \frac{1}{|G|} \sum_{g \in G} w = w. \]

So \( \rho|_W = id_W \) and \( \rho|_W \) is an isomorphism. Hence 1.2.3(c) shows that \( V \cong W \). As \( \rho \) is \( \mathbb{K}G \)-linear, \( \ker \rho \) is a \( G \)-submodule of \( V \). Hence \( W \) is a direct summand of \( V \) as an \( \mathbb{K}G \)-module.

\[ \square \]

Example 1.3.2. Let \( \mathbb{K} \) be a field and let
\[ g := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \text{GL}_2(\mathbb{K}) \quad \text{and} \quad G := \langle g \rangle \leq \text{GL}_2(\mathbb{K}) \]

Then \( \mathbb{K}^2 \) is a \( \mathbb{K}G \)-module. Define \( p := \text{char} \mathbb{K} \) and let \( n \in \mathbb{N} \).
\[ g^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}. \]

So \( G \) has order \( p \) if \( p > 0 \) and \( G \) has infinite order of \( p = 0 \).

We claim that \( \mathbb{K} \times 0 \) is the unique non-zero proper \( \mathbb{K}G \)-submodule of \( \mathbb{K}^2 \). Indeed \( \mathbb{K} \times 0 \) is a \( \mathbb{K}G \)-submodule. Let \( W \) be a non-zero proper \( \mathbb{K}G \)-submodule of \( \mathbb{K}^2 \). Then \( \dim_{\mathbb{K}} W = 1 \) and each \( 0 \neq w \in W \) in an eigenvector of \( g \). Note that \( 1 \) is the only eigenvalue for \( g \) and \( \mathbb{K} \times 0 \) is the corresponding eigenspace for \( g \). Thus \( W = \mathbb{K} \times 0 \).

So \( \mathbb{K} \times 0 \) is the unique non-zero proper \( \mathbb{K}G \)-submodule of \( \mathbb{K}^2 \). It follows that \( \mathbb{K}^2 \) is not semisimple as an \( \mathbb{K}G \)-module. Hence Maschke’s Theorem can fail if \( 0 \neq \text{char} \mathbb{K} || |G| \) or if \( \text{char} \mathbb{K} = 0 \) and \( G \) is infinite.
1.4 Jacobson Radical

**Definition 1.4.1.** Let $M$ be an $R$-module, $S \subseteq R$ and $N \subseteq M$.

(a) $SN$ is the additive subgroup of $M$ generated by $\{sn \mid s \in S, n \in N\}$

(b) $A_{S}(N) := \{s \in S \mid sN = 0\}$. $A_{S}(N)$ is called the annihilator of $N$ in $S$.

(c) $A_{N}(S) = \{n \in N \mid Sn = 0\}$

(d) $M$ is a faithful $R$-module if $A_{R}(M) = 0$.

(e) $M$ is a cyclic $R$-module if $M = Rm$ for some $m \in M$.

(f) $M$ is a finitely generated $R$-module if $M = RN$ for a finite subset $N$ of $M$.

(g) $N$ is $S$-invariant if $sn \in N$ for all $s \in S, n \in N$.

**Lemma 1.4.2.** Let $M$ be an $R$-module and $N \subseteq M$ with $M = RN$. Then

$$\phi: \bigoplus_{n \in N} R/ A_{R}(n) \to M, \quad (r_n + A_{R}(n))_{n \in N} \mapsto \sum_{n \in N} r_n \cdot n$$

is a well defined onto $R$-linear homomorphism. If $|N| = 1$, then $\phi$ is an isomorphism.

**Proof.** Readily verified. \qed

**Definition 1.4.3.** Let $M$ be an $R$-module, $N \subseteq M$ and $I \subseteq R$.

(a) $N$ is called $R$-closed in $M$ if $N = A_{M}(J)$ for some $J \subseteq R$.

(b) $N^\circ := A_{M}(A_{R}(N))$ is the closure of $N$ in $M$ with respect to $R$.

(c) $I$ is $M$-closed in $R$ if $I = A_{R}(U)$ for some $U \subseteq M$.

(d) $I^\circ := A_{R}(A_{M}(I))$ is the closure of $I$ in $R$ with respect to $M$.

**Lemma 1.4.4.** Let $M$ be an $R$-module, $I \subseteq R$ and $N \subseteq M$.

(a) $0$ and $M$ are closed in $M$.

(b) $R$ is closed and $0^\circ = A_{R}(M)$.

(c) Let $J \subseteq R$ with $I \subseteq J$ and $P \subseteq M$ with $N \subseteq P$. Then $A_{R}(J) \subseteq A_{R}(I)$ and $A_{R}(P) \subseteq A_{R}(N)$. In particular, $I^\circ \subseteq J^\circ$ and $N^\circ \subseteq P^\circ$.

(d) $I \subseteq A_{R}(N)$ if and only if $in = 0$ for all $i \in I, n \in N$ and if and only if $N \subseteq A_{M}(I)$.

(e) $A_{R}(N)$ is a left ideal in $R$.

(f) If $N$ is $R$-invariant, then $A_{R}(N)$ is an ideal in $R$.

(g) $A_{M}(I)$ is an an $\text{End}_{R}(M)$-submodule of $M$.

(h) If $I$ is right ideal, then $A_{M}(I)$ is an $R$-submodule of $M$.

(i) $A_{R}(N) = A_{R}(N^\circ)$ and $N^{\circ \circ} = N^\circ$. 

1.4. Jacobson Radical
(j) $A_M(I) = A_M(I^0)$ and $I^{00} = I^0$.

(k) $N^0$ is a smallest $R$-closed subset of $M$ containing $N$. In particular, $N$ is $R$-closed in $M$ if and only if $N = N^0$.

(l) $I^0$ is the smallest $M$-closed subset of $R$ containing $I$. In particular, $I$ is $M$-closed in $R$ if and only if $I = I^0$.

(m) $A_R(RN)$ is the largest right ideal of $R$ contained in $A_R(N)$.

(n) $M$ is faithful module for $R/ A_R(M)$ via $(r + A_R(M)) \cdot m = rm$.

Proof. Let $s, \tilde{s} \in A_R(N), m, \tilde{m} \in A_M(I)$ and $r \in R$.

(a): $0 = A_M(1)$ and $M = A_M(0)$.

(b): $R = A_R(0)$ and since $A_M(0) = M$, $0^0 = A_R(M)$.

(c): This follows immediately from the definition of $A_R(N)$ and $A_M(I)$.

(d): Obvious.

(e): Since $(s \pm \tilde{s})N \subseteq sN + \tilde{s}N = 0$, $A_R(N)$ is closed under addition and additive inverses. Moreover, $(rs)N = r(sN) = r0 = 0$ and so $A_R(N)$ is a left ideal.

(f): If $N$ is $R$-invariant we have $(sr)N = s(rN) \subseteq aN = 0$ and so (e) holds.

(g): $I(m \pm \tilde{m}) \subseteq Im + I\tilde{m} = 0$. Also $0 \in A_M(I)$. Let $\alpha \in \text{End}_R(M)$. Then Moreover, $I(\alpha(m)) = \alpha(Im) = \alpha(0) = 0$ and so (d) holds.

(h): $I(Im) = (Ir)m \subseteq Im = 0$.

(i): From $N^0 = A_M(A_R(N))$ we get $N \subseteq N^0$ and $A_R(N) \subseteq A_R(N^0)$. Form $N \subseteq N^0$ we get $A_R(N^0) \subseteq A_R(N)$. Thus $A_R(N) = A_R(N^0)$ and so

$$N^0 = A_M(A_R(N^0)) = A_M(A_R(N)) = N^0.$$

(j): From $I^0 = A_R(A_M(I))$ we get $I \subseteq I^0$ and $A_M(I) \subseteq A_M(I^0)$. From $I \subseteq I^0$ we get $A_M(I^0) \subseteq A_M(I)$. Thus $A_R(N) = A_R(N^0)$ and so

$$I^{00} = A_R(A_M(I^0)) = A_R(A_M(I)) = I^0.$$

(k): By definition of $N^0$ we have $N^0 = A_M(A_R(N))$. Thus $N^0$ is closed. Suppose $N \subseteq W$ for some $R$-closed subset $W$ of $M$. Then $W = A_M(J)$ for some $J \subseteq R$. Hence $N \subseteq W = A_M(J)$ and (j) gives $J \subseteq A_R(N)$. Now (e) implies $A_M(A_R(N)) \subseteq A_M(J)$, that is $N^0 \subseteq W$.

(l). By definition of $I^0$ we have $I^0 = A_R(A_M(I))$. Thus $I^0$ is closed. Suppose that $I \subseteq J$ for some $M$-closed subset $J$ of $R$. Then $J = A_R(W)$ for some $W \subseteq M$. Hence $I \subseteq J = A_R(W)$ and (j) gives $W \subseteq A_R(I)$. Now (e) implies $A_R(A_M(I)) \subseteq A_R(W)$, that is $I^0 \subseteq J$.

(m) Since $RN$ is an $R$-submodule, (j) implies that $A_R(RN)$ is an ideal in $R$. Now let $J$ be a right ideal of $R$ with $J \subseteq A_R(N)$. Then $N \subseteq A_R(J)$. By (j) $A_M(J)$ is an $R$-submodule of $M$ and so $RN \subseteq A_M(J)$. Thus $J \subseteq A_R(RN)$.

(n) Readily verified. \hfill \Box

**Corollary 1.4.5.** Let $M$ be an $R$-module. Then
Theorem 1.4.8. Let \( R \) be a non-zero ring and let \( A_R \) of \( R \).

Proof. Let \( N \subseteq M \) and \( I \subseteq R \) be closed. By 1.4.4(m) \( N = N^\circ = A_M(A_R(N)) \) and by 1.4.4(m) \( I = I^\circ = A_R(A_M(I)) \). So the maps given in the corollary are inverse to each other. If \( N \) is an \( R \)-submodule, \( A_R(N) \) is an ideal. If \( I \) is an ideal, when \( A_M(I) \) is an \( R \)-submodule.

Lemma 1.4.6. Let \( M \) be an \( R \)-module and \( 0 \neq m \in M \).

(a) If \( M \) is simple, then \( M = Rm \).

(b) If \( M = Rm \), then \( M \cong R/\mathbb{A}_R(m) \).

(c) If \( M = Rm \), then \( M \) is a simple \( R \)-module if and only if \( \mathbb{A}_R(m) \) is a maximal left ideal in \( R \).

(d) If \( M \) is a simple \( R \)-module, then \( M \cong R/\mathbb{A}_R(m) \) and \( \mathbb{A}_R(m) \) is a maximal left ideal in \( R \).

Proof. (a): Just observe that \( Rm \) is a non-zero \( R \)-submodule of \( M \).

(b): Just recall from 1.4.2 that \( Rm \cong R/\mathbb{A}_R(m) \).

(c): By (b) \( M \cong R/\mathbb{A}_R(m) \). Also the \( R \)-submodules of \( R/\mathbb{A}_R(m) \) are all the \( J/\mathbb{A}_R(m) \), where \( J \) a left ideal of \( R \) containing \( \mathbb{A}_R(m) \).

(d): By (a) \( M = Rm \). So (d) follows from (a) and (c).

Definition 1.4.7. Let \( R \) be a ring.

(a) \( J(R) \) is the intersection of the maximal left ideals in \( R \). \( J(R) \) is called the Jacobson radical of \( R \).

(b) Let \( M \) be an \( R \)-module. Then \( \mathcal{J}_M(R) \) is the intersection of the maximal \( R \)-submodules of \( M \), with \( \mathcal{J}_M(R) = M \) if \( M \) has no maximal \( R \)-submodule.

Note that \( \mathcal{J}_R(R) = J(R) \).

Theorem 1.4.8. Let \( R \) be a non-zero ring and let \( S(R) \) be the class of simple \( R \)-modules. Then

\[
J(R) = \bigcap_{S \in S(R)} \mathbb{A}_R(S).
\]

In particular, \( J(R) \) is proper ideal of \( R \).

Proof. Let \( \mathcal{M}(R) \) be the set of maximal left ideal of \( R \) and for \( S \in S(R) \) let \( \mathcal{M}_S(R) := \{\mathbb{A}_R(s) \mid 0 \neq s \in S\} \).

By 1.4.6 \( \mathbb{A}_R(s) \) is a maximal ideal of \( R \) for each \( 0 \neq s \in S \) and so \( \mathcal{M}_S(R) \subseteq \mathcal{M}(R) \). Note also that \( \mathbb{A}_R(S) = \bigcap \mathcal{M}_S(R) \).

Let \( I \in \mathcal{M}(R) \). Then \( R/I \in S(R) \). Note that \( \mathbb{A}_R(1 + I/I) = \{r \in R \mid r + I = I\} = I \) and so \( \mathbb{A}_R(R/I) \subseteq \mathbb{A}_R(1 + I/I) = I \). Hence

\[
\bigcap_{I \in \mathcal{M}(R)} \mathcal{M}(R) = \bigcap_{I \in \mathcal{M}(R)} I \supseteq \bigcap_{I \in \mathcal{M}(R)} \mathbb{A}_R(R/I) \supseteq \bigcap_{S \in S(R)} \mathbb{A}_R(S) = \bigcap_{S \in S} \mathcal{M}_S(R) \supseteq \mathcal{M}(R).
\]

Thus equality holds everywhere and the lemma is proved.
Lemma 1.4.9. Suppose that $M$ is a semisimple $R$-module, then $I_M(R) = 0$.

Proof. Let $S$ be a set of simple $R$-submodules of $M$ with $M = \bigoplus S$. For $S \in S$ define $S^* := \sum_{S \neq T \in S}T$. Then $M/S^* \cong S$ and so $S^*$ is a maximal $R$-submodule of $M$. Observe that $\bigcap_{S \in S} S^* = 0$ and so $I_M(R) = 0$. \hfill \Box$

Lemma 1.4.10. Let $M$ and $R$-module and $F$ a finite set of maximal $R$-submodules of $M$ with $\bigcap F = 0$. Choose $N \subseteq F$ minimal with $\bigcap N = 0$. Then the map

$$\phi : M \to \bigoplus_{N \in M} M/N, \quad m \mapsto (m + N)_{N \in N}$$

is an $R$-isomorphism. In particular, $M$ is semisimple.

Proof. Clearly $\phi$ is $R$-linear. Let $N \in N$ and $m \in \ker \phi$. Then $m + N = 0_{M/N} = N$ and so $m \in N$. Thus $m \in \bigcap N = 0$ and $\phi$ is 1-1. Let $N \in N$ and $N^* := \bigcap\{T \in N \mid T \neq N\}$ with $N^* := M$ if $N = \{N\}$. The minimality of $N$ implies $N^* \neq 0$. Since $\phi$ is 1-1, also $\phi(N^*) \neq 0$. Put $W := \bigoplus_{N \in N} M/N$ and view $M/N$ as a subgroup of $W$. Note that $0 \neq \phi(L^*) \leq M/L$ and the simplicity of $M/L$ implies $M/L = \phi(L^*) \leq \text{Im} \phi$. Thus $W = \sum_{N \in N} M/N \leq \text{Im} \phi$ and so $\phi$ is onto. \hfill \Box

Corollary 1.4.11. Let $M$ be an $R$-module with DCC. Then $M$ is semisimple if and only if $I_M(R) = 0$.

Proof. If $M$ is semisimple, then $I_M(R) = 0$ by Lemma 1.4.9. Suppose now that $I_M(R) = 0$ and let $M$ be the set of maximal $R$-submodules of $M$. Then $\bigcap M = I_M(R) = 0$. Since $M$ fulfills DCC, 1.2.8 implies that there exists a finite subset $F$ of $M$ with $\bigcap F = \bigcap M = 0$. So Lemma 1.4.10 shows that $M$ is semisimple. \hfill \Box

1.5 Simple modules for algebras

Lemma 1.5.1 (Schur I). Let $M, N$ be simple $R$-modules and $f \in \text{Hom}_R(M, N)$. If $f \neq 0$, then $f$ is $R$-isomorphism. In particular, $\text{End}_R(M)$ is a division ring.

Proof. Since $f \neq 0$, $\ker f \neq M$. Also $\ker f$ is an $R$-submodule and so $\ker f = 0$ and $f$ is 1-1. Similarly, $\text{Im} f \neq 0$, $\text{Im} f = N$ and so $f$ is onto. So $f$ is a bijection and has an inverse $f^{-1}$. An easy computation shows that $f^{-1} \in \text{Hom}_R(N, M)$. Choosing $N = M$ we see that $\text{End}_R(M)$ is a division ring. \hfill \Box

Definition 1.5.2. Let $R$ be ring.

(a) An $R$-ring is pair $(A, \cdot)$ such that $(A, \cdot)$ is an $R$-module, $A$ is a ring and $r(ab) = (ra)b$ for all $r \in R$, $a, b \in A$.

(b) An $R$-algebra is pair $(A, \cdot)$ such that $(A, \cdot)$ is an $R$-ring and $(ra)b = a(rb)$ for all $r \in R, a, b \in A$.

Example 1.5.3. Let $R$ be a commutative ring and $M$ an $R$-module. Then $\text{End}_R(M)$ is an $R$-algebra via $(ra)(m) = r(a(m))$ for all $r \in R, a \in \text{End}_R(M)$ and $m \in M$. If $M$ is a free $R$-module with finite basis $B$, then $\text{End}_B(M)$ is a free $R$-module with basis $(\phi_{ab})_{(a, b) \in B \times B}$ where $\phi_{ab} : M \to M$ is the unique $R$-linear function with $\phi_{ab}(c) = \delta_{ac}b$ for all $c \in B$.

Lemma 1.5.4. Let $R$ be a ring and $A$ an $R$-ring.

(a) The function $\rho : R \to A, r \mapsto r1_A$ is a homomorphism of rings.

(b) $A$ is a right $R$-module via $ar := a(r1_A)$ for all $a \in A, r \in R$. 
\[ ra = (r1_A)a, (ar)b = a(rb), a(br) = (ab)r \text{ and } (ra)s = r(as) \text{ for all } r, s \in R, a, b \in A. \]

(d) Suppose \( A \) is an \( R \)-algebra. Then \( \text{Im} \rho \subseteq Z(A) \).

(c) Let \( M \) be a simple \( R \)-module, \( N \) an \( R \)-closed subset of \( M \) and \( m \in M \). Since \( M \) is simple this gives (c).

(e) Let \( \phi : A \to \text{End}(M) \) be the homomorphism corresponding the \( A \)-module \( M \). Then \( \phi \circ \rho : R \to \text{End}(M) \) is a homomorphism and so (e) holds.

**Definition 1.5.5.** Let \( R \) ring and \( M \) an \( R \)-module. Then \( R \)-subspaces of \( M \) are \( R \)-closed.

\[ \text{dim}_R \mathbb{K} = n \text{ is finite dimensional over } \mathbb{K}, \text{ so is } \mathbb{D}. \]

\[ \mathbb{D} : \text{By 1.5.1} \mathbb{D} \text{ is a division ring. Since } \mathbb{K} \text{ is finite, then also } \mathbb{D} \text{ is finite. Thus Wedderburn’s Theorem implies that } \mathbb{D} \text{ is a field.} \]

\[ \mathbb{E} : \text{Let } d \in \mathbb{D}. \text{ Put } \mathbb{F} := \mathbb{K}|M \cong \mathbb{K} \text{ and let } \mathbb{F} \text{ be the subring of } \mathbb{D} \text{ generated by } \mathbb{E} \text{ and } d. \text{ Then } \mathbb{F} \text{ is a } \mathbb{A} \text{-field extension of } \mathbb{E}. \text{ Since } \mathbb{K} \text{ is algebraically closed, we conclude that } \mathbb{F} = \mathbb{E}. \text{ So } d \in \mathbb{E} \text{ and } \mathbb{K} = \mathbb{E}. \]

**Lemma 1.5.7.** Let \( M \) be a simple \( R \)-module, \( N \) an \( R \)-closed subset of \( M \) and \( m \in M \setminus \mathbb{A} \). Put \( J := \mathbb{A}_R(N) \) and \( m \in M \setminus \mathbb{N} \). Then \( M = \mathbb{J}m \) and the function
\[ J/\mathbb{A}_J(m) \to M, \quad j + \mathbb{A}_J(m) \mapsto jm \]
is a well defined \( R \)-isomorphism.

**Proof.** Since \( N \) is closed, \( N = N^\circ = \mathbb{A}_R(J) \). Since \( m \notin N \) this gives \( Jm \neq 0 \). By 1.4.4 \( J \) is a left ideal in \( R \) and so \( Jm \) is an \( R \)-submodule of \( M \). Since \( M \) is simple this gives \( M = Jm \). It follows that \( \rho : J \to M, j \mapsto jm \) is an onto \( R \)-linear function with \( \ker \rho = \mathbb{A}_J(m) \).

**Lemma 1.5.8.** Let \( M \) be simple \( R \)-module and \( \mathbb{D} := \text{End}_R(M) \). Let \( V \leq W \) be \( \mathbb{D} \)-submodules of \( M \) with \( \text{dim}_\mathbb{D}(W/V) \) finite. If \( V \) is \( R \)-closed in \( M \), also \( W \) is \( R \)-closed in \( M \). In particular, all finite dimensional \( \mathbb{D} \)-subspaces of \( M \) are \( R \)-closed.
Proof. By induction on \( \dim_\mathbb{D} W/V \) we may assume that \( \dim_\mathbb{D} W/V = 1 \). Then \( W = V + Dw \) for some \( w \in W/V \). Let \( I = A_R(V) \) and \( J = A_J(w) \). We will show that \( W = A_M(J) \). So let \( m \in A_M(J) \). Then \( J \subseteq A_I(m) \) and hence the map \( \alpha : I/J \to M, i + J \to im \) is well-defined and \( R \)-linear. By [1.5.7] the map \( \beta : I/J \to M, i + J \to iw \) is an \( R \)-isomorphism and so has an inverse. Put \( \delta = \alpha \beta^{-1} \). Then \( \delta : M \to M \) is \( R \)-linear and \( \delta(iw) = im \) for all \( i \in I \). Hence \( \delta \in \mathbb{D} \) and

\[
  i(\delta(w) - m) = i\delta(w) - im = \delta(iw) - im = 0
\]

for all \( i \in I \). Since \( V \) is \( R \)-closed in \( M \) we know that \( V = A_M(I) \) and so \( \delta(w) - m \in V \). Thus \( m \in \delta(w) + V \subseteq W \). Clearly \( W \leq A_M(J) \) and so indeed, \( W = A_W(J) \) is closed.

**Definition 1.5.9.** Let \( M \) be an \( R \)-module and \( \mathbb{D} \subseteq \text{End}_R(M) \) a division ring. Then we say that \( R \) is dense on \( M \) with respect to \( \mathbb{D} \) if for each tuple \( \mathbb{D} \)-linear independent tuple \( (m_i)_{i=1}^n \subseteq M^n \) and each \( (w_i)_{i=1}^n \subseteq M^n \), there exists \( r \in R \) with \( rm_i = w_i \) for all \( 1 \leq i \leq n \).

**Theorem 1.5.10** (Jacobson’s Density Theorem). Let \( M \) be simple \( R \)-module and put \( \mathbb{D} := \text{End}_R(M) \). Then \( R \) is dense on \( M \) with respect to \( \mathbb{D} \).

**Proof.** Let \( (m_i)_{i=1}^n \subseteq M^n \) be \( \mathbb{D} \)-linear independent and \( (w_i)_{i=1}^n \subseteq M^n \). By induction on \( n \) we will show that there exists \( r \in R \) with \( rm_i = w_i \) for all \( 1 \leq i \leq n \). For \( n = 0 \) there is nothing to prove. By induction there exists \( s \in R \) with \( sm_i = w_i \) for all \( 1 \leq i < n \). Let \( V = \sum_{i=1}^{n-1} \mathbb{D}m_i \). Then by [1.5.8] \( V \) is closed and so by [1.5.7] there exists \( t \in A_R(V) \) with \( tm_i = w_i - sm_i \). Put \( r = s + t \). For \( 1 \leq i < n \) \( rm_i = sm_i + tm_i = sm_i \). Also \( rm_i = sm_i + tm_i = sm_i + (w_i - sm_i) = w_i \) and the theorem is proved.

**Corollary 1.5.11.** Let \( M \) be a simple \( R \)-module, \( \mathbb{D} := \text{End}_R(M) \) and \( W \) a finite dimensional \( \mathbb{D} \)-submodule of \( M \). Put \( N_R(W) = \{ r \in R \mid rw \subseteq W \} \). Then \( N_R(W) \) is a subring of \( W \), \( W \) is a \( N_R(W) \)-submodule of \( M \), \( A_R(W) \) is an ideal in \( N_R(W) \) then

\[
N_R(W)/A_R(W) \cong N_R(W)|_W = \text{End}_\mathbb{D}(W).
\]

**Proof.** All but the very last statement in are readily verified. Clearly \( N_R(W)|_W \) is contained in \( \text{End}_\mathbb{D}(W) \). Let \( \phi \in \text{End}_\mathbb{D}(W) \) and choose a \( \mathbb{D} \)-basis \( (v_i)_{i=1}^n \) for \( W \). By [1.5.10] there exists \( r \in R \) with \( rv_i = \phi(v_i) \). Then \( rW \subseteq W \) and so \( r \in N_R(W) \). The image of \( r \) in \( \text{End}(W) \) is \( \phi \). Thus \( \phi \in N_R(W)|_W \) and so \( N_R(W)|_W = \text{End}_\mathbb{D}(W) \).

**Definition 1.5.12.** A ring with no proper ideals is called simple. A direct sum of simple ring is called semisimple.

**Corollary 1.5.13.** Let \( R \) be a simple ring. Then there exists a simple \( R \)-module \( M \). Moreover, if \( M \) is a simple \( R \)-module and \( \mathbb{D} = \text{End}_R(M) \), then \( R \) is isomorphic to a dense subring of \( \text{End}_\mathbb{D}(M) \).

**Proof.** Let \( I \) be a maximal left ideal, then \( R/I \) is a simple \( R \)-module. Now let \( M \) be any simple \( R \)-module. Since \( R \) is simple, \( A_R(M) = 0 \). Thus \( R \cong R^M \) and by [1.5.10] \( R \) and so also \( R^M \) is dense on \( M \).

**Proposition 1.5.14.** Let \( M \) be faithful, simple \( R \)-module and put \( \mathbb{D} = \text{End}_R(M) \). Suppose that \( \dim_\mathbb{D} M \) is finite.

(a) \( R \cong R|_M = \text{End}_\mathbb{D}(M) \) as rings.

(b) Let \( n := \dim_\mathbb{D} M \). Then \( R \cong M^n \) as a left \( R \)-module.

(c) Let \( I \) be a maximal left ideal in \( R \). Then \( I = A_R(m) \) for some \( 0 \in m \in M \) and \( R/I \cong M \)

(d) The \( M \)-closed subsets of \( R \) are exactly the left ideal of \( R \).
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(c) The $R$-closed subsets of $M$ are exactly the $\mathcal{D}$-subspaces of $M$.

(f) The map $I \rightarrow A_R(I)$ is a bijection between the left ideals of $R$ and the $\mathcal{D}$-subspaces of $M$ with inverse $N \rightarrow A_M(N)$.

(g) Each simple $R$-module is isomorphic to $M$.

(h) $R$ is a simple ring.

Proof. (a) Note that $N_R(M) = R$ and $A_R(M) = 0$ and so (a) follows from 1.5.11.

(b) Observe first that $M$ is a simple $R$-module. Let $\mathcal{B}$ be a basis for $M$ over $\mathcal{D}$ and let $b \in \mathcal{B}$. Then by 1.4.6 $A_R(b)$ is a maximal ideal in $R$ and $R/ A_R(b) \cong M$. Let $E$ be a subset of $\mathcal{B}$.

By 1.5.8 $\mathcal{E}$ is $R$-closed in $M$. Note that also $M$ is $R$-closed.

So $\mathcal{B}$ is minimal in $\mathcal{B}$ with respect to $\bigcap_{b \in \mathcal{B}} A_R(b) = 0$. Since $M$ is a simple $R$-module we know that $M \cong R/ A_R(e)$ and $A_R(e)$ is a maximal left ideal in $R$. Thus 1.4.10 implies

$$R \cong \bigoplus_{b \in \mathcal{B}} R/ A_R(b) \cong M^n.$$

(c) By (b) $R \cong M^n$ and so 1.1.26(d) shows that $R/I \cong M^k$ for some $1 \leq k \leq n$. Since $R/I$ is a simple $R$-module we get $k = 1$. So $R/I \cong M$. Let $\psi : R/I \rightarrow M$ be an $R$-isomorphism and put $m = \phi(1 + I / I)$. Then

$$I = A_R(1 + I / I) = A_R(m).$$

(d) By 1.4.4(e) any $M$-closed subset of $R$ is an ideal in $R$. Conversely, let $I$ be an $M$-closed ideal in $R$. Then by 1.4.4(b) $A_R(N)$ is a maximal ideal in $R$ containing $I$. Since $R$ is a semisimple $R$-module 1.4.7 show that $R/I$ is semisimple. Thus 1.4.9 implies that $I_{R/I} = 0$. Hence $\bigcap_{I \in \mathcal{I}} = I$. By (c), for each $J \in M$ there exists $m_j \in M$ with $J = A_R(m_j)$. Put $N := \{x \in M | J \in \mathcal{I}\}$. Then

$$A_R(N) = \bigcap_{J \in \mathcal{I}} A_R(m_j) = \bigcap_{J \in \mathcal{I}} J = I.$$
Proof. Without loss $M \neq 0$ and $D = D|M$. Put $F := \text{End}_{\text{End}D}(M)$. Note that $M$ is a simple $\text{End}D(M)$-module and so $\text{[1.5.1]}$ is a division ring.

(a) Let $B$ be a $\mathbb{D}$-basis for $M$. Since $M \neq 0$ we can choose $m \in B$. Define $\phi \in \text{End}D(M)$ by $\phi(b) = \delta_{bm}m$ for all $b \in B$. Then $\phi(M) = Dm$. Let $f \in F$. Then

$$f(\phi(M)) = \phi(f(M)) \leq \phi(M) = Dm.$$  

Thus $fm = dm$ for some $d \in F$. Hence $(f - d)m = 0$. Note that $m \neq 0$, $f - d \in F$ and $F$ is a division ring. Thus $f - d = 0$. Hence $f = d$ and $F = D$.  

(b) Since $\mathbb{D} \subseteq F$ and $M$ is finite dimensional over $\mathbb{D}$ we see that $M$ is finite dimensional over $\mathbb{F}$. Also $M$ is a faithful simple $\text{End}_{\mathbb{D}}(M)$ and so $\text{[1.5.14]}$ shows that $\text{End}_{\mathbb{D}}(M)$ is a simple rings.  

\begin{definition}
A ring $R$ is called Artinian if it fulfills the DCC on left ideals.
\end{definition}

\begin{lemma}
Let $R$ be an Artinian ring and $M$ a simple $R$-module. Then $M$ is finite dimensional over $\mathbb{D}D = \text{End}_{\mathbb{D}}(M)$.
\end{lemma}

\begin{proof}
Suppose for a contradiction that $M$ is infinite dimensional over $\mathbb{D}$. Then there exists an infinite strictly ascending chain

$$M_1 \leq M_2 \leq M_3 \leq \ldots$$  

of finite dimensional $\mathbb{D}$-subspaces. By $\text{[1.5.8]}$ each $M_i$ is closed. Thus by $\text{[1.4.9]}$

$$A_R(M_1) \geq A_R(M_2) \geq A_R(M_3) \geq \ldots$$  

is a strictly descending chain of left ideals in $R$, contradicting the DCC conditions on Artinian rings.  

\end{proof}

\begin{lemma}[Chinese Remainder Theorem]
Let $R$ be ring and $I$ a finite set of ideals in $R$. Suppose that $\bigcap I = \emptyset$ and $I + J = R$ for all $I \neq J \in I$. Let $I \in I$ and put $I^* := \bigcap \{I \setminus \{I\}$.  

(a) $R = I \bigoplus I^*$.  

(b) $I = \bigoplus_{I \in I \setminus \{I\}} I^*$.  

(c) $R = \bigoplus_{I \in I} I^*$.  

(d) The function $I^* \to R/I$, $r \mapsto r + I$ is an isomorphisms of rings.  

(e) The function $R \to \bigoplus_{I \in I} R/I$, $r \mapsto (r + I)_{I \in I}$ is an isomorphism of rings.
\end{lemma}

\begin{proof}
(a) and (b): Let $J := I \setminus \{I\}$ and let $\emptyset \neq N \subseteq J$. We claim that $R = I + \bigcap N$. If $|N| = 1$, this holds by assumption. Let $J \in N$ and put $S := \bigcap N \setminus \{J\}$. By induction $R = I + S = I + J$. As $R$ has an identity ,

$$R = R^2 = (I + S)(I + J) = I^2 + SJ + IJ + S J.$$  

Since $I$ is an ideal, $I^2 + SJ + IJ \leq I$. Since $S$ and $J$ are ideals $SJ \leq S \cap J = \bigcap N$. So $R = I + \bigcap N$, proving the claim.

For $N = J$ we conclude that $R = I + I^*$. Since $I \cap I^* = \bigcap I = 0$ we have $R = I \oplus I^*$. Thus (c) holds. In particular, $I$ has an identity and so is $I$ is a ring. Note that $\bigcap_{J \in J}(I \cap J) = \bigcap I = 0$ and $J^* = \bigcap_{K \in J \setminus \{J\}} (I \cap J)$. Moreover for distinct, $I, J, K \in I$ we have

$$I = RI = (J + K)I = JI + KI = (I \cap J) + (I \cap K).$$  

By induction (c) holds for $R$ replaced by $I$ and $I$ by $\{I \cap J \mid J \in J\}$. This gives (b).
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(c) follows from (a) and (b).
(d) follows from (a).
(e): Since $\mathfrak{I}$ is 1-1, the function is 1-1. Note that $I^* + J/I = 0$. Also by (a) $I^* + I/I = R/I$ hence the image of $I^*$ in $\bigoplus_{I \in T} R/I$ is $R/I$. Thus the function is onto.  

\[ \square \]

**Lemma 1.5.19.** Let $S$ be a finite set rings and $R = \bigoplus S$. For $S \in S$ put $S^* := \bigoplus S \setminus \{S\}$.

(a) Let $I$ be left ideal in $R$. Then $I = \bigoplus_{S \in S} I \cap S$.

(b) Let $I$ be a minimal left ideal of $R$. Then $I \subseteq S$ for some $S \in S$. Moreover, $I$ is a minimal left ideal of $S$.

(c) Let $I$ be a maximal left ideal of $R$. Then $S^* \subseteq I$ for some $S \in S$. Moreover, $I = (I \cap S) \oplus S^*$ and $I \cap S$ is a maximal ideal in $S$.

(d) Let $M$ be a simple $R$-module. Then $S^* \subseteq A_R(M)$ for some $S \in S$. Moreover, $M$ is a faithful simple $R$-module.

(e) $J(R) = \bigoplus_{S \in S} J(S)$.

**Proof.**

(a): Note that $I = RI = \bigoplus_{S \in S} SI$. It follows that $I \cap S = SI$ and so (a) holds.

(b) and (c) follow immediately from (a).

(d): By (1.4.6) $M \cong R/I$ for some maximal left ideal $I$ in $R$. By (c), $S^* \subseteq I$ for some $S \in S$. Note that $S^* R \subseteq S^* \subseteq I$ and so $S^* \subseteq A_R(R/I) = A_R(M)$.

(e) Follows from (c).  

\[ \square \]

**Corollary 1.5.20.** Let $R$ be a semisimple ring and let $S$ be a (finite) set of simple rings with $R = \bigoplus S$. For $S \in S$ put $S^* := \bigoplus S \setminus \{S\}$. Then

(a) Let $I$ be an ideal in $R$. Put $I := \{S \in S \mid S \subseteq I\}$. Then $I \subseteq S$ and $I = \bigoplus I$.

(b) $S$ is the set of minimal ideals of $R$.

(c) $\{S^* \mid S \in S\}$ is the set of maximal ideals of $R$.

(d) Let $M$ be a simple $R$-module, then $A_R(M) = S^*$ for some $S \in S$. In particular, $M$ is a faithful simple $S$-module.

(e) $J(R) = 0$.

**Proof.**

(a) Let $I$ be an ideal in $R$ and $S \in S$. Then $S \cap I$ is an ideal in the simple ring $S$. Thus either $S \cap I = 0$ or $S \cap I = S$. Hence $I = \{S \cap I \mid S \cap I \neq 0\}$. By (1.5.19a) $R = \bigoplus_{S \in S} I \cap S$ and so (a) holds.

(b) and (c) follows immediately from (a).

(d) By (1.5.19d) $S^* \subseteq A_R(M)$ for some $S \in S$. Since $A_R(M)$ is a proper ideal of $R$ and $S^*$ is a maximal ideal this gives $A_R(M) = S^*$.

(e) Let $S \in S$. By (1.4.8) $J(S)$ is a proper ideal of $R$. Since $S$ is simple this gives $J(S) = 0$. By (1.5.19) we have $J(R) = \bigoplus_{S \in S} J(S)$ and so $J(R) = 0$.  

\[ \square \]
Theorem 1.5.21 (Wedderburn-Artin). Let $R$ be an Artinian ring with $J(R) = 0$. Let $\mathcal{M}$ be a set of representatives for the isomorphism classes of simple $R$-modules and let $M \in \mathcal{M}$. Put

$$A_R^*(M) := \bigcap \{A_R(P) \mid M \neq P \in \mathcal{M}\} \quad \text{and} \quad D_M := \text{End}_R(M).$$

Then

(a) $\mathcal{M}$ is finite.

(b) $M$ is finite dimensional over $D_M$.

(c) $A_R^*(M) \cong R|_M = \text{End}_{D_M}(M)$.

(d) $R = \bigoplus_{M \in \mathcal{M}} A_R^*(M) \cong \bigoplus_{M \in \mathcal{M}} R|_M = \bigoplus_{M \in \mathcal{M}} \text{End}_{D_M}(M)$

(e) $A_R(M) = \bigoplus_{P \in M \setminus \{M\}} A_R^*(P)$.

(f) $R$ is a semisimple ring.

Proof. Note that $M$ is a faithful simple $R/A_R(M)$-module. By [1.5.17] $M$ is finite dimensional over $D_M$. Thus [1.5.14] shows that

$$R/A_R(M) \cong R|_M = \text{End}_{D_M}(M),$$

$R/A_R(M)$ is a simple ring, and $M$ is, up to isomorphism, the unique simple $R/A_R(M)$-module. In particular, $A_R(M)$ is a maximal ideal of $R$.

Let $N \in \mathcal{M}$ with $A_R(M) \subset A_R(N)$. Since $A_R(M)$ is a maximal ideal of $R$ this gives $A_R(M) = A_R(N)$. Hence both $N$ and $M$ are simple $R/A_R(M)$-modules. It follows that $N \cong R$ and so $N = M$.

So if $N \in \mathcal{M}$ with $N \neq M$, then $A_R(M) \subsetneq A_R(N)$ and since $A_R(N)$ is a maximal ideal of $R$ we get $A_R(M) + A_R(N) = R$.

By [1.4.8] $\bigcap_{M \in \mathcal{M}} A_R(M) = J(R) = 0$. Since $R$ is Artinian, DCC holds for the left ideals in $R$ and so [1.2.8] shows there exists a finite subset $\mathcal{N}$ of $\mathcal{M}$ with $\bigcap_{N \in \mathcal{N}} A_R(N) = \bigcap_{M \in \mathcal{M}} A_R(M) = 0$. We proved that the hypothesis of [1.5.18] is fulfilled for $J := \{A_R(N) \mid N \in \mathcal{N}\}$. For $N \in \mathcal{N}$ put $N^* := \bigcap_{P \in \mathcal{N} \setminus \{N\}} A_R(P)$. Then

$$R = \bigoplus_{N \in \mathcal{N}} N^*, \quad N^* \cong R/A_R(N), \quad \text{and} \quad A_R(N) = \bigoplus_{P \in \mathcal{N} \setminus \{N\}} P^*.$$

In particular, $N^*$ is a simple ring for each $N \in \mathcal{N}$ and $R$ is semisimple. Since $R = \bigoplus_{N \in \mathcal{N}} N^*$ and $A_R(M)$ is a maximal ideal of $R$ we conclude from [1.5.20] that

$$A_R(M) = \bigoplus_{P \in \mathcal{N} \setminus \{N\}} P^* = A_R(N)$$

for some $N \in \mathcal{N}$. As seen above this implies $M = N$. Hence $\mathcal{N} = \mathcal{M}$ and so $\mathcal{M}$ is finite. Moreover, $A_R^*(M) = M^*$ and the lemma is proved. $\square$

Proposition 1.5.22. Let $R$ be a ring. Then the following statements are equivalent.

(a) Every $R$-module is semisimple.

(b) $R$ is semisimple as a left $R$-module.

(c) $R$ is Artinian and $J(R) = 0$.

(d) $R$ is an Artinian and semisimple ring.
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Proof. (a) \implies (b): Obvious.

(b) \implies (a): Suppose \( R \) is semisimple as a left \( R \)-module and let \( M \) be an \( R \)-module. Let \( m \in M \). Then \( Rm \cong R/A_R(m) \). Since \( R \) is semisimple as an \( R \)-module, also \( R/A_R(m) \) is semisimple, see \[1.1.28\]. Thus \( Rm \) is a semisimple \( R \)-module and so the sum of its simple \( R \)-submodules. Since this holds for all \( m \in R \), also \( R \) is the sum of its simple \( R \)-submodules. Thus \[1.1.27\] implies that \( M \) is semisimple.

(c) \implies (d): Since \( R \) is a semisimple \( R \)-module, \[1.4.9\] shows that \( J(R) = J_R(R) = 0 \). Moreover, \( R \) is the sum of its simple \( R \)-submodules and so there exists a finite set \( I \) of simple \( R \)-submodules with \( 1 \in \sum_i c_i I_i \). But then \( R = RI \leq \sum_i I_i \) and so \( R = \sum_i I_i \). Let \( W \) be an \( R \)-submodule of \( R \). The by \[1.1.26\] \( W \cong \bigoplus_i J_i \) for some \( J_i \subseteq I_i \) and so there exists a set of simple \( R \)-submodules \( \mathcal{A} \) of \( W \) with \( W = \bigoplus \mathcal{A} \). Each \( A \in \mathcal{A} \) is simple and so directly indecomposable. Also \( \text{End}_R(A) \) is a division ring and so a local ring. Thus \[1.2.14\] shows that \( |\mathcal{A}| \) is independent of the choice of \( \mathcal{A} \). Define \( d_W := |\mathcal{A}| \). If \( V \) is a proper \( R \)-submodules of \( R \), then \( 1.1.26 \) implies that \( W = V \oplus U \) for some non-zero \( R \)-submodules of \( U \). It follows that \( d_W = d_V + d_U \) and so \( d_V < d_W \). It follows that \( R \) fulfills DCC on \( R \)-submodules and so is an Artinian ring.

(c) \implies (b): Since \( R \) is Artinian, \( R \) fulfills DCC on \( R \)-submodules of \( R \). Note that \( J_R(R) = J(R) = 0 \) and so \[1.4.11\] shows that \( R \) is a semisimple \( R \)-module.

(c) \implies (d): This is \[1.5.21\].

(d) \implies (c): By \[1.5.20\] we know that \( J(R) = 0 \) for any semisimple ring \( R \).

Corollary 1.5.23. Let \( R \) be a simple ring. Then the following are equivalent.

(a) \( R \) is Artinian.

(b) \( R \cong \text{End}_D(M) \) for some division ring \( D \) and some finite dimensional \( D \)-space \( M \).

(c) \( R \) has a minimal left ideal.

(d) \( R \) is semisimple as a left \( R \)-module.

Proof. (a) \implies (b): Since \( R \) is a simple ring and \( J(R) \) is a proper ideal in \( R \) we know that \( J(R) = 0 \). Hence \( R \cong \bigoplus_{M \in \mathcal{M}} \text{End}_D(M) \) for a finite set of simple \( R \)-modules \( \mathcal{M} \). Since \( R \) is a simple rings we get \( |\mathcal{M}| = 1 \) and so \( b \) holds.

(b) \implies (a): Without loss \( R = \text{End}_D(M) \). By \[1.5.15\] we have \( D = \text{End}_D(M) \). Since \( R \) is simple ring, \( M \) is a faithful \( R \)-module. As \( \text{dim}_D M \) is finite we can apply rf[unique simple] \( e \) and conclude that \( U \to A_R(U) \) is an inclusion reversing bijection between the \( D \)-subspaces of \( M \) and the left ideal of \( R \). Note that there exists a maximal \( D \)-subspace of \( M \). Then \( A_R(U) \) is a minimal left ideal in \( R \).

(c) \implies (d): Let \( I \) be a minimal left ideal in \( R \). Then \( I \) is a simple \( R \)-module. For each \( r \in R \), the function \( \alpha : I \to R, i \mapsto ir \) is \( R \)-linear. Since \( I \) is simple \( R \)-module either ker \( \alpha = I \) or ker \( \alpha = 0 \). Hence either \( Ir = 0 \) or \( Ir \cong I \). In the latter case \( Ir \) is a simple \( R \)-submodule of \( R \). Thus \( IR \) is a sum of simple \( R \)-submodules. Hence \[1.1.26\] \( IR \) shows that \( IR \) is a semisimple \( R \)-module. Note that \( IR \) is non-zero ideal in \( R \) and since \( R \) is simple we get \( IR = R \). So \( R \) is a semisimple \( R \)-module.

(d) \implies (a): By \[1.5.22\] any ring \( R \), which semisimple as a left \( R \)-module, is Artinian.

Since there do exist non-Artinian simple rings, we conclude that there exists simple rings without minimal left ideals and in particular that semisimple rings do not need to be semisimple as a left \( R \)-module.

Corollary 1.5.24. Let \( K \) be a field and \( G \) a finite group with char \( K \nmid |G| \). Then \( K[G] \) is a semisimple, Artinian ring with \( 1(K[G]) = 0 \).
Proof. By 1.3.1 all \( \mathbb{K}G \)-modules are semisimple. So the Corollary follows from 1.5.22. \( \square \)

**Corollary 1.5.25.** Let \( \mathbb{K} \) be an algebraically closed field and \( R \) a finite dimensional \( \mathbb{K} \)-algebra with \( 1(R) = 0 \). Let \( M \) be a set of representatives for the isomorphism classes of simple \( R \)-modules. Then \( M \) is finite, each \( M \in M \) is finite dimensional over \( \mathbb{K} \) and

\[
R \cong \bigoplus_{M \in M} \text{End}_\mathbb{K}(M).
\]

**Proof.** Since \( \dim_{\mathbb{K}} R \) is finite, \( R \) is Artinian. Let \( M \in M \) and \( 0 \neq m \in M \). Then \( M \cong R/\text{Ann}_R(m) \). Thus \( \dim_{\mathbb{K}} M \leq \dim_{\mathbb{K}} R < \infty \). Since \( \mathbb{K} \) is algebraically closed 1.5.6 shows that \( \text{End}_\mathbb{K}(M) = \mathbb{K} \). The Corollary follows from 1.5.21. \( \square \)

**Proposition 1.5.26.** Let \( R \) be an Artinian ring. Then \( J(R) \) is nilpotent, that is \( J(R)^n = 0 \) for some \( n \in \mathbb{N} \).

**Proof.** Put \( J := J(R) \) and choose \( n \in \mathbb{N} \) with \( K := J^n \) minimal. If \( K^2 = 0 \), then \( J^{2n} = 0 \) and \( J \) is nilpotent. So suppose for a contradiction that \( K^2 \neq 0 \). Put \( A := \{ a \in K \mid Ka = 0 \} = \text{Ann}_K(K) \). Then \( A \) is an ideal in \( K \) with \( A \neq K \) and we can choose a left ideal \( L \) of \( R \) minimal with \( A \leq L \leq K \). Then \( L/A \) is a simple \( R \)-module. Hence by 1.4.8 \( J \leq \text{Ann}_R(L/A) \) and so \( JL \leq A \). It follows that \( KJL = 0 \). By minimality of \( K \), \( K = J^{n+1} = KJ \). Thus \( KL = 0 \), so \( L \leq A \) contrary to the choice of \( L \). \( \square \)

### 1.6 Tensor Products

**Definition 1.6.1.** Let \( R \) be a commutative ring and \( W \) an \( R \)-module. Furthermore let \( (M_i)_{i \in I} \) be a family of \( R \)-modules and \( f : \bigtimes_{i \in I} M_i \to M \) a function. Let \( J \) and \( K \) be subset of \( I \) with \( I = J \cup K \) and \( J \cap K = \emptyset \).

(a) \( M_J := \bigtimes_{i \in J} M_i \).

(b) \( \iota_{J,K} : M_J \times M_K \to M_{J}, ((m_j), (m_k)) \to (m_i) \)

(c) For \( m \in M_K \) define \( f_m : M_J \to W, n \to f(\iota_{J,K}(n,m)) \).

(d) \( f \) is called \( R \)-multilinear if for all \( i \in I \) all \( m \in M_{I_{\neq i}} \), \( f_m : M_i \to M \) is \( R \)-linear.

We usually will identify \((n,m) \in M_J \times M_K \) with \( \iota_{J,K}(n,m) \). In particular, we will write \( f(n,m) \) for \( f(\iota_{J,K}(n,m)) \) and so \( f_m(n) = f(n,m) \).

**Definition 1.6.2.** Let \( R \) be a commutative ring, \( (M_i)_{i \in I} \) a family of \( R \)-modules and \( f : M_i \to W \) an \( R \)-multilinear function. Suppose that for all \( R \)-multilinear functions \( g : M_i \to U \) there exists unique \( R \)-linear map \( h : W \to U \) with \( g = h \circ f \). Then \( f \) is called a tensor product of \((M_i)_{i \in I} \) over \( R \).

**Lemma 1.6.3.** Let \( R \) be a commutative ring and \( (M_i)_{i \in I} \) a family of \( R \)-modules. Then there exists a tensor map \( f : M_i \to W \) over \( R \) and \( f \) is unique up to isomorphism.

**Proof.** Let \( F \) be a free \( R \)-module with basis \((a(m))_{m \in M_i} \). Let \( Z \) be the \( R \)-submodule generated by all the \( a(ru + sv, n) - ra(u, n) - sa(v, n), r, s \in R, i \in I, u, v \in M_i \) and \( n \in M_{I_{\neq i}} \). Put \( W := F/Z \) and define \( f(m) = a(m) + Z \) for all \( m \in M_i \). Then clearly \( f \) is \( R \)-multilinear. Now let \( g : M_i \to N \) be \( R \)-multilinear. Then there exists a unique \( R \)-linear function \( \tilde{h} : F \to N \) with \( \tilde{h}(a(m)) = g(m) \) for all \( m \in M_i \). Since \( g \) is \( R \)-multilinear, \( Z \leq \ker \tilde{h} \) and so there exists an \( R \)-linear \( h : M \to N \) with \( h(a(m) + W) = g(m) \) for all \( m \in M_i \). Hence \( g = h \circ f \). The uniqueness of \( h \) is readily verified. So \( f \) is tensor product of \((M_i)_{i \in I} \) over \( R \).

That \( f \) is unique up to isomorphism is obvious. \( \square \)
1.6. TENSOR PRODUCTS

Notation 1.6.4. Let $R$ be a commutative ring, and $f : M_I \to W$ be a tensor product for the family of $R$-modules $(M_i)_{i \in I}$. Then

(a) $\bigotimes_{i \in I} M_i := \bigotimes M_i := W$.

(b) Let $m \in M_I$. Then $\otimes m := \bigotimes_{i \in I} m_i := f(m)$.

(c) Let $g : M_I \to N$ be $R$-multilinear and $h$ the unique $R$-linear function $W \to N$ with $g = h \circ f$. Then $\otimes g := h$.

Lemma 1.6.5. Let $R$ be a commutative ring and a $(M_i)_{i \in I}$ a finite family of $R$-modules.

(a) If $M_i = R$ for all $i \in I$, then $\bigotimes_{i \in I} R \to R, (r_i) \mapsto \prod_{i \in I} r_i$ is a tensor product of $(M_i, i \in I)$.

(b) Suppose for each $i \in I$, $M_i = \bigoplus \mathcal{W}_i$ for some set $\mathcal{W}_i$ of $R$-submodules of $M_i$.

\[
\bigotimes_{i \in I} M_i \cong_{R} \bigoplus_{i \in I} \left( \bigotimes_{i \in I} \mathcal{W}_i \biggm{\text{where } (W_i)_{i \in I} \in \bigotimes_{i \in I} \mathcal{W}_i} \right) .
\]

(c) Suppose that for each $i \in I$, $M_i$ is a free $R$-module with basis $\mathcal{B}_i$. Then $\bigotimes_{i \in I} M_i$ is a free $R$-module with basis

\[\bigotimes_{i \in I} \mathcal{B}_i := (\otimes b \mid b \in \bigotimes \mathcal{B}_i).\]

(d) If $I = \emptyset$, then $f : \bigotimes_{I} R \to R, () \mapsto 1$ is a tensor product of $()$.

(e) If $I = \{i\}$, then $\text{id}_M$ is a tensor product for $(M_i)_{i \in I}$ over $R$.

Proof. (a) Define

\[f : R^I \to R, \ (r_i) \mapsto \prod_{i \in I} r_i .\]

Then $f$ is $R$-multilinear. Let $g : R^I \to N$ be $R$-multilinear. Define $n := g((1)_{i \in I})$ and

\[h : R \to N, \ r \mapsto rn .\]

Then $g = h \circ f$ and $h$ is unique with this property. So (a) holds.

(b) For $i \in I$ and $W_i \in \mathcal{W}_i$ let $\pi_{W_i} : M_i \to W_i$ be the projection according to $M_i = \bigoplus \mathcal{W}_i$. Define

\[f : M_I \to \bigoplus \left( \bigotimes_{i \in I} W_i \biggm{\text{where } (W_i)_{i \in I} \in \bigotimes_{i \in I} \mathcal{W}_i} \right), \ (m_i) \mapsto \left( \bigotimes_{i \in I} \pi_{W_i}(m_i) \biggm{\text{where } (W_i)_{i \in I} \in \bigotimes_{i \in I} \mathcal{W}_i} \right) .\]

Note that for given $i \in I$ and $m_i \in M_i$, $\pi_{W_i}(m_i) = 0$ for almost all $W_i \in \mathcal{W}_i$. Hence also $\bigotimes_{i \in I} \pi_{W_i}(m_i) = 0$ for almost all $(W_i)_{i \in I} \in \bigotimes_{i \in I} \mathcal{W}_i$. Thus $f$ is a well-defined $R$-multilinear function.

Now let $g : M_I \to N$ be $R$-multilinear. For $W = (W_i)_{i \in I} \in \bigotimes_{i \in I} \mathcal{W}_i$ let $g_W$ be the restriction of $g$ to $\bigotimes_{i \in I} W_i$. Then there exists a unique $R$-linear function $h_W : \bigotimes_{i \in I} W_i \to N$ with $g_W(w) = h_W(\otimes w)$ for all $w \in \bigotimes_{i \in I} W_i$. Define

\[h : \bigoplus \left( \bigotimes_{i \in I} W_i \biggm{\text{where } W \in \bigotimes_{i \in I} \mathcal{W}_i} \right) \to N, \ \left( a_W \right)_{W \in \bigotimes_{i \in I} \mathcal{W}_i} \mapsto \sum_{W \in \bigotimes_{i \in I} \mathcal{W}_i} h_W(a_W) .\]
Then it is easy to check that $g = h \circ f$ and $h$ is unique with this property. So (b) holds.

(c) Let $b_i \in B_i$, then by \( \bigoplus_{i \in I} Rb_i \) is a free $R$-module of rank 1 with basis $\bigoplus_{i \in I} b_i$. Moreover, $M_i = \bigoplus_{b \in B_i} Rb$ and so (c) follows from (b).

(d) Note here that the direct product $\bigtimes_{\varnothing}$ of the empty family of sets is a set with one element, namely the empty tuple $()$. Given an $R$-module $N$ and $g : \bigtimes_{\varnothing} \to N, () \to n$. Define $h : R \to N, r \to rn$. Then $g = h \circ f$.

(e) is readily verified. \( \square \)

**Lemma 1.6.6.** Let $R$ be a commutative ring, $(M_i)_{i \in I}$ a family of $R$-modules and $\Delta$ be a partition of $I$. Then there exists a unique $R$-linear function

\[
\rho : \bigotimes_{i \in I} M_i \to \bigotimes_{D \in \Delta} \left( \bigotimes_{i \in D} M_i \right) \quad \text{with} \quad \otimes_{i \in I} m_i \mapsto \otimes_{D \in \Delta} (\otimes_{i \in D} m_i).
\]

Moreover, if $\Delta$ is finite, then $\rho$ is an isomorphism.

**Proof.** Observe that the function

\[
M_I \to \bigotimes_{D \in \Delta} \left( \bigotimes_{i \in D} M_i \right) \quad (m_i)_{i \in I} \mapsto \otimes_{D \in \Delta} (\otimes_{i \in D} m_i)
\]

is $R$-multilinear. Thus the uniqueness and existence of $\rho$ follows from the definition of the tensor product.

To show that $\rho$ is an isomorphism, if $\Delta$ is finite we may assume by induction that $|\Delta| = 2$. Say $\Delta = \{J, K\}$. Then $I = J \cup K$ and $J \cap K = \varnothing$. Let $q \in M_K$. Then

\[
f_q : M_J \to \bigotimes_{i \in J} M_i, \quad p \mapsto \otimes (p, q)
\]

is $R$-multilinear and we obtain an $R$-linear function

\[
\otimes f_q : \bigotimes_{i \in J} M_i \to \bigotimes_{i \in J} M_i, \quad \otimes p \mapsto \otimes (p, q).
\]

Hence for $a \in \bigotimes M_J$ we can define a $R$-multilinear function

\[
g_a : M_K \to \bigotimes_{i \in K} M_i, \quad q \mapsto (\otimes f_q)(a)
\]

and obtain an $R$-linear function

\[
\otimes g_a : \bigotimes_{i \in K} M_i \to \bigotimes_{i \in K} M_i, \quad \otimes q \mapsto (\otimes f_q)(a).
\]

This now gives rise to a $R$-bilinear function

\[
h : \bigotimes M_J \times \bigotimes M_K \to \bigotimes M_I, \quad (a, b) \mapsto (\otimes g_a)(b)
\]

and then a $R$-linear function

\[
\otimes h : \bigotimes M_J \otimes \bigotimes M_K \to \bigotimes M_I, \quad a \otimes b \mapsto (\otimes g_a)(b).
\]

For $p \in M_J, q \in M_K$ we have

\[
(\otimes h)(\otimes p \otimes \otimes q) = (\otimes g_{\otimes p})(\otimes q) = (\otimes f_q)(\otimes p) = \otimes (p, q)
\]

and so $\otimes h$ is inverse to $\rho$. \( \square \)
1.7. INDUCED AND COINDUCED MODULES

For finite Δ we will usually identify \( \bigotimes_{i \in \Delta} (\bigotimes_{j \in J} M_i) \) with \( \bigotimes_{i \in I} M_i \) via the map \( \rho \) of the preceding lemma. In particular, if \( I = J \cap K \) with \( J \cap K = \emptyset \) and \( \rho = (m_i)_{i \in J} \times (m_j)_{j \in J} \) and \( q = (m_k)_{k \in K} \in \bigotimes_{i \in I} M_i \) then

\[
(\otimes \rho) \otimes (\otimes q) = \otimes (\rho, q) = \otimes_{i \in I} m_i.
\]

**Definition 1.6.7.** Let \( R \) be a ring, \( X \) a right \( R \)-module and \( Y \) a left \( R \)-module, \( Z \) a \( \mathbb{Z} \)-module and \( f : X \times Y \to Z \) a function.

(a) \( f \) is called \( R \)-balanced if it is \( \mathbb{Z} \)-multilinear and for all \( x \in X, y \in Y \) and \( r \in R \), \( f(xr, y) = f(x, ry) \).

(b) Suppose that \( f \) is balanced and that for each \( R \)-balanced map \( g : X \times Y \to N \), there exists a unique \( \mathbb{Z} \)-linear map, \( h : Z \to N \) with \( g = h \circ f \). Then \( f \) is called a tensor map of \( X \) and \( Y \) over \( R \) and \( Z \) is called a tensor product of \( X \) and \( Y \) over \( R \).

**Lemma 1.6.8.** Let \( X \) be a right- and \( Y \) a left \( R \)-module. Then there exists a tensor product for \( X \) and \( Y \) over \( R \) and \( Z \) is a \( \mathbb{Z} \)-module and \( f : X \times Y \to Z \) a function.

Proof. Let \( F := X \otimes \mathbb{Z} Y \) and \( W \) the \( \mathbb{Z} \)-subspace generated by the \( (xr) \otimes y - x \otimes (ry), x \in X, y \in Y \) and \( r \in R \). Put \( Z := F / W \) and define \( f : X \times Y \to X, (x, y) \mapsto x \otimes y + W \). \( \square \)

We denote the tensor product of \( X \) and \( Y \) over \( R \) by \( X \otimes_R Y \). We reader might convince themselves that in the case of a commutative ring our two notation of tensor product agree. More precisely suppose that \( X \) and \( Y \) are left \( R \)-modules. Let \( \hat{X} \) be a the right \( R \)-module, define by \( \hat{X} = X \) as abelian groups and \( xr = rx \) for all \( x \in X, r \in R \). Then the tensor product for \( X \) and \( Y \) over \( R \) is also the tensor product for \( \hat{X} \) and \( Y \) over \( R \).

**Lemma 1.6.9.** Let \( R \) be a ring, \( X \) a free right \( R \)-module with basis \( B \) and \( Y \) a left \( R \)-module. Then the function

\[
Y_B \to X \otimes \mathbb{S} Y \quad (\gamma_b)_{b \in B} \mapsto \sum_{b \in B} b \otimes y_b
\]

is a \( \mathbb{Z} \)-isomorphism.

Proof. Since \( B \) is an \( S \)-basis for the right \( S \)-module \( X \) we have \( X = \bigoplus_{b \in B} bS \) and the function \( S \to bS, \ s \mapsto bs \) is an isomorphism of \( S \)-modules. Moreover, by \( S \otimes W = W \) with \( s \otimes w = sw \). Hence we obtained a sequence of \( \mathbb{Z} \)-isomorphism

\[
\begin{align*}
X \otimes_S Y &= (\bigoplus_{b \in B} bS) \otimes_R Y \cong \bigoplus_{b \in B} bS \otimes_S Y \cong \bigoplus_{b \in B} S \otimes_S Y = \bigoplus_{b \in B} Y = Y_B \\
\sum_{b \in B} b \otimes y_b &\mapsto (b1 \otimes y_b)_{b \in B} \mapsto (1 \otimes y_b)_{b \in B} = (y_b)_{b \in B}
\end{align*}
\]

\( \square \)

### 1.7 Induced and Coinduced Modules

**Definition 1.7.1.** Let \( S \) be a ring, \( R \) an \( S \)-ring, \( W \) an \( S \)-module and \( M \) an \( R \)-module. Recall from 1.5.4 that \( M \) is an \( S \)-module.

(a) Let \( f : W \to M \) be \( S \)-linear. We say that \( f \) is the function induced from \( W \) to \( R \) provided that whenever \( N \) is an \( R \)-module and \( g : W \to N \) is \( S \)-linear, then there exists a unique \( R \)-linear function \( h : M \to N \) with \( g = h \circ f \). In this case \( M \) is called the \( R \)-module induced from \( W \) to \( R \) and is denoted by \( W \downarrow^R \). \( f \) is denoted by \( f^S \) and \( h \) by \( g^S \).


(b) Let \( f : M \rightarrow W \) be \( S \)-linear. We say that \( f \) is coinduced from \( W \) to \( R \) provided that whenever \( N \) is an \( R \)-module and \( g : N \rightarrow W \) is \( S \)-linear, then there exists a unique \( R \)-linear function \( h : N \rightarrow M \) with \( g = f \circ h \). In this case \( M \) is called the \( R \)-module coinduced from \( W \) to \( R \) and is denoted by \( W \otimes_{S} R \). \( f \) is denoted by \( \pi_{S}^{R}(W) \) and \( h \) by \( g \otimes_{S} R \).

**Lemma 1.7.2.** Let \( S \) be a ring, \( R \) an \( S \)-ring and \( W \) an \( S \)-module. Recall from 1.5.4(b) that \( R \) is a right \( S \)-module.

(a) There exists a unique \( R \)-module structure

\[
R \times R \otimes_{S} W \rightarrow R \otimes_{S} W \quad \text{with} \quad (r, t \otimes w) \mapsto rt \otimes w
\]

for all \( r, t \in R, w \in W \).

(b) The function

\[
f : W \rightarrow R \otimes_{S} W, \quad w \mapsto 1 \otimes w
\]

is induced from \( W \) to \( R \).

(c) Any function induced from \( W \) to \( R \) is isomorphic to \( f \).

**Proof.**

(a) Let \( r \in R \) and define

\[
\alpha_{r} : R \times W \rightarrow R \otimes_{S} W, \quad (t, w) \mapsto rt \otimes w.
\]

Let \( t \in R, s \in S \) and \( w \in W \). Then

\[
\alpha_{r}(ts, w) = r(ts) \otimes w = (rt)S \otimes w = rt \otimes sw = \alpha_{r}(t, sw)
\]

and so \( \alpha_{r} \) is \( S \)-balanced. Hence the universal property of the tensor product gives rise to uniquely determined function

\[
\otimes \alpha_{r} : R \otimes_{S} W \rightarrow R \otimes_{S} W, \quad t \otimes w \mapsto rt \otimes w
\]

This yields a function

\[
f : R \times R \otimes_{S} W \rightarrow R \otimes_{S} W, \quad (r, a) \mapsto \alpha_{r}(a).
\]

Then \( f(r, t \otimes w) = \alpha_{r}(t \otimes w) = rt \otimes w \) and (a) holds.

(b) Let \( N \) be an \( R \)-module and \( g : W \rightarrow N \) an \( S \)-linear function. Then the function \( R \times W \rightarrow N, (r, w) \mapsto r(g(w)) \) is \( S \)-balanced. So by definition of the tensor product there exists a \( \mathbb{Z} \)-linear function

\[
h : R \otimes_{S} W \rightarrow N, \quad r \otimes w \mapsto r(g(w)).
\]

Then \( h(f(w)) = h(1 \otimes w) = 1(g(w)) = g(w) \) and so \( h \circ f = g \). Moreover

\[
r(h(t \otimes w)) = r(t(g(w))) = (rt)(g(w)) = h(rt \otimes w) = h(r(t \otimes w)).
\]

and so \( h \) is \( R \)-linear. Thus (b) holds.

(c) is obvious. \( \square \)

**Lemma 1.7.3.** Let \( S \) be a ring, \( R \) an \( S \)-ring and \( W \) an \( S \)-module. Recall that \( R \) is an \( S \)-module.
1.7. INDUCED AND COINDUCED MODULES

(a) $\text{Hom}_S(R, W)$ is an $R$-module via $(ta)(r) = \alpha(rt)$ for all $r, t \in R, \alpha \in \text{Hom}_S(R, W)$.

(b) The function

$$f: \text{Hom}_S(R, W) \rightarrow W, \quad \alpha \mapsto \alpha(1)$$

is $S$-linear and coinduced from $W$ to $R$.

(c) Any function coinduced from $W$ to $R$ is isomorphic to $f$.

Proof. (a) We first will verify that $ta$ is $S$-linear. Let $s \in S$ and $r \in R$. Then

$$(ta)(sr) = \alpha((sr)t) = \alpha(s(rt)) = s(\alpha(rt)) = s((ta)(r)).$$

So indeed $ta \in \text{Hom}_S(R, W)$. To check that this is an $R$-module structure let $u \in R$. Then

$$(ut)\alpha(r) = \alpha(r(ut)) = \alpha((ru)r) = (ta)(ru) = (u(\alpha))(r).$$

So $(ut)\alpha = u(\alpha)$ and (a) is proved.

(b) We have

$$f(sa) = (sa)(1) = \alpha(s1) = s(\alpha(1)) = s(f(\alpha)),$$

and so $f$ is $S$-linear. Let $N$ be an $R$-module and $g: N \rightarrow W$ be $S$-linear. Define

$$h: N \rightarrow \text{Hom}_S(R, W) \quad \text{by} \quad (h(n))(r) = g(rn).$$

for all $n \in N$ and $r \in R$. For $n \in N, r \in R$ and $s \in S$ we have

$$(h(n))(sr) = g((sr)n) = g(srn) = s(g(rn)) = s((h(n))(r))$$

and so $h(n)$ is indeed $S$-linear. Also

$$f(h(n))(1) = g(1n) = g(n)$$

and so $f \circ h = g$.

(c) Obvious. □

Proposition 1.7.4 (Frobenius Reciprocity). Let $S$ be a ring, $R$ an $S$-ring, $W$ an $S$-module and $V$ an $R$-module.

(a) The function $\text{Hom}_R(W \uparrow^R_S, V) \rightarrow \text{Hom}_S(W, V), \alpha \mapsto \alpha \circ \iota^R_S(W)$ is a $\mathbb{Z}$-isomorphism with inverse $\beta \mapsto \beta \uparrow^R_S$.

(b) The function $\text{Hom}_R(V, W \downarrow^R_S) \rightarrow \text{Hom}_S(V, W), \alpha \mapsto \pi^S_R(W) \circ \alpha$ is a $\mathbb{Z}$-isomorphism with inverse $\beta \mapsto \beta \downarrow^R_S$.

Proof. The functions in question are clearly $\mathbb{Z}$-linear.

(a) Put $\iota := \iota^R_S(W)$ and $M := W \uparrow^R_S$. By definition of an induced function, for each $\beta \in \text{Hom}_S(W, V)$ there exists a unique $\alpha \in \text{Hom}_S(W, V)$ with $\beta = \alpha \circ \iota$, namely $\alpha = \beta \uparrow^R_S$. This gives (a).

(b) Put $\pi := \pi^S_R(W)$ and $M := W \downarrow^R_S$. By definition of a coinduced function, for each $\beta \in \text{Hom}_S(V, W)$ there exists a unique $\alpha \in \text{Hom}_S(V, W)$ with $\beta = \pi \circ \alpha$, namely $\alpha = \beta \downarrow^R_S$. This gives (b). □
Lemma 1.7.5. Let S be a ring, R an S-ring and W an S-module.

(a) Suppose that R, as a right S-module, is free with basis \( B \). Put \( \iota := \iota^S(W) \). Then the function

\[
W_B \rightarrow W^*_S, \quad (w_b)_{b \in B} \rightarrow \sum_{b \in B} b(\iota(w_b))
\]

is a \( \mathbb{Z} \)-isomorphism. In particular, \( \iota \) is 1-1.

(b) Suppose that R, as an S-module, is free with basis \( B \). Put \( \pi := \pi^S(W) \). Then the function

\[
W \rightarrow W^*_S, \quad \alpha \mapsto (\pi(\alpha))_{b \in B}
\]

is a \( \mathbb{Z} \)-isomorphism. In particular, \( \pi \) is onto.

Proof. (a) By 1.7.2 we may assume that \( W^*_S = R \otimes_S W \) and \( \iota(w) = 1 \otimes w \). Then \( b(\iota(w_b)) = b(1 \otimes w_b) = b \otimes w_b \) and the function in (a) becomes

\[
W_B \rightarrow R \otimes_S W, \quad (w_b)_{b \in B} \rightarrow \sum_{b \in B} b \otimes w_b
\]

By 1.6.9 this function is a \( \mathbb{Z} \)-isomorphism.

(b) By 1.7.3 we may assume that \( W = \text{Hom}_S(R, W) \) and \( \pi(\alpha) = \alpha(1) \) for \( \alpha \in \text{Hom}_S(R, W) \). For \( b \in B \) we get \( \pi(b \alpha) = (b \alpha)(1) = \alpha(1b) = \alpha(b) \) and so the function in (b) becomes

\[
\text{Hom}_S(R, W) \rightarrow W^*_S, \quad \alpha \mapsto (\alpha(b))_{b \in B}
\]

This function is clearly \( \mathbb{Z} \)-linear and by definition of a free \( S \)-module, for each \( (w_b)_{b \in B} \) there exists a unique \( \alpha \in \text{Hom}_S(R, W) \) with \( \alpha(b) = w_b \) for all \( b \in B \). So the function is a bijection. \( \square \)

1.8 Absolutely Simple Modules

Lemma 1.8.1. Let \( R \) be commutative ring and let \( A \) and \( B \) be \( R \)-algebras. Then there exists a unique \( R \)-multilinear binary operation

\[
A \otimes_R B \times (A \otimes_R B) \rightarrow A \otimes_R B \text{ with } (a \otimes b) \cdot (c \otimes d) = (ac) \otimes (bd)
\]

for all \( a, c \in A, b, d \in B \). Moreover, \( A \otimes_R B \) is \( R \)-algebra.

Proof. For a fixed \( (c, d) \in A \times B \), the function

\[
A \times B \rightarrow A \otimes B, \quad (a, b) \rightarrow (ac) \otimes (bd)
\]

is \( R \)-multilinear and we obtain a uniquely determined \( R \)-linear map

\[
f_{cd} : A \otimes B \rightarrow A \otimes B, \quad a \otimes b \rightarrow (ac) \otimes (bd).
\]

The map \( A \times B \rightarrow \text{Hom}_R(A \otimes B, A \otimes B), \quad (c, d) \rightarrow f_{cd} \) is \( R \)-multilinear and so we obtain a uniquely determined \( R \)-linear map

\[
f : A \otimes B \rightarrow \text{Hom}_R(A \otimes B, A \otimes B), \quad c \otimes d \rightarrow f_{cd}.
\]

For \( x, y \in A \otimes B \) define \( xy = f(y)(x) \). The lemma is now readily verified. \( \square \)
Lemma 1.8.2. Let $R$ be a commutative ring, $B$ an $R$-algebra, $A$ and $R$-module and $M$ an $A$-module.

(a) $A \otimes_R M$ is an $B$-module via
\[ T \times A \otimes_R M \rightarrow A \otimes_R M \quad \text{with} \quad (b, a \otimes m) \mapsto a \otimes bm. \]
for all $a \in A, b \in B, m \in M$.

(b) Suppose $M$ is a simple $B$-module. Then $A \otimes_R M$ is a semisimple $B$-module and each simple $B$-submodule of $A \otimes_R M$ is isomorphic to $M$.

(c) Suppose that
(i) $M$ and $N$ are simple $B$-modules.
(ii) $A \otimes_R M \neq 0$
(iii) $A \otimes_R M$ and $A \otimes_R N$ are isomorphic $B$-modules.

Then $M$ and $N$ are isomorphic $B$-modules.

Proof. (a) is readily verified.

Let $a \in A$. Then the function
\[ \phi_a : M \rightarrow A \otimes_R M, m \mapsto a \otimes m. \]

is a $B$-linear. Since $M$ is a simple $B$-module, we conclude that either $\text{Im} \, \phi_a = \text{Im} \, \phi_a \cong_B M$. Put $S := \{\text{Im} \, \phi_a \mid a \in A, \text{Im} \, \phi_a \neq 0\}$. Then $A \otimes_R M = \sum S$ and each element of $S$ is a simple $B$-module isomorphic to $M$. Hence $[1.1.26g]$ shows that $A \otimes_R M$ is a semisimple $B$-module. Moreover, by $[1.1.26h]$ each simple $B$-submodule of $A \otimes_R M$ is isomorphic to an element of $S$ and so isomorphic to $M$.

By (b) $A \otimes_R M$ is a semisimple $B$-module and by hypothesis, $A \otimes_R M \neq 0$. Hence $A \otimes_R M$ contains a simple $B$-submodule $S$. By (b) $S \cong_B M$. As $A \otimes_R M \cong_B A \otimes_R N$ we know that $S$ is isomorphic to a simple $B$-submodule of $A \otimes_R N$. Thus (b) shows that $S \cong_B N$ and so also $M \cong_B N$. \[\square\]

Lemma 1.8.3. Let $F \leq \mathbb{K}$ be a field extension, $A$ a finite dimensional $F$-algebra and $B := \mathbb{K} \otimes_F A$.

(a) Let $T$ be a simple $B$-module. Then $T$ is a semisimple $A$-module and any two simple $T$-submodules of $S$ are isomorphic.

(b) Let $S$ be a simple $A$-module. Then there exists a simple $B$-module with $S$ as an $A$-submodule.

(c) The function
\[ S(B) \rightarrow S(A), \quad T \mapsto [S] \]

where $T \in \mathcal{T}$, $S$ a simple $A$-submodule of $T$ and $[S]$ is the class of $A$-modules isomorphic to $S$, is a well-defined and surjective.

(d) The number of isomorphism classes of simple $A$-modules is less or equal to the number of isomorphism classes of simple $B$-modules.
Let \( 0 \neq t \in T \). Since \( \dim_T A \) is finite, also \( \dim_T At \) is finite. Let \( S \) be a non-zero \( A \)-submodule of \( At \) with \( \dim_T S \) minimal. Then \( S \) is a simple \( B \)-submodule of \( T \). Observe that \( \sum_{k \in K} kS \) is non-zero \( B \)-submodule of \( T \) and so \( T = \sum_{k \in K} kS \). Note also that each \( kS \), \( k \in K \) is isomorphic to \( S \) as \( A \)-module. Hence (a) holds.

Put \( U := K \otimes_T S \). Since \( S \) is a simple \( A \)-module and \( \dim_T S \) is finite, also \( \dim_K U \) is finite and there exist a simple \( B \)-submodule \( T \) of \( U \). By (a) \( T \) contains a simple \( A \)-submodule \( S \). Also by 1.8.2(b) any simple \( A \)-submodule of \( U \) is isomorphic to \( U \). So \( S_A \) and (b) follows.

By (c) the function is well-defined and by (b) the function is surjective. 

\[ \square \]

**Lemma 1.8.4.** Let \( R \) be a commutative ring, let \( A \) and \( B \) be \( R \)-algebras and let \( M \) be \( B \)-module. Then \( A \otimes_R M \) is an \( A \otimes_R B \)-module via

\[ A \otimes_R B \times A \otimes_R M \to A \otimes_R M \quad \text{with} \quad (a \otimes b, c \otimes m) \mapsto ac \otimes bm. \]

for all \( a, c \in S, b \in B, m \in M \).

**Proof.** Readily verified. \[ \square \]

**Definition 1.8.5.** Let \( K \) be a field, \( A \) a \( K \)-algebra and \( M \) an \( A \)-module. Then \( M \) is called an absolutely simple \( A \)-module over \( K \) provided that \( F \otimes_K M \) is a simple \( F \otimes_K A \)-module for all fields \( F \) with \( K \leq F \).

**Lemma 1.8.6.** Let \( K \) be a field, \( A \) an \( K \)-algebra and \( M \) a simple \( A \)-module. Then \( M \) is absolutely simple over \( K \) if and only if \( \text{End}_A(M) = K \).

**Proof.** \( \implies \): Suppose \( M \) is absolutely simple over \( K \) and let \( F \) be a subfield of \( \text{End}_A(M) \) with \( K \leq F \). Note that the function \( F \otimes_K M \to M, (f, m) \mapsto f(m) \) is \( K \)-multilinear and so we obtain a \( K \)-linear function

\[ \alpha : F \otimes_K M \to M, \quad \text{with} \quad f \otimes m \mapsto f(m). \]

Also observe that \( M \) is an \( F \otimes_K A \)-module via \( (f \otimes a) \cdot m = f(am) \) for all \( f \in F, a \in A, m \in M \). We claim that \( \alpha \) is \( F \otimes_K A \)-linear. For this let \( f, g \in F, a \in A \) and \( m \in M \). Then

\[ \alpha((f \otimes a)(g \otimes m)) = \alpha(fg \otimes am) = (fg)(am) = f(g(am)) = f((f \otimes a)(g \otimes m)) = (f \otimes a)(g \otimes m). \]

So \( \alpha \) is indeed \( F \otimes_K A \)-linear. In particular, \( \ker \alpha \) is an \( F \otimes_K A \)-submodule of \( F \otimes_K M \). Let \( 1 \otimes M = \{ 1 \otimes m \mid m \in M \} \) and observe that \( f|_{1 \otimes M} \) is onto. Thus \( F \otimes_K M = 1 \otimes M + \ker \alpha \). Since \( M \) is absolutely simple over \( K \) we know that \( F \otimes_K M \) is a simple \( F \otimes_K A \)-module. It follows that \( \ker \alpha = 0 \) and so \( F \otimes M = 1 \otimes M \). Let \( B \) be a \( F \)-basis for \( M \). Then \( 1 \otimes B \) is an \( F \)-basis for \( F \otimes_K M \) and spans \( 1 \otimes M \) as a \( K \)-space. Hence \( F = K|_M \). Let \( d \in D := \text{End}_A(M) \) and \( F(d) \) be the subdivision ring of \( D \) generated by \( K|_M \) and \( d \). Since \( K|_M \leq Z(D) \), \( F \) is a field. So \( d \in F = K|_M \) and thus \( D = K|_M \).

\( \iff \): Suppose that \( \text{End}_A(M) = K|_M \). Then Jacobson’s Density Theorem shows that \( A \) is dense on \( M \) with respect to \( K \). Let \( F \) be a field extension of \( K \) and let \( v, w \in F \otimes_K M \) with \( v \neq 0 \). We will show that \( w = bv \) for some \( b \in F \otimes_K A \). For this choose \( e_1, \ldots, e_n, f_1, \ldots, f_m \) in \( F \) and \( v_1, \ldots, v_n, w_1, \ldots, w_m \in M \) with

\[ v = \sum_{i=1}^n e_i \otimes v_i \quad \text{and} \quad w = \sum_{j=1}^m f_j \otimes w_j. \]
We choose \( n \) minimal with respect to these properties. Then \( (v_i)_{i=1}^n \) is linearly independent over \( \mathbb{K} \) and \( e_1 \neq 0 \). Let \( 1 \leq j \leq m \). Since \( A \) is dense on \( M \) with respect to \( \mathbb{K} \), there exist \( a_j \in A \) with \( a_j v_1 = w_j \) and \( a_j v_i = 0 \) for all \( 2 \leq i \leq n \). Put \( b := \sum_{j=1}^{m} \frac{v_j}{e_1} \otimes a_j \in \mathbb{F} \otimes_{\mathbb{K}} A \). Then

\[
 bv = \left( \sum_{j=1}^{m} \frac{f_j}{e_1} \otimes a_j \right) \left( \sum_{i=1}^{n} e_i \otimes v_i \right) = \sum_{j=1}^{m} \sum_{i=1}^{n} \left( \frac{f_j}{e_1} \otimes a_j \right) (e_i v_i) = \sum_{j=1}^{m} \sum_{i=1}^{n} \left( \frac{f_j}{e_1} e_i \right) (a_j v_i) = \sum_{j=1}^{m} f_j w_j = w,
\]

and so \( w \in (\mathbb{F} \otimes_{\mathbb{K}} A)v \). We conclude that \( (\mathbb{F} \otimes_{\mathbb{K}} A)v = \mathbb{F} \otimes_{\mathbb{K}} M \) and so \( \mathbb{F} \otimes_{\mathbb{K}} M \) is a simple \( \mathbb{F} \otimes_{\mathbb{K}} A \)-module. \( \square \)

**Corollary 1.8.7.** Let \( \mathbb{K} \) be a field, \( A \) a \( \mathbb{K} \)-algebra and \( M \) a simple \( A \)-module. Let \( \mathbb{F} \) be a maximal subfield of \( \text{End}_A(M) \). Then \( \mathbb{K}|M \leq \mathbb{F} \) and \( M \) is an absolutely simple \( \mathbb{F} \otimes_{\mathbb{K}} A \)-module over \( \mathbb{F} \).

**Proof.** Let \( \mathbb{D} = \text{End}_A(M) \). We have \( \text{End}_{\mathbb{F} \otimes_{\mathbb{K}} A}(M) = \text{End}_A(M) \cap \text{End}_\mathbb{F}(M) = C_\mathbb{F}(\mathbb{F}) \). Note that \( \mathbb{F} \leq \mathbb{Z}(C_\mathbb{F}(\mathbb{F})) \) and the maximality of \( \mathbb{F} \) implies that \( C_\mathbb{F}(\mathbb{F}) = \mathbb{F} \). It follows that \( \mathbb{K}|M \leq \mathbb{F} \) and \( \text{End}_{\mathbb{F} \otimes_{\mathbb{K}} A}(M) = \mathbb{F} \). Now 1.8.6 shows that \( M \) is an absolutely simple \( \mathbb{F} \otimes_{\mathbb{K}} A \)-module over \( \mathbb{F} \). \( \square \)

**Corollary 1.8.8.** Let \( \mathbb{K} \) be a algebraically closed field, \( A \) a \( \mathbb{K} \)-algebra and \( M \) a simple \( A \)-module. If \( M \) is finite dimensional over \( \mathbb{K} \), then \( M \) is an absolutely simple \( A \)-module over \( \mathbb{K} \).

**Proof.** By 1.5.6 \( \text{End}_A(M) = \mathbb{K} \) and so 1.8.6 implies that \( M \) is absolutely simple over \( \mathbb{K} \). \( \square \)

### 1.9 Systems of Imprimitivity and Clifford Theory

**Definition 1.9.1.** Let \( R \) be a ring, \( G \) a group and \( M \) an \( R[G] \)-module.

(a) A system of imprimitivity for \( R[G] \) on \( M \) is a tuple \( (M_b)_{b \in \mathcal{B}} \) such that

(i) \( \mathcal{B} \) is a \( G \)-set.

(ii) For \( b \in \mathcal{B} \), \( M_b \) is a non-zero \( R \)-submodule of \( M \).

(iii) \( gM_b = M_{gb} \) for all \( g \in G, b \in \mathcal{B} \).

(iv) \( M = \bigoplus_{b \in \mathcal{B}} M_b \).

(b) A system of imprimitivity is called proper if \( |\mathcal{B}| > 1 \).

(c) An \( R[G] \)-module with a proper system of imprimitivity is called imprimitive.

(d) An \( R[G] \)-module \( M \) is called primitive if \( M \) is simple and not imprimitive.

**Example 1.9.2.** Let \( G \) be a group acting on a set \( \mathcal{B} \) and let \( R \) be a ring. Let \( M \) be free \( R \)-module with basis \( (m_b)_{b \in \mathcal{B}} \). Note that \( M \) is a \( R[G] \)-module via \( gm_b = m_{gb} \) for \( g \in G \) and \( b \in \mathcal{B} \). Then \( (Rm_b)_{b \in \mathcal{B}} \) is a system of imprimitivity for \( R[G] \) on \( M \).

**Remark 1.9.3.** Let \( G \) be group, \( R \) a ring and \( M \) an \( R[G] \)-module. Let \( \mathcal{B} \) be a \( G \)-invariant set of non-zero \( R \)-submodules of \( M \) with \( M = \bigoplus \mathcal{B} \). Then \( (B)_{b \in \mathcal{B}} \) is a system of imprimitivity for \( R[G] \) on \( M \).

**Lemma 1.9.4.** Let \( R \) be a ring, \( G \) a group, \( M \) an \( R[G] \)-module and \( (M_b)_{b \in \mathcal{B}} \) a system of imprimitivity for \( R[G] \) on \( M \). Fix \( a \in \mathcal{B} \) and let \( W_a \) be non-zero \( R[C_G(a)] \)-module of \( M_a \). Put \( W := R[G]W_a \). Then

(a) \( W \cap M_{ga} = gW_a \) for all \( g \in G \).

(b) \( W \leq \sum_{b \in Ga} M_b \).
Let $b \in G$ and choose $g \in G$ with $b = ga$. Define $W_g := gW_a$. Since $C_G(a)W_a = W_a$, this is well defined. By definition of system of imprimitivity $M_b = gM_a$ and so $W_b \leq M_b$. Thus

$$W = R[G]W_a = GW_a = \sum_{g \in G} gW_a = \sum_{b \in G} W_b = \bigoplus_{b \in G} W_b$$

Hence $W \cap M_b = W_b$ for all $b \in \mathcal{B}$ and the lemma is proved. \qed

**Notation 1.9.5.** Let $R$ be ring, $G$ a group, $H \leq G$ and $W$ an $R_rG_s$-module. Then we write $W \twoheadrightarrow_{H}^G$ for $W \twoheadrightarrow_{R}^G \twoheadrightarrow_{H}^G$.  

**Lemma 1.9.6.** Let $R$ be ring, $G$ a group, $H \leq G$ and $W$ is a non zero $R_rG_s$-module. Put $V := W \twoheadrightarrow_{H}^G$ and $\iota := \iota_H^G(W)$. Let $T$ be a transversal to $H$ in $G$.

(a) $T \iota(W) = g\iota(W)$ for all $T = gH \in G/H$.

(b) Let $T$ be a transversal to $H$ in $G$. Then the function $\alpha : W_T \rightarrow V$, $(w_t)_{t \in T} \mapsto \sum_{t \in T} (t(w_t))$

is a $\mathbb{Z}$-isomorphism

(c) $\iota$ is 1-1.

(d) $V = \bigoplus_{t \in T} t(W)$ and $(T \iota(W))_{T \in G/H}$ is a system of imprimitivity for $R[G]$ on $V$.

**Proof.** (b): Since $\iota$ is $R[H]$-linear, we have $(gH)\iota(W) = g(\iota(HW)) = g\iota(W)$.

(c) and (a): Note that

$$R[G] = \bigoplus_{g \in G} Rg = \bigoplus_{t \in T} \bigoplus_{h \in H} Rh = \bigoplus_{t \in T} \left( \bigoplus_{h \in H} Rh \right) = \bigoplus_{t \in T} tR[H]$$

and so $T$ is a basis for $R[G]$ as a right $R[H]$-module. Hence (c) and (a) follows from 1.7.5(a)

(d): For $t \in T$ define $W_t := \{ w \in W_T \mid w_s = 0 \text{for all } s \in T \setminus \{t\} \}$. Then $W_T \bigoplus_{t \in T} W_t$ and $\alpha(W_t) = t(W)$. Since $\alpha$ is a $\mathbb{Z}$-isomorphism this gives

$$V = \bigoplus_{t \in T} t(W)$$

Since $T \rightarrow G/H, \rightarrow tH$ is a bijection and, by (a), $t(W) = (tH)\iota(W)$ this gives

$$V = \bigoplus_{T \in G/H} T \iota(W)$$

\qed

**Lemma 1.9.7.** Let $R$ be ring, $G$ a group, $V$ an $R[G]$-module and $(V_b)_{b \in \mathcal{B}}$ a system of imprimitivity for $R[G]$ on $V$. Let $b \in \mathcal{B}$.

(a) $V_b$ is an $R[C_G(b)]$-submodule of $V$. 

(b) Since $\iota$ is $R[H]$-linear, we have $(gH)\iota(W) = g(\iota(HW)) = g\iota(W)$.

(c) and (a): Note that

$$R[G] = \bigoplus_{g \in G} Rg = \bigoplus_{t \in T} \bigoplus_{h \in H} Rh = \bigoplus_{t \in T} \left( \bigoplus_{h \in H} Rh \right) = \bigoplus_{t \in T} tR[H]$$

and so $T$ is a basis for $R[G]$ as a right $R[H]$-module. Hence (c) and (a) follows from 1.7.5(a)

(d): For $t \in T$ define $W_t := \{ w \in W_T \mid w_s = 0 \text{for all } s \in T \setminus \{t\} \}$. Then $W_T \bigoplus_{t \in T} W_t$ and $\alpha(W_t) = t(W)$. Since $\alpha$ is a $\mathbb{Z}$-isomorphism this gives

$$V = \bigoplus_{t \in T} t(W)$$

Since $T \rightarrow G/H, \rightarrow tH$ is a bijection and, by (a), $t(W) = (tH)\iota(W)$ this gives

$$V = \bigoplus_{T \in G/H} T \iota(W)$$

\qed
Proof. Put \( \rho := \mathcal{C}_G(b)(V_b) \). Then there exists a unique \( R[G] \)-linear function

\[
\rho : \quad V_b \mathcal{C}_G(b) \rightarrow V \quad \text{with} \quad \rho(v) \rightarrow v
\]

for all \( v \in V_b \).

(c) \( \rho \) is 1-1 and \( \text{Im} \rho = \sum_{a \in Gb} V_a \).

(d) Suppose \( G \) acts transitively on \( \mathcal{B} \). Then \( \rho \) is an isomorphism of \( R[G] \)-modules.

\[\text{Proof.} \quad \text{Put} \ H := \mathcal{C}_G(b) \text{ and } W := V_b \text{ and } \mathcal{T} \text{ be a left transversal for } H \text{ on } G. \]

(1): For \( h \in H \) we have \( hV_b = V_{hb} = V_b \). Also by definition of a system of imprimitivity we know that \( V_b \) is an \( R \)-submodule for \( V \). Hence \( W \) is an \( R[H] \)-submodule of \( V \).

(2): Let \( j : W \rightarrow V \), \( w \rightarrow w \) be the inclusion function. The uniqueness and existence of \( \rho \) follows from the definition of the induced module, namely \( \rho = j \mathcal{C}_{G} \).

(3) Let \( u \in W \, \mathcal{C}_{G} \). By \text{1.9.6 b} we know that \( u = \sum_{r \in \mathcal{T}} t(\iota(w_r)) \) for some \( (w_r)_{r \in \mathcal{T}} \in W_\mathcal{T} \). So

\[
(*) \quad \rho(u) = \rho \left( \sum_{r \in \mathcal{T}} t(\iota(w_r)) \right) = \sum_{r \in \mathcal{T}} \rho(t(\iota(w_r))) = \sum_{r \in \mathcal{T}} tw_r.
\]

Since \( w_r \in V_b \) we have \( tw_r \in V_{yb} \). Thus \( (*) \) shows that \( \text{Im} \rho \subseteq \sum_{a \in Gb} V_a \). Let \( a \in Gb \) and \( m \in V_a \). Then \( a = sb \) for some \( s \in \mathcal{T} \). Note that \( t^{-1}m \in V_b \). By \( (*) \) \( \rho(t(\iota(t^{-1}m))) = tt^{-1}m = m \). Hence \( V_a \subseteq \text{Im} \rho \) and so \( \text{Im} \rho = \sum_{a \in Gb} V_a \).

Suppose that \( u \in \ker \rho \). Since \( \mathcal{T} \) is transversal to \( \mathcal{C}_G(b) \) we have \( tb \neq sb \) for all \( t, s \in \mathcal{T} \) with \( t \neq s \). Recall that \( V = \bigoplus_{g \in \mathcal{B}} V_b \), \( tw_r \in V_{yb} \) and \( \rho(u) = 0 \). Hence \( (*) \) shows that \( tw_r = 0 \) for all \( r \in \mathcal{T} \). Hence also \( w_r = 0 \) and \( u = 0 \). So \( \rho \) is 1-1.

(4): If \( G \) acts transitively on \( \mathcal{B} \), then \( Gb = \mathcal{B} \) and so \( (3) \) shows that \( \text{Im} \rho = \sum_{a \in \mathcal{B}} V_a = V \). So \( \rho \) is onto. By \( (1) \) \( \rho \) is also 1-1 and so \( (4) \) holds.

\[\text{Definition 1.9.8.} \quad \text{Let } R \text{ be a ring and } M \text{ an } R\text{-module.} \]

(a) Let \( \alpha \in \text{Aut}(R) \). Then \( \alpha M \) denotes the \( R \)-module with \( M = \alpha M \) as an abelian group and

\[
r \alpha \cdot m := \alpha^{-1}(r)m
\]

for all \( r \in R, m \in M \).

(b) Let \( G \text{ group, } N \cong G, W \text{ an } R[N]\text{-module and } g \in G \). Then \( \alpha N \) := \( \alpha N \text{ where } \alpha \text{ is unique automorphism of } R[N] \) with \( \alpha(r) = r \) and \( \alpha(g) = g \alpha^{-1}g^{-1} \) for all \( r \in R, n \in N \). So

\[
\alpha \left( \sum_{n \in N} r_n n \right) = \sum_{n \in N} r_{g^{-1}ng} n, \quad r \alpha \cdot w = r w, \quad n \alpha \cdot w = (g^{-1}ng)w
\]

for all \( r \in R, n \in N \) and \( w \in W \).

(c) \( [M] \) denotes the isomorphism class of \( M \), that is the class of all \( R \)-modules isomorphic to \( M \).

(d) Let \( M \) be an isomorphism class of \( R \)-modules and \( (a, \in) \text{Aut}(R) \). Then \( \alpha M := \{ \alpha M \mid M \in M \} \).

\[\text{Remark 1.9.9.} \quad \text{Let } R \text{ be an ring, let } V \text{ and } W \text{ be } R\text{-modules and let } \alpha, \beta \in \text{Aut}(R). \text{ Then} \]

\[\ldots\]
(a) \( \text{Hom}_R(V, W) = \text{Hom}_R(\mathbb{V}, \mathbb{W}) \). In particular, \( V \cong_R W \) if and only if \( \varphi V \cong_R \varphi W \).

(b) Let \( \mathcal{M} \) be a isomorphism class of \( R \)-modules, \( a \in \text{Aut}(R) \) and \( M \in \mathcal{M} \). Then \( a\mathcal{M} = [aM] \).

(c) \( \rho(\mathbb{V}) = \varphi \mathbb{V} \).

(d) \( \text{Aut}(R) \times \mathbb{S}(R) \to \mathbb{S}(R), (\alpha, \mathcal{S}) \to \alpha\mathcal{S} \) is a well-defined action of \( \text{Aut}(R) \) on \( \mathbb{S}(R) \).

**Proof.** (a): Should be obvious.

(b): Follows from (a).

(c): \( r \varphi \beta \cdot v = (\alpha \beta)^{-1}(r)w = (\beta^{-1}(\alpha^{-1}(r)))v = \alpha^{-1}(r) \beta \cdot v = r \varphi(\beta) \cdot v \).

(d) follows from (a) and (c). \( \square \)

**Lemma 1.9.10.** Let \( G \) be a group, \( N \trianglelefteq G \), \( V \) an \( RG \)-module, \( W \) an \( RN \)-submodule of \( V \) and \( g \in G \). Then \( gW \) is an \( R[N] \)-submodule of \( V \) isomorphic to \( \varphi W \).

**Proof.** Define \( \rho : \varphi W \to V \), \( w \to gw \). Then clearly \( \rho \) is 1-1, \( \text{Im} \rho = gW \) and \( \rho \) is \( R \)-linear. Now let \( n \in N \) and \( w \in W \). Then
\[
\rho(n \cdot w) = \rho((g^{-1}ng)w) = gg^{-1}ngw = ngw = n(\rho(w)).
\]
Thus \( \rho \) is \( R[N] \)-linear. Note that \( gW = \text{Im} \rho \) and so \( gW \) is an \( R[N] \)-submodule of \( V \) isomorphic to \( \varphi W \). \( \square \)

**Theorem 1.9.11** (Clifford). Let \( R \) be a ring, \( G \) a group, \( N \trianglelefteq G \) and \( M \) an \( R[G] \)-module. Let
\[
\mathbb{S} := \{ S \in \mathbb{S}(R[N]) \mid M_S \neq 0 \}.
\]
Recall here that for an isomorphism class \( S \) of simple \( R[N] \)-module, \( M_S = \sum_{S \in \mathbb{S}} S \in \mathbb{S} \mid S \cong_{R[N]} M \).

(a) \( gM_S = M_S \) for all \( S \in \mathbb{S} \) and \( g \in G \).

(b) \( \mathbb{S} \) is a \( G \)-invariant subset of \( \mathbb{S}(R) \). In particular, \( G \) acts on \( \mathbb{S} \).

(c) \( (M_S)_{S \in \mathbb{S}} \) is a system of imprimitivity for \( G \) on \( M_S \). Here \( M_S = \sum_{S \in \mathbb{S}} M_S \) is the sum of simple \( R[N] \)-submodule of \( V \), and so the largest semisimple \( R[N] \)-submodule of \( M \).

(d) \( N_G(M_S) = N_G(S) = \{ g \in G \mid S \cong_{R[N]} gS \} \) for all \( S \in \mathbb{S} \) and \( S \in \mathbb{S} \).

(e) Suppose that \( \mathbb{S} \neq \emptyset \) and let \( S \in \mathbb{S} \). Then \( M \) is a simple \( RG \)-module if and only if each of the following holds:

\( i \). \( M \) is a semisimple \( R[N] \)-module.

\( ii \). \( G \) acts transitively on \( \mathbb{S} \).

\( iii \). \( M_S \) is a simple \( R[N_G(S)] \)-module.

(f) Suppose that \( M \) is a simple \( RG \)-module, \( \mathbb{S} \neq \emptyset \) and \( S \in \mathbb{S} \). Then \( M \cong_{R[G]} M_S^G \).

**Proof.** (a) Let \( S \in \mathbb{S} \) and \( S \in \mathbb{S} \). By 1.9.10, \( gS \cong gS \) and so \( gS \in gS \). Thus \( gS \leq M_S \) and so \( gM_S \leq M_S \).

By 1.9.9, \( \varphi^{-1}(\mathbb{S}) = \mathbb{S} \) and so also \( gM_S \leq M_S \). Thus (a) holds.

(b) Let \( S \in \mathbb{S} \) and \( g \in G \). By (a) \( M_S = M_S \neq 0 \) and so \( gS \in \mathbb{S} \).

(c) Note that \( M_S \) is a semisimple \( R \)-module and \( (M_S)_S = M_S \). Hence 1.1.30 shows that
Let \( M_S = \bigoplus_{S \in \mathcal{S}(R[N])} M_S = \bigoplus_{S \in \mathcal{S}} M_S \).

Together with (a) and (b) this gives (c).

Let \( g \in G \) and \( S \in \mathcal{S} \). Then \( gM_S \subseteq M_S \) if and only if \( M_S \subseteq M_S \), see (a). As \( M_S \cap M_T = 0 \) for \( T \in \mathcal{S} \), the latter holds if and only if \( gS = S \).

Suppose first that \( M \) is a simple \( RG \)-module. By assumption \( \mathcal{S} \neq \emptyset \) and so \( M_S \neq 0 \). From (a) we know that \( M_S \) is \( G \)-invariant and so an \( R[G] \)-submodule of \( M \). Since \( M \) a simple for \( R[G] \)-module this gives \( M_S = M \). In particular, \( M \) is a semisimple \( R[N] \)-module. Let \( S \in \mathcal{S} \) and let \( W_S \) a nonzero \( N_G(S) \)-submodule of \( M_S \). Put \( W := R[G]W_S \). Since \( M \) is a simple \( R[G] \)-module we get \( W = M \). By 1.9.4 \( M = W \leq \sum_{T \in \mathcal{S}} M_T \). As \( M = M_S = \bigoplus_{T \in \mathcal{S}} M_T \) this gives \( S \cong G \). Thus \( G \) acts transitively on \( \mathcal{S} \). By 1.9.4 \( W_S = W \cap M_S = M_S \) and so \( M_S \) is a simple \( R[N_G(S)] \)-module.

Suppose now that (e:i)-(e:iii) hold. Let \( W \) be a zero \( R[G] \)-submodule of \( M \). By (e:i), \( M \) and so also \( W \) is a semisimple \( R[N] \)-module. In particular, there exist a simple \( R[N] \)-submodule \( T \) of \( W \). By (e:ii) \( G \) acts transitively on \( \mathcal{S} \), so \( T \) is \( S \) for some \( g \in G \). Thus \( gT \leq W \cap M_S \). It follows that \( W \cap M_S \neq 0 \). By (e:iii) \( M_S \) is a simple \( R[N_G(S)] \)-module and we conclude that \( M_S = W \cap M_S \leq W \). As \( G \) acts transitively on \( \mathcal{S} \) this implies \( M_T \leq W \) for all \( T \in \mathcal{S} \). Since \( M \) is a semisimple \( R[N] \)-module, we conclude that \( M = M_S = \sum_{T \in \mathcal{S}} M_T \leq W \). Thus \( M = W \) and \( M \) is a simple \( RG \)-module.

By (e:iv) \( M \) is a semisimple \( R[N] \)-module. So \( M = M_S \) and (c) shows that \( (M_S)_{S \in \mathcal{S}} \) is a system of imprimitivity for \( R[G] \) on \( M \). Now 1.9.7 implies that \( M \cong R[G] \bigoplus_{N_G(S)} M_S \).

\[ \square \]

1.10 The number of simple modules of finite group

**Definition 1.10.1.** Let \( p \) be an integer. An element \( g \) in a group \( G \) is called \( p \)-singular if \( p \) divides \( |g| \). Otherwise \( g \) is called \( p \)-regular. A conjugacy class is called \( p \)-regular if all its elements are \( p \)-regular.

The goal of this section is to show that if \( K \) is an algebraically closed field, \( G \) is a finite group and \( p := \text{char } K \) then the number of isomorphism classes of simple \( K[G] \)-modules equals the number of \( p \)-regular conjugacy classes.

**Definition 1.10.2.** Let \( R \) be ring and \( p := \text{char } R \). Then

\[ S(R) = \langle xy - yx \mid x, y \in R \rangle_{\mathbb{Z}}. \]

Let \( \overline{p} = p \) if \( p \neq 0 \) and \( \overline{p} = 1 \) if \( p = 0 \).

\[ T(R) := \{ r \in R \mid r^{\overline{m}} \in S(R) \text{ for some } m \in \mathbb{N} \} . \]

**Lemma 1.10.3.**

(a) Let \( G \) be a group, \( n \in \mathbb{Z}^+ \) and \( a_1, \ldots, a_n \in G \). Then for all \( i \in \mathbb{N} \) \( a_{i+1} a_{i+2} \ldots a_{i+n} \) is conjugate \( a_1 a_2 \ldots a_n \) in \( G \).

(b) Let \( R \) be a ring, \( n \in \mathbb{Z}^+ \) and \( a_1, \ldots, a_n \in R \). Then for all \( i \in \mathbb{N} \),

\[ a_{i+1} a_{i+2} \ldots a_{i+n} \equiv a_1 a_2 \ldots a_n \pmod{S(R)} . \]

**Proof.** (a) We have \( a_i^{-1} a_1 a_2 \ldots a_n a_1 = a_2 \ldots a_n a_1 \). So (a) follows by induction on \( n \).

(b) \( a_1 \cdot a_2 \ldots a_n - a_2 \ldots a_n \cdot a_1 \in S(R) \). So (b) follows by induction on \( n \).  \[ \square \]
**Lemma 1.10.4.** Let \( R \) be a commutative ring and \( G \) a group. Then \( S(R[G]) \) is the set of all \( a = \sum_{g \in G} a_g g \in R[G] \) with \( \sum_{g \in C} a_g = 0 \) for each conjugacy class \( C \) of \( G \).

**Proof.** Let \( U \) be the set of all \( a \in R[G] \) with \( \sum_{g \in C} a_g = 0 \) for each conjugacy class \( C \) of \( G \). Note that both \( S(R) \) and \( U \) are \( R \)-submodules of \( R[G] \).

As an \( R \)-modules \( S(R) \) is generated by \( gh - hg, g, h \in G \). By 1.10.3 \( gh \) and \( hg \) are conjugate in \( G \). Thus \( gh - hg \in U \) and so \( S(R) \subseteq U \).

As an \( R \)-module \( U \) is generated by \( g - h, g \) and \( h \) conjugate elements of \( G \). Let \( h = aga^{-1} \) with \( a \in G \). Then

\[
g - h = a^{-1} \cdot ag = ag \cdot a^{-1}
\]

Thus \( g - h \in S(R) \) and so \( U \subseteq S(R) \). \( \square \)

**Lemma 1.10.5.** Let \( R \) be a ring with \( p := \text{char } R \) a prime.

(a) \( (a + b)^p \equiv a^p + b^p \pmod{S(R)} \) for all \( a, b \in R \) and \( m \in \mathbb{N} \).

(b) \( T(R) \) is \( Z(R) \)-submodule of \( R \).

(c) Suppose that \( R = \bigoplus_{i=1}^s \mathbb{R}i_{i=1} \) as a ring. Then \( S(R) = \bigoplus_{i=1}^s S(R_i) \) and \( T(R) = \bigoplus_{i=1}^s T(R_i) \).

(d) Let \( I \) be an ideal in \( R \). Then \( S(R/I) = (S(R) + I)/I \).

(e) Let \( I \) be a nilpotent ideal in \( R \). Then \( I \leq T(R) \), \( T(R/I) = T(R)/I \) and \( R/T(R) \equiv (R/I)/T(R/I) \).

**Proof.**

(a) Let \( H = \langle h \rangle \) be cyclic group of order \( p \) and \( D := \times_{i=1}^p \{a, b\} \). Note that \( H \) acts on the set \( D \) via \( h(d_i)_{i=1}^p = (d_{i+1})_{i=1}^p \). Then \( H \) has exactly two fixed-points on \( D \) namely the constant sequences \( (a)^n_{i=1} \) and \( (b)^n_{i=1} \). Since the length of any orbit of \( H \) divides \( |H| \), all other orbits have length \( p \). Let \( C \) be an orbit of length \( p \) for \( H \) on \( D \). For \( d = (d_1, \ldots, d_p) \in D \) put \( \prod d = d_1d_2 \ldots d_p \in R \). Then by 1.10.3

\[
\prod c \equiv \prod d \pmod{S(R)}
\]

for all \( c, d \in C \) and so

\[
\sum_{c \in C} \prod c = p \prod d \equiv 0 \pmod{S(R)}
\]

Let \( O \) be the set of orbits of \( H \) on \( D \). Then

\[
(a + b)^p \equiv \sum_{c \in D} \prod c \equiv \sum_{C \in O} \sum_{c \in C} \prod c \equiv a^p + b^p \pmod{S(R)}
\]

Hence (a) holds for \( m = 1 \). Now (a) now follows by induction on \( m \).

(b) Follows from (a).

(c) Obvious.

(d) Obvious.

(e) Since \( I \) is nilpotent, \( I^k = 0 \) for some positive integer \( k \). Choose \( m \) with \( p^m \geq k \). Then for all \( i \in I \), \( i^{p^m} = 0 \in S(R) \) and so \( i \in T(R) \). Thus \( I \leq T(R) \). Since \( (S(R) + I)/I = S(R)/I \) we have \( T(R)/I \leq T(R/I) \).

Conversely let \( t + I \in T(R/I) \). Then \( t^{p^l} \in S(R) + I \) for some \( l \in \mathbb{Z}^+ \). Since both \( S(R) \) and \( I \) are in \( T(R) \), (b) shows that \( S(R) + I \leq T(R) \). Thus \( t^{p^l} \in T(R) \) and so also \( t \in T(R) \). \( \square \)
1.10. THE NUMBER OF SIMPLE MODULES OF FINITE GROUP

Lemma 1.10.6. Let \( \mathbb{F} \) be an integral domain and let \( G \) be a periodic group. Put \( p := \text{char} \mathbb{F} \) and let \( C_p \) be the set of \( p \)-regular conjugacy classes of \( G \). For \( C \in C_p \) choose \( g_C \in C \). Then \( \mathbb{F}[G]/T(\mathbb{F}[G]) \) is a free \( \mathbb{F} \)-module with basis

\[
( g_C + T(\mathbb{F}[G]) )_{C \in C_p}
\]

In particular, if \( \mathbb{F} \) is a field, then \( \dim_{\mathbb{F}} \mathbb{F}[G]/T(\mathbb{F}[G]) \) is the number of \( p \)-regular conjugacy classes of \( G \).

Proof. Let \( g \in G \) and let \( a, b \in G \) such that \( g = ab \), \([a, b] = 1\), \( a^{p^n} = 1 \) and \( b \) is \( p \)-regular. Then \( g^{p^n} = b^{p^n} \) and so,

\[
(g - b)^{p^n} \equiv 0 \pmod{S(\mathbb{F}[G])}
\]

Thus \( g - b \in T(\mathbb{F}[G]) \). Put \( C := \langle b \rangle \). By Lemma 1.10.4 \( b - g_C \equiv g_C \in S(\mathbb{F}[G]) \subseteq T(\mathbb{F}[G]) \). It follows that

\[
g \equiv g_C \pmod{T(\mathbb{F}[G])}
\]

and so

\[
( g_C + T(\mathbb{F}[G]) )_{C \in C_p}
\]

generates \( \mathbb{F}[G]/T(\mathbb{F}[G]) \) as an \( \mathbb{F} \)-module.

Let \( r \in \mathbb{F} \) with \( \sum_{C \in C_p} r_C g_C \in T(\mathbb{F}[G]) \). Then there exists \( m \in \mathbb{N} \) with

\[
( \sum_{C \in C_p} r_C g_C )^{p^n} \in S(\mathbb{F}[G])
\]

Since \( g_C \) is \( p \)-regular, \( p \nmid |g_C| \). Since \( \mathbb{F} \) is an integral domain either \( p = 0 \) or \( p \) is a prime. Hence \( \gcd(p, |g_C|) = 1 \). So \( \bar{p} \) is invertible in \( \mathbb{Z}/|g_C|\mathbb{Z} \). Thus \( \bar{p} \) has finite order in the groups of multiplicative units \( \mathbb{Z}/|g_C|\mathbb{Z} \) and so there exists \( m_C \in \mathbb{Z}^+ \) such that

\[
|g_C| (|\bar{p}|^m - 1).
\]

Put \( k := m \prod_{C \in C_p} m_C \). It follows that \( m_C \mid k \), \( (|\bar{p}|^m - 1) \mid (|\bar{p}|^k - 1) \), \( |g_C|^{|\bar{p}|^k - 1} \) and so \( g_C^{\bar{p}^k} = g_C \). Also \( m \mid k \) and \( (\ast) \) implies that \( (\sum_{C \in C_p} r_C g_C)^{\bar{p}^k} \in S(\mathbb{F}[G]) \). Since

\[
\sum_{C \in C_p} r_C^{\bar{p}^k} g_C = \sum_{C \in C_p} r_C g_C^{\bar{p}^k} \equiv 0 \pmod{S(\mathbb{F}[G])}
\]

Thus \( \sum_{C \in C_p} r_C^{\bar{p}^k} g_C \in S(\mathbb{F}[G]) \). Now \( 1.10.4 \) shows that \( r_C^{\bar{p}^k} = 0 \) for all \( C \in C_p \). Since \( \mathbb{F} \) is an integral domain this gives \( r_C = 0 \) and so

\[
( g_C + T(\mathbb{F}[G]) )_{C \in C_p}
\]

is linearly independent over \( \mathbb{F} \).

\( \Box \)

Lemma 1.10.7. Let \( R \) be a commutative ring and \( n \in \mathbb{Z}^+ \). Put \( p := \text{char} R \). Let \( M_n(R) \) be the ring of \( n \times n \) matrices with coefficients in \( R \).

(a) \( S(M_n(R)) \) consists of the trace zero matrices and \( M_n(R)/S(M_n(R)) \cong_R R \).

(b) Suppose \( p \) is a prime. Then \( T(M_n(R)) = \{ a \in M_n(R) \mid \text{tr}(a)^{\bar{p}^m} = 0 \text{ for some } m \in \mathbb{N} \} \).

(c) If \( R \) is an integral domain, then \( S(M_n(R)) = T(M_n(R)) \) and \( M_n(R)/T(M_n(R)) \cong_R R \).
Proof. (a): Let $x, y \in M_n(R)$. Since $R$ is commutative, $\text{tr}(xy) = \text{tr}(yx)$ and so $\text{tr}(xy - yx) = 0$. Thus $S(M_n(R)) \leq \ker \text{tr}$.

For $1 \leq i, j \leq n$ put $E_{ij} = (\delta_{ij} \delta_{ji})$. Then $\ker \text{tr}$ is, as an $R$-module, generated by the matrices $E_{ij}$ and $E_{ii} - E_{jj}$ with $i \neq j$. Note that

$$E_{ij} = E_{ii}E_{ij} - E_{ij}E_{ii} \quad \text{and} \quad E_{ii} - E_{jj} = E_{ij}E_{ji} - E_{ji}E_{ij}.$$ 

So both $E_{ij}$ and $E_{ii} - E_{jj}$ are in $S(M_n(R))$. Thus $S(M_n(R)) \leq \ker \text{tr}$. Observe that $M_n(R) = RE_{11} \oplus \ker \text{tr}$ as an $R$-module and so $M_n(R)/\ker \text{tr} \cong_R RE_{11} \cong_R R$.

(b): Suppose now that $p$ is a prime and let $a \in M_n(R)$. Put

$$b := \text{tr}(a)E_{11} \quad \text{and} \quad c := a - b.$$ 

Then $\text{tr}(a) = \text{tr}(b)$ and $\text{tr}(c) = 0$, so $c \in S(M_n(R)) \subseteq T(M_n(R))$. By $1.10.5(b)$ $T(M_n(R))$ is an additive subgroup of $M_n(R)$. Thus $a \in T(M_n(R))$ if and only if $b \in T(M_n(R))$. Note that $b^{p^n} = \text{tr}(a)^{p^n}E_{11}$, so $\text{tr}(b^{p^n}) = \text{tr}(a)^{p^n}$. It follows that $b \in T(M_n(R))$ if and only if $b^{p^n} \in S(M_n(R))$ (for some $m \in \mathbb{N}$) if and only if $\text{tr}(a)^{p^n} = 0$. So (b) holds.

(c): Suppose $R$ is an integral domain. Then $\text{tr}(a)^{p^n} = 0$ for some $m \in \mathbb{N}$ if and only if $\text{tr}(a) = 0$. Together with (a) and (b) this gives (c). \qed

Theorem 1.10.8. Let $G$ be a finite group and $K$ an algebraically closed field. Put $p := \text{char } K$. Then the number of isomorphism classes of simple $K[G]$-modules equals the number of $p$-regular conjugacy classes.

Proof. By $1.10.6$ the number of $p$-regular conjugacy classes of $G$ is $\dim_K K[G]/T(K[G])$.

Put $A := K[G]/J(K[G])$. By $1.4.8 A \leq A_{K[G]}(M)$ for each simple $K[G]$-modules $M$ and it follows that the number of isomorphism classes of simple $K[G]$-modules is equal to the number of isomorphism classes of simple $A$-modules.

By $1.5.26$ $J(K[G])$ is nilpotent and so by $1.10.5(c)$, $K[G]/T(K[G]) \cong A/T(A)$. Observe that $A$ is an Artinian ring with $J(A) = 0$. Hence $1.5.25$ gives

$$A \cong \bigoplus_{i=1}^{n} M_{d_i}(K)$$

where $n$ is the number of isomorphism classes of simple $A$ and $d_i \in \mathbb{Z}^+$ and so by $1.10.5(c)$

$$T(A) = \bigoplus_{i=1}^{n} T(M_{d_i}(K)).$$

Thus

$$A/T(A) \cong \bigoplus_{i=1}^{n} M_{d_i}(K)/T(M_{d_i}(K)).$$

By $1.10.7(c)$, $M_{d_i}(K)/T(M_{d_i}(K)) \cong K$ and so $A/T(A) \cong K^n$. Thus

$$\dim_K K[G]/T(K[G]) = \dim_K A/T(A) = n$$

and the theorem is proved. \qed
Chapter 2

Representations of the Symmetric Groups

2.1 The Symmetric Groups

Notation 2.1.1. For \( n \in \mathbb{Z}^+ \) let \( I_n := \{1, 2, \ldots, n\} \) and \( \text{Sym}(n) := \text{Sym}(I_n) \). Let \( g \in \text{Sym}(n) \). Let \( O(g) := \{O_1, \ldots, O_k\} \) be the sets of orbits for \( \langle g \rangle \) on \( I_n \). Let \( n_i = |O_i| \) and choose notation such that \( n_1 \geq n_2 \geq n_3 \geq \ldots \geq n_k \). Define \( n_i = 0 \) for all \( i > k \). Then the sequence \( \{n_i\}_{i=1}^{\infty} \) is called the cycle type of \( g \).

Pick a \( i_0 \in O_i \) and for \( j \in \mathbb{Z} \) define \( a_{ij} := g^j(a_{i0}) \). Note that \( a_{ij} = a_{ik} \) if and only if \( j \equiv k \pmod{n_i} \). We denote the element \( g \) by

\[
g = (a_{11}, a_{12}, \ldots, a_{1n_1}) (a_{21}, a_{22}, \ldots, a_{2n_2}) \cdots (a_{k1}, a_{k2}, \ldots, a_{kn_k}).
\]

Lemma 2.1.2. Let \( n \in \mathbb{Z}^+ \). Two elements in \( \text{Sym}(n) \) are conjugate if and only if they have the same cycle type.

Proof. Let \( g \) be as above and \( h \in \text{Sym}(n) \). Then

\[
hgh^{-1} = (h(a_{11}), h(a_{12}), \ldots, h(a_{1n_1}))(h(a_{21}), h(a_{22}), \ldots, h(a_{2n_2})) \cdots (h(a_{k1}), h(a_{k2}), \ldots, h(a_{kn_k}))
\]

and the lemma is readily verified \( \square \)

Definition 2.1.3. Let \( p, n \in \mathbb{Z}^+ \).

(a) For \( B \subseteq \mathbb{Z} \) let \( B^\infty := B^\mathbb{Z}^+ \) be set of all infinite sequence \( b = (b_i)_{i=1}^\infty \) with \( b_i \in B \). Let \( B_{\mathbb{Z}^+} = B_{\mathbb{Z}^+}^\mathbb{Z}^+ \) be set of all infinite sequence \( b \in B^\mathbb{Z}^+ \) with \( b_i = 0 \) for almost all \( i \in \mathbb{Z}^+ \).

(b) \( \lambda \in \mathbb{N}_\infty \) is called non-increasing if \( \lambda_i \geq \lambda_{i+1} \) for all \( i \in \mathbb{Z}^+ \).

(c) An (additive) partition of \( n \) is a non-increasing sequence \( \lambda \in \mathbb{N}_\infty \) with \( n = \sum_{i=1}^\infty \lambda_i \).

(d) An partition \( \lambda \) of \( n \) is called \( p \)-singular, if there exists \( i \in \mathbb{N} \) with \( 0 \neq \lambda_{i+1} = \lambda_{i+2} = \ldots = \lambda_{i+p} \). Otherwise \( \lambda \) is called \( p \)-regular. Any additive partition is called 0-regular.

(e) A multiplicative partition of \( n \) is a sequence \( \mu \in \mathbb{N}_\infty \) with \( n = \prod_{i=1}^\infty \mu_i \).

(f) A multiplicative partition \( \mu \) of \( n \) is called \( p \)-singular if \( \mu_i \geq p \) for some \( i \in \mathbb{Z}^+ \). \( \mu \) is called \( p \)-regular if \( \mu_i \leq p - 1 \) for all \( i \in \mathbb{Z}^+ \). Any multiplicative partition is called 0-regular.
(g) If \( \lambda \in \mathbb{N}_\infty \), then \( \hat{\lambda} \in \mathbb{N}_\infty \) is defined by \( \hat{\lambda}_i = |\{ j \in \mathbb{Z}^+ \mid \lambda_j = i\}| \) for each \( i \in \mathbb{Z}^+ \).

**Example 2.1.4.** \( \lambda = (4, 4, 4, 3, 1, 1, 1, 0, 0, 0, \ldots) \) is a partition of 22. We denote this partition by \((4^3, 3^2, 1^4)\)
\[
\hat{\lambda} = (4, 0, 2, 3, 0, 0, \ldots) \text{ is a multiplicative partition of 22.}
\]
\( \lambda \) and \( \hat{\lambda} \) are \( p \)-singular for \( p = 1, 2, 3 \) and 4 and \( p \)-regular for \( p = 0 \) and for any \( p \geq 5 \).

The cycle-type of any elements of Sym\( (n) \) is partition of \( n \).

**Remark 2.1.5.** Let \( p \in \mathbb{N} \) and \( n \in \mathbb{Z}^+ \). The function \( \lambda \rightarrow \hat{\lambda} \) is bijection the partition of \( n \) and the multiplicative partition of \( n \). A partition \( \lambda \) of \( n \) is \( p \)-regular if and only if \( \hat{\lambda} \) is \( p \)-regular.

**Lemma 2.1.6.** Let \( p = 0 \) or a positive prime. Let \( n \in \mathbb{Z}^+ \). Then the number of \( p \)-regular conjugacy classes of Sym\( (n) \) equals the number of \( p \)-regular (multiplicative) partitions of Sym\( (n) \).

**Proof.** For \( p = 0 \) this is obvious. So suppose \( p \) is prime. Let \( g \in \text{Sym}(n) \) and \( \lambda \) its cycle-type. Then \( g \) is \( p \)-regular if and only if none of positive \( \lambda_j \)’s are divisible by \( p \) and if and only if \( \hat{\lambda}_i = 0 \) for all \( i \in \mathbb{Z}^+ \) with \( p | i \).

Put \( A := \mathbb{Z}^+ \setminus p \mathbb{Z}^+ \). We conclude that the \( p \)-regular conjugacy classes of Sym\( (n) \) are in 1-1 correspondence with the sequences \( j \in \mathbb{N}_A \) with \( \sum_{i=1}^{\infty} i j_i = n \).

Put \( B := \{0, \ldots, p - 1\} \). Observe that \( p \)-regular multiplicative partition of \( n \) is just a sequence \( j \in B_\infty \) with \( \sum_{i=1}^{\infty} i j_i = n \).

Let
\[
f := \prod_{i=1}^{\infty} (1 - x^{p i}) \prod_{i=1}^{\infty} (1 - x^i)
\]
viewed as an element of \( \mathbb{Z}[[x]] \), the integral domain of formal integral power series.

We compute \( f \) in two different ways:

1. \[
f = \prod_{i=1}^{\infty} (1 - x^{p i}) \prod_{i=1}^{\infty} (1 - x^i) = \prod_{i \in A} \frac{1}{1 - x^i} = \prod_{i \in A} \sum_{j=0}^{\infty} x^{ij} = \prod_{j \in B_\infty} \sum_{i \in A} x^{ij} = \sum_{j \in B_\infty} \sum_{i=1}^{\infty} x^{ij}.
\]

Thus the coefficient of \( x^n \) in \( f \) is the number \( j \in \mathbb{N}_A \) with \( \sum_{i=1}^{\infty} i j_i = n \) and so equal to number of \( p \)-regular conjugacy classes in Sym\( (n) \).

2. Let \( B = \{0, 1, \ldots, p - 1\} \).

\[
f = \prod_{i=1}^{\infty} \frac{1 - x^{ip}}{1 - x^{i}} = \prod_{i=1}^{\infty} \sum_{j=0}^{p-1} x^{ij} = \sum_{j \in B_\infty} \prod_{i=1}^{\infty} x^{ij} = \sum_{j \in B_\infty} x^{\sum_{i=1}^{\infty} ij_i}.
\]

Thus the coefficient of \( x^n \) in \( f \) is the number of \( j \in B_\infty \) with \( \sum_{i=1}^{\infty} i j_i = n \) and so equal to the \( p \)-regular multiplicative partitions of \( n \).

\[\square\]

**Corollary 2.1.7.** Let \( n \in \mathbb{Z}^+ \) and \( \mathbb{K} \) an algebraically closed field. Put \( p := \text{char} \mathbb{K} \). Then the following numbers are equal:

(a) The numbers of \( p \)-regular partitions of \( n \).

(b) The numbers of \( p \)-regular conjugacy classes of Sym\( (n) \).
2.1. THE SYMMETRIC GROUPS

(c) The number of isomorphism classes of simple $\mathbb{K}[\text{Sym}(n)]$-modules.

Proof. By $2.1.6$ the first two numbers are equal. By $1.10.8$ the last two numbers are equal. □

Our goal now is to find an explicit 1-1 correspondence between of $p$-regular partitions of $n$ and the simple $\mathbb{K}[\text{Sym}(n)]$-modules. We start by associating a $\mathbb{K}[\text{Sym}(n)]$-module $M_\lambda$ to each partition $\lambda$ of $n$. But this modules is not simple. In later section we will determine a simple section of $M_\lambda$.

Definition 2.1.8. Let $I$ be a set of size $n$ and $\lambda$ a partition of $n$. A $\lambda$-partition of $I$ is a sequence

$$\Delta = (\Delta_i)_{i=1}^\infty$$

of subsets of $I$ such that

(i) $I = \bigcup_{i=1}^\infty \Delta_i$

(ii) $\Delta_i \cap \Delta_j = \emptyset$ for all $1 \leq i < j < \infty$.

(iii) $|\Delta_i| = \lambda_i$.

Example 2.1.9. $\{1, 3, 5\}, \{2, 4\}, \{6\}, \emptyset, \emptyset, \ldots$ is a $(3, 2, 1)$ partition of $I_6$. We will denote such a partition as

\[
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
6
\end{array}
\]

The lines in this array are a reminder that the order of the elements in the row does not matter. So

\[
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
6
\end{array} = \begin{array}{ccc}
3 & 1 & 5 \\
4 & 2 \\
1
\end{array}
\]

On the otherhand since sequences are ordered

\[
\begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & 6 \\
1 & 3 & 5
\end{array} \neq \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 \\
6
\end{array}
\]

Notation 2.1.10. Let $n \in \mathbb{Z}^+$, $\lambda$ a partition of $n$ and $F$ a commutative ring. $M^\lambda$ is the set of $\lambda$-partitions of $I_n$. Note that $\text{Sym}(n)$ acts on $M^\lambda$ via

$$\pi \Delta = (\pi(\Delta_i))_{i=1}^\infty$$

for $\pi \in \text{Sym}(n)$ and $\Delta \in M_\lambda$.

$M^\lambda := M^\lambda_F := F_M^\lambda$. We identify $\Theta \in M_\lambda$ with $(\delta_{\Delta\Theta})_{\Delta \in M_\lambda} \in M^\lambda$. Then $M^\lambda$ is a free $F$-module with basis $M^\lambda$ and $v = \sum_{\Delta \in M_\lambda} v_{\Delta}$ for all $v \in M^\lambda$.

Recall from Example 1.1.17 that group action of $\text{Sym}(n)$ on $M_\lambda$ extends to $F[\text{Sym}(n)]$-module structure on $M^\lambda$.

$\langle \cdot \mid \cdot \rangle = \langle \cdot \mid \cdot \rangle F$ is the unique $F$-bilinear form on $M^\lambda$ with orthonormal basis $M^\lambda$. So $\langle v \mid w \rangle = \sum_{\Delta \in M_\lambda} v_{\Delta} w_{\Delta}$ for all $v, w \in M^\lambda$.

Remark 2.1.11. Let $n \in \mathbb{Z}^+$, $\lambda$ a partition of $n$ and $F$ a commutative ring.
(a) \( \langle \cdot | \cdot \rangle \) is Sym(n)-invariant, that is \( \langle \pi v | \pi w \rangle = \langle v | w \rangle \) for all \( \pi \in \text{Sym}(n) \), \( v, w \in M^A \).

(b) \( \langle \cdot | \cdot \rangle \) is symmetric, that is \( \langle v | w \rangle = \langle w | v \rangle \) for all \( v, w \in M^A \).

(c) \( v = \sum_{\Delta \in M^P} \langle v | \Delta \rangle \Delta \) for all \( v \in M^A \).

(d) \( \langle \cdot | \cdot \rangle \) is non-degenerate, that is \( \langle v | w \rangle = 0 \) for all \( v \in M_A \) such that \( \langle v | w \rangle = 0 \) for all \( w \in M^A \), then \( v = 0 \).

### 2.2 Diagrams, Tableaux and Tabloids

*Definition 2.2.1.* Let \( D \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \)

(a) Let \( (i, j), (k, l) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \). Then \( (i, j) \leq (k, l) \) provided that \( i \leq k \) and \( j \leq l \)

(b) \( D \) is called a Ferrers diagram if \( e \in D \) for all \( d \in D \) and \( e \in \mathbb{Z}^+ \times \mathbb{Z}^+ \) with \( e \leq d \).

(c) The elements of a Ferrers diagram are called the nodes of the diagram.

(d) The \( i \)-th row of \( D \) is \( D_i := D \cap ([i] \times \mathbb{Z}^+) \) and the \( j \)-column of \( D \) is \( D^j := D \cap (\mathbb{Z}^+ \times \{j\}) \).

*Definition 2.2.2.*

For \( \lambda \in \mathbb{Z}^{+\infty} \) define

\[
\boxed{[\lambda] := \{(i, j) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid 1 \leq j \leq \lambda_i\}}.
\]

*Lemma 2.2.3.*

(a) Let \( D \) be a Ferrers diagram and \( i, j \in \mathbb{Z}^+ \). Then \( (i, j) \in D \) if and only if \( j \leq \lambda(D)_i \) and if and only if \( i \leq \lambda'(D)_j \).

(b) The function \( D \to \lambda(D) \) is a bijection between the Ferrers diagrams of size \( n \) and the partitions of \( n \) with inverse \( \lambda \to [\lambda] \).

(c) Let \( \lambda \) be a partition of \( n \) and let \( i, a, b \in \mathbb{Z}^+ \). If \( a \leq i \) and \( b \leq \lambda_i \), then \( (a, b) \in [\lambda] \). If \( a \geq i \) and \( b > \lambda_i \), then \( (a, b) \notin D \).

*Proof.* (a): Let \( k \) be maximal in \( \mathbb{Z}^+ \) with \( (i, k) \in D \) (with \( k = 0 \) if \( D_i = \emptyset \)). If \( j \leq k \), then \( (i, j) \leq (i, k) \) and so \( (i, j) \in D \). If \( j > k \), the maximal choice of \( k \) shows that \( (i, j) \notin D \). So

\[
(i, j) \in D \iff j \leq k.
\]

It follows that \( k = |D_i| = \lambda(D)_i \). So the first statement in (a) holds. By symmetry, also the second one holds.

(b): Let \( D \) be a Ferrers diagram of size \( n \). Put \( \mu = \lambda(D) \). We will first show \( \mu \) is partition of \( n \). Note that

\[
\sum_{i=1}^{\infty} \mu_i = \sum_{i=1}^{\infty} |D_i| = |D| = n.
\]

To show that \( \mu \) is non-increasing, let \( i, k \in \mathbb{Z}^+ \) with \( k \leq i \). Then \( (k, \mu_i) \leq (i, \mu_i) \). Since \( \mu_i \leq \mu_k \), (a) shows that \( (i, \mu_i) \in D \). Hence the definition of a Ferrers diagram implies \( (k, \mu_i) \in D \). By (a) this means \( \mu_i \leq \mu_k \). Hence \( \mu \) is non-increasing and so \( \mu \) a partition of \( n \).

Let \( \lambda \) be partition of \( n \). Let \( (i, j) \in [\lambda] \) and \( (a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \) with \( (a, b) \leq (i, j) \). Then

\[
a \leq i \leq \lambda_j \leq \lambda_b.
\]
and so \((a, b) \in [\lambda]\). Thus \([\lambda]\) is a Ferrers diagram. Observe that \(|[\lambda]|_i = \lambda_i\) and so \(\lambda([\lambda]) = \lambda\). By \((a)\), \([\lambda(D)] = D\) and so \((b)\) is proved.

\((c)\): Suppose \(a \leq i\) and \(b \leq \lambda_i\). Since \(\lambda\) is non-increasing we have \(\lambda_i \leq \lambda_{ia}\). As \(b \leq \lambda_i\) this gives \(b \leq \lambda_{ia}\) and so \((a, b) \in [\lambda]\).

Suppose \(a \geq i\) and \(b > \lambda_i\). Since \(\lambda\) is non-increasing we have \(\lambda_i \geq \lambda_{ia}\). As \(b > \lambda_i\) this gives \(b > \lambda_{ia}\) and so \((a, b) \notin [\lambda]\). □

**Example 2.2.4.**

![Diagram]

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 \\
1 & x & x & x & x & x \\
2 & x & x & x & & \\
3 & x & x & & & \\
4 & x & x & & & \\
5 & x & & & & \\
6 & x & & & & \\
7 & & & & & \\
\end{array}
\]

\([5, 3^2, 2^1, 1] =

**Definition 2.2.5.** Let \(\lambda\) and \(\mu\) be partitions of \(n\).

(a) We say that \(\lambda\) dominates \(\mu\) and write \(\lambda \trianglerighteq \mu\) if

\[
\sum_{i=1}^{j} \lambda_i \geq \sum_{i=1}^{j} \mu_i
\]

for all \(j \in \mathbb{Z}^+\).

(b) We write \(\lambda > \mu\) provided that there exists \(i \in \mathbb{Z}^+\) with \(\lambda_i > \mu_i\) and \(\lambda_j = \mu_j\) for all \(1 \leq j < i\). ‘\(>\)’ is called the lexicographic ordering.

**Remark 2.2.6.** (a) \(\trianglerighteq\) is a partial ordering on the partitions of \(n\).

(b) ‘\(>\)’ is a total ordering on the partitions of \(n\).

(c) Let \(\lambda\) and \(\mu\) be partitions of \(n\) with \(\lambda \triangleright \mu\). Then \(\lambda > \mu\).

**Proof.** \((a)\) and \((b)\) are obvious. For \((c)\) let \(i \in \mathbb{Z}^+\) be minimal with \(\lambda_i \neq \mu_i\). Then

\[
\sum_{j=1}^{i-1} \lambda_j = \sum_{j=1}^{i-1} \mu_j \quad \text{and} \quad \sum_{j=1}^{i} \lambda_j > \sum_{j=1}^{i} \mu_i.
\]

It follows that \(\lambda_i > \mu_i\), \(\lambda_j > \mu_j\) and \(\lambda > \mu\). □

**Example 2.2.7.** The dominant ordering for \(n = 6\):
Definition 2.2.8.  (a) Let \( D \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+ \). Then \( D' := \{(j, i) \mid (i, j) \in D\} \). \( D' \) is called the conjugate of \( D \).

(b) Let \( \lambda \) be a partition of \( n \). Then \( \lambda' := \lambda'([\lambda]) \), so \( \lambda'_j \) is the number of nodes in column \( j \) of \([\lambda]\). \( \lambda' \) is called the conjugate of \( \lambda \).

Example 2.2.9. Let \( \lambda = (5, 3^3, 2^2, 1) \) and \( D = [\lambda] \). Then

\[
\begin{array}{cccccc}
  & 1 & 2 & 3 & 4 & 5 \\
1 & x & x & x & x & x \\
2 & x & x & x \\
3 & x & x & x \\
4 & x & x & x \\
5 & x & x \\
6 & x & x \\
7 & x \\
\end{array}
\]

\[
\lambda = (5, 3^3, 2^2, 1) = \lambda(D)
\]

\[
\lambda' = \lambda'(\lambda) = (7, 6, 4, 1^2) = \lambda(D')
\]

Lemma 2.2.10. Let \( D \) be a Ferrers diagram of size \( n \) and \( \lambda \) a partition of \( n \).

(a) \( D' \) is a Ferrers diagram of size \( n \) and \( D'' = D \).

(b) \( (D_i)' = (D'_i)' \) and \( (D''_i)' = (D'_i)' \), for all \( i \in \mathbb{Z}^+ \).
(c) \( \lambda(D') = \lambda'(D) \) and \( \lambda(D) = \lambda'(D') \).

(d) \( \lambda' = \lambda([\lambda']) \) and \( [\lambda'] = [\lambda]' \). In particular, \( \lambda' \) is a partition of \( n \).

**Proof.**

(a): Follows immediately from the definition of a Ferrers diagram.

(b): Is obvious.

(c): Follows from (d).

(d) By definition, \( \lambda' = \lambda'([\lambda]) \) and by (c) \( \lambda'([\lambda]) = \lambda([\lambda']) \). By [2.2.3(b)] \( \lambda \) is a Ferrers diagram, so by (a) \( [\lambda'] \) is a Ferrers diagram and then by [2.2.3(b)] \( \lambda([\lambda']) = [\lambda]' \) is a partition of \( n \).

By (d) \( \lambda([\lambda']) = \lambda'([\lambda]) = \lambda' \) and so using [2.2.3(b)] \( r \lambda s = r \lambda s 1 \). □

**Lemma 2.2.11.** Let \( \lambda \) and \( \mu \) be partitions of \( n \). Then \( \lambda \trianglerighteq \mu \) if and only if \( \lambda' \trianglerighteq \mu' \).

**Proof.** Fix \( j \in \mathbb{Z}^+ \) and put \( i = \mu_j \). Define the following subsets of \( \mathbb{Z}^+ \times \mathbb{Z}^+ \):

\[
\begin{align*}
\text{Top} & := \{(a, b) \mid a \leq i\} & \text{Bottom} & := \{(a, b) \mid a > i\} \\
\text{Left} & := \{(a, b) \mid b \leq j\} & \text{Right} & := \{(a, b) \mid b > i\}
\end{align*}
\]

Since \( \lambda \) dominates \( \mu \):

(1) \[ |\text{Top} \cap [\lambda]| = \sum_{a=1}^i \lambda_i \geq \sum_{a=1}^i \mu_i = |\text{Top} \cap [\mu]| \]

Let \((a, b) \in \text{Top} \cap \text{Left} \). Then \( a \leq i = \mu_j \) and \( b \leq j \). Thus by [2.2.3(c)] \( (b, a) \in [\mu'] = [\mu] \) and so \( (a, b) \in [\mu] \). Hence \( \text{Top} \cap \text{Left} \subseteq [\mu] \) and so

(2) \[ |\text{Top} \cap \text{Left} \cap [\lambda]| \leq |\text{Top} \cap \text{Left}| = |\text{Top} \cap \text{Left} \cap [\mu]| \]

Let \((a, b) \in \text{Bottom} \cap \text{Right} \). Then \( a > i = \mu_j \) and \( b > j \). Thus by [2.2.3(c)] \( (b, a) \notin [\mu'] = [\mu] \) and so \( (a, b) \notin [\mu] \). Hence \( \text{Bottom} \cap \text{Right} \cap [\mu] = \emptyset \) and so

(3) \[ |\text{Bottom} \cap \text{Right} \cap [\lambda]| = 0 = |\text{Bottom} \cap \text{Right} \cap [\mu]| \]

From (1) and (2) we conclude

(4) \[ |\text{Top} \cap \text{Right} \cap [\lambda]| = |\text{Top} \cap \text{Right} \cap [\mu]| \]

(3) and (4) imply:

\[ |\text{Right} \cap [\lambda]| \geq |\text{Right} \cap [\mu]| \]

Since \( |[\lambda]| = n = |[\mu]| \) we conclude

\[ |\text{Left} \cap [\lambda]| \geq |\text{Left} \cap [\mu]| \]

Thus \( \sum_{c=1}^l \lambda'_c \leq \sum_{c=1}^l \mu'_c \) and \( \lambda' \trianglelefteq \mu' \). □
Definition 2.2.12. Let $\lambda$ be a partition of $n$. A $\lambda$-tableau is a bijection $t : [\lambda] \to I_n$.

Notation 2.2.13.

\[
\begin{array}{c}
5 \\
1 \\
4 \\
2 \\
3
\end{array}
\]

denotes the $(3, 2)$-tableau $t$ with

\[
(1, 1) \mapsto 4, \quad (1, 2) \mapsto 1, \quad (1, 3) \mapsto 4, \quad (2, 1) \mapsto 2, \quad (2, 2) \mapsto 3.
\]

Definition 2.2.14. Let $t$ be a $\lambda$-tableau. Then

\[
\Delta(t) := \left(t([\lambda]_i)\right)_{i=1}^{\infty} \text{ and } \Delta'(t) := \left(t([\lambda]'_i)\right)_{i=1}^{\infty}.
\]

$\Delta(t)$ is called the row partition of $t$ and $\Delta'(t)$ the column partition of $t$. $t([\lambda]_i)$ is called the $i$-th row of $t$ and $t([\lambda]'_j)$ the $j$-th column of $t$.

The functions row, col, row, and col, are defined by

\[
\begin{aligned}
\text{row} : & \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+, \quad (i, j) \mapsto i, \\
\text{col} : & \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Z}^+, \quad (i, j) \mapsto i
\end{aligned}
\]

\[
\text{row}_t := \text{row} \circ t^{-1} : I_n \to \mathbb{Z}^+, \quad \text{and} \quad \text{col}_t := \text{col} \circ t^{-1} : I_n \to \mathbb{Z}^+
\]

$\leq_t$ is the partial ordering on $I_n$ defined by $a \leq b$ if $t^{-1}(a) \leq t^{-1}(b)$.

Example 2.2.15. If

\[
t = \begin{array}{c}
2 \\
4 \\
3 \\
6 \\
1 \\
5 \\
7
\end{array}
\]

then

\[
\Delta(t) = \begin{array}{c}
2 \\
4 \\
3 \\
6 \\
1 \\
5 \\
7
\end{array} \quad \text{and} \quad \Delta'(t) = \begin{array}{c}
2 \\
4 \\
3 \\
6 \\
1 \\
4 \\
7
\end{array}
\]

Remark 2.2.16. Let $t$ be a $\lambda$-tableau.

(a) $\Delta(t)$ is a $\lambda$-partition of $I_n$ and $\Delta'(t)$ is a $\lambda'$-partition of $I_n$.

(b) $\leq_t$, when restricted to row or column of $t$, is a total ordering.

(c) Let $i, j \in \mathbb{Z}^+$ and $a \in I_n$. Then

\[
\begin{aligned}
\text{row}_t(a) & = i \text{ if and only if } a \in \Delta(t)_i, \\
\text{col}_t(a) & = j \text{ if and only if } a \in \Delta'(t)_j.
\end{aligned}
\]

Definition 2.2.17. Let $s, t$ be $\lambda$-tableaux.

(a) $s$ and $t$ are called row-equivalent if $\Delta(t) = \Delta(s)$. Any equivalence class of this relation is called a tabloid and the tabloid containing $t$ is denoted by $\text{tab}(t)$.
2.2. DIAGRAMS, TABLEAUX AND TABLOIDS

Example 2.2.18. If \( t = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \) then
\[
\pi(t) = \begin{pmatrix} 1 & 4 & 4 & 1 \\ 2 & 3 & 3 & 2 \end{pmatrix}
\]

Lemma 2.2.19. Let \( \lambda \) be partition of \( n \), let \( \pi \in \text{Sym}(n) \) and let \( s, t \) be \( \lambda \)-tableaux.

(a) \( \text{Sym}(n) \) acts transitively on the set of \( \lambda \)-tableaux via \( \pi t = \pi \cdot t \).

(b) \( \pi(\Delta(t)) = \Delta(\pi t) \).

(c) \( s \) and \( t \) are row-equivalent if and only if \( \pi s \) and \( \pi t \) are row-equivalent. In particular, \( \text{Sym}(n) \) acts on the set of \( \lambda \)-tabloids via \( \pi \pi t = \pi t \).

(d) \( \text{row}_{\pi t} = \text{row}_t \circ \pi^{-1} \) and \( \text{col}_{\pi t} = \text{col}_t \circ \pi^{-1} \). In particular, \( \text{row}_{\pi t} \circ \pi = \text{row}_t \) and \( \text{col}_{\pi t} \circ \pi = \text{col}_t \).

(e) \( s \) and \( t \) are row-equivalent if and only if \( \text{row}_s = \text{row}_t \).

Proof. (a) Clearly \( \pi t = \pi \cdot t \) defines an action of \( \text{Sym}(n) \) on the set of \( \lambda \)-tableaux. Since \( s \) and \( t \) are bijections from \( [\lambda] \rightarrow I_n \), we see that \( \rho := s \circ t^{-1} \in \text{Sym}(n) \). Then \( \rho \circ t = s \) and so the action is transitive.

(b) Set \( D := [\lambda] \). Then \( \Delta(t) = (t(D_i))_{i=1}^{\infty} \) and
\[
\pi(\Delta(t)) = \pi\left((t(D_i))_{i=1}^{\infty}\right) = \left((\pi t(D_i))_{i=1}^{\infty}\right) = \Delta(\pi t).
\]

(c) \( s \) is row-equivalent to \( t \) iff \( \Delta(s) = \Delta(t) \) and so iff \( \pi(\Delta(s)) = \pi(\Delta(t)) \). So by (b) iff \( \Delta(\pi s) = \Delta(\pi t) \) and iff \( \pi t \) and \( \pi s \) are row-equivalent.

(d) \( \text{row}_{\pi t} = \text{row} \circ (\pi \circ t)^{-1} = (\text{row} \circ t^{-1}) \circ \pi^{-1} = \text{row}_t \circ \pi^{-1} \).

(e) Note that \( \text{row}_t = \text{row}_s \) if and only if \( \text{row}^{-1}_t(i) = \rho_s(i)^{-1} \) for all \( i \in \mathbb{Z}^+ \). Also \( \Delta(t) = \Delta(s) \) if and only if \( \Delta(t)_i = \Delta(s)_i \) for all \( i \in \mathbb{Z}^+ \).

Let \( i \in \mathbb{Z}^+ \) and \( a \in I_n \). By 2.2.16(a) we have \( \text{row}_t(a) = i \) if and only if \( a \in \Delta(t)_i \). Hence \( \text{row}^{-1}_t(i) = \Delta(t)_i \) and by (c) holds. \( \square \)

Remark 2.2.20. Let \( \Delta = (\Delta_i)_{i=1}^{\infty} \) be a \( \lambda \)-partition of \( I_n \). Let \( \pi \in \text{Sym}(n) \). Then
\[
C_{\text{Sym}(n)}(\Delta) = \{ \pi \in \text{Sym}(n) \mid \pi \Delta = \Delta \} = \{ \pi \in \text{Sym}(n) \mid \pi(\Delta_i) = \Delta_i \text{ for all } i \in \mathbb{Z}^+ \}
\]
\[
= \bigcap_{i=1}^{\infty} N_{\text{Sym}(n)}(\Delta_i) = \times_{i=1}^{\infty} \text{Sym}(\Delta_i).
\]

In particular, \( C_{\text{Sym}(n)}(\Delta) \) has order
\[
\lambda! := \prod_{i=1}^{\infty} \lambda_i!.
\]
Definition 2.2.21. Let $t$ be a tableau. Then

$$R_t := C_{\text{Sym}(\Delta(t))} \quad \text{and} \quad C_t := C_{\text{Sym}(\Delta'(t))}.$$ 

$R_t$ is called the row-stabilizer and $C_t$ the column-stabilizer of $t$.

Example 2.2.22. If

$$t = \begin{bmatrix} 2 & 4 & 3 \\ 6 & 1 \\ 5 & 7 \end{bmatrix}$$

then

$$R_t = \text{Sym}\{2, 3, 4\} \times \text{Sym}\{1, 6\} \times \text{Sym}\{5, 7\}, \quad \text{and} \quad C_t = \text{Sym}\{2, 5, 6\} \times \text{Sym}\{4, 1, 7\} \times \text{Sym}\{3\}.$$ 

Lemma 2.2.23. Let $s$ and $t$ be $\lambda$-tableaux. Then $s$ and $t$ are row-equivalent if and only if $s = \pi t$ for some $\pi \in R_t$.

Proof. Then by 2.2.19(a), $s = \pi t$ for some $\pi \in \text{Sym}(n)$. Then $s$ is row-equivalent to $t$ if and only if $\Delta(t) = \Delta(\pi t)$. By 2.2.19(b), $\Delta(\pi t) = \pi(\Delta(t))$ and so $s$ and $t$ are row equivalent iff $\Delta(t) = \pi(\Delta(t))$, that is if $\pi \in R_t$. \hfill $\square$

Lemma 2.2.24. Let $A$ and $B$ be totally ordered sets and $\rho : A \to B$ be a 1-1 functions. Suppose that $A$ is finite. Then there exists a unique $\alpha \in \text{Sym}(A)$ such that $\rho \circ \alpha$ is strictly increasing.

Proof. Without loss $B = \rho(A)$. Then $\rho$ is a bijection. Put $n = |A| = |B|$ and let $A = \{a_1, \ldots, a_n\}$ with $a_1 < a_2 < \ldots < a_n$ and $B = \{b_1, \ldots, b_n\}$ with $b_1 < b_2 < \ldots < b_n$. Define $\beta : A \to B, a_i \mapsto b_i$ and observe that $\beta$ is the unique strictly increasing function from $A$ to $B$. Let $\alpha \in \text{Sym}(n)$. Then $\rho \circ \alpha$ is strictly increasing if and only if $\rho \circ \alpha = \beta$ and so if and only if $\alpha = \rho^{-1} \circ \beta$. \hfill $\square$

Lemma 2.2.25. Let $\lambda$ and $\mu$ be partitions of $n$, $t$ a $\lambda$-tableau and $s$ a $\mu$-tableau. Suppose that $|C \cap R| \leq 1$ for any column $C$ of $s$ and any row $R$ of $t$. (That is, no two entrees from the same column of $s$ lie in the same row of $t$.)

(a) $\mu \succeq \lambda$.

(b) Suppose in addition that $\lambda = \mu$. Then there exists a $\lambda$-tableau $r$ such that $t$ is row-equivalent to $r$, and $r$ is column-equivalent to $s$.

Proof. Let $C_j := \Delta'(s)_j$ be column of $s$. Let $a, b \in C_j$ with $\text{row}_i(a) = \text{row}_i(b) =: i$. Then $a$ and $b$ are both in $C_j \cap \Delta(t)_i$. By hypothesis $|C_j \cap \Delta(t)_i| \leq 1$ and $a = b$. Thus the restriction $\text{row}_i|_{C_j}$ 1-1. Recall from 2.2.16 that $(C_j, \preceq_s)$ is a totally ordered set. Since also $\mathbb{Z}^+$ is totally ordered we conclude from 2.2.24 that there exists a (unique) $\beta_j \in \text{Sym}(C_j)$ such that $\rho_s \circ \beta_j$ is increasing. Define $\beta \in \text{Sym}(n)$ by $\beta(a) := \beta_j(a)$ if $a \in C_j$. Put $r := \beta s$. 

2.2. DIAGRAMS, TABLEAUX AND TABLOIDS

To illustrate this construction we compute \( r \) in an example:

\[
\begin{array}{cccccccc}
3 & 6 & 7 & 1^3 & 2^3 & 3^1 & 4^2 & 7^1 & 6^1 & 3^1 & 4^2 \\
\hline
r : & 5 & 4 & s : & 5^2 & 6^1 & r : & 5^2 & 2^3 \\
& 2 & 1 & & 7^1 & & & 1^3
\end{array}
\]

here the superscript \( i \) in \( a' \) indicates that \( a \) appears in Row \( i \) of \( t \).

Note that \( \beta \in C_s \), so \( r \) is column equivalent to \( s \). For \( a \in I_n \), define \( a' = \beta^{-1}(a) \). Then

\[
r^{-1}(a) = (\beta s)^{-1}(a) = s^{-1}(\beta^{-1}(a)) = s^{-1}(a')
\]

Considering the first coordinate, gives row, \( r(a) = \text{row}_s(a') \). Let \( a, b \in C_j \). It follows that

\[
\text{row}_r(a) < \text{row}_r(b) \iff \text{row}_s(a') < \text{row}_s(b')
\]

Since \( \beta \in C_s \), we have \( a', b' \in C_j \). Hence \( s^{-1}(a') = (\text{row}_s(a'), j) \) and \( s^{-1}(b') = (\text{row}_s(b'), j) \). Hence

\[
\text{row}_s(a') < \text{row}_s(b') \iff s^{-1}(a') < s^{-1}(b')
\]

By definition of \( <_s \)

\[
s^{-1}(a') < s^{-1}(b') \iff a' <_s b'.
\]

Restricted to \( C_j \), we have \( \text{row}_r \circ \beta = \text{row}_s \circ \beta \) and so \( \text{row}_r \circ \beta \) is increasing on \( C_j \). Thus

\[
a' <_s b' \iff (\text{row}_r \circ \beta)(a') < (\text{row}_r \circ \beta)(b')
\]

Using that \( \beta(a') = a \) and \( \beta(b') = b \) we conclude that

\[
\text{row}_r(a) < \text{row}_r(b) \iff \text{row}_r(a) < \text{row}_r(b)
\]

Since \( \text{row}_r(r(i, j)) = i \) for all \( 1 \leq i \leq \mu_j \) we have

\[
\text{row}_r(r(1, j)) < \text{row}_r(r(2, j)) < \ldots < \text{row}_r(r(\mu_j, j))
\]

and so

\[
\text{row}_r(r(1, j)) < \text{row}_r(r(2, j)) < \ldots < \text{row}_r(r(\mu_j, j)).
\]

A trivial induction argument shows that \( \text{row}_r(r(i, j)) \geq i = \text{row}_r(r(i, j)) \). Since \( r \) is a bijection and so onto, this shows \( \text{row}_r(a) \geq \text{row}_r(a) \) for all \( a \in I_n \). Let \( a \in \bigcup_{i=1}^{k} \Delta(t)_i \). Then \( \text{row}_r(a) \leq k \). Put \( i := \text{row}_r(a) \). Then \( a \in \Delta(t)_i \) and \( i = \text{row}_r(a) \leq \text{row}(a) \leq k \) and \( a \in \Delta(t)_i \). Thus

\[
(*) \quad \bigcup_{i=1}^{k} \Delta(t)_i \subseteq \bigcup_{i=1}^{k} \Delta(r)_i
\]

Since \( \Delta(t) \) is a \( \lambda \)-partition and \( \Delta(r) = \Delta(s) \) is a \( \mu \)-partition, \( |\Delta(t)_i| = \lambda_i \) and \( |\Delta(r)_i| = \mu_i \). So computing the size of the sets (*) we get
\[
\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i
\]

So indeed \( \mu \succeq \lambda \).

(b) Suppose that \( \lambda = \mu \). Then also \( \sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \mu_i \) and the sets in (*) have equal size. Hence (*) implies that

\[
\bigcup_{i=1}^{k} \Delta(t)_i = \bigcup_{i=1}^{k} \Delta_r(i)
\]

Since \( \Delta(t) \) and \( \Delta(r) \) are partitions this gives

\[
\Delta(t)_k = \left( \bigcup_{i=1}^{k} \Delta(t)_i \right) \setminus \left( \bigcup_{i=1}^{k-1} \Delta(t)_i \right) = \left( \bigcup_{i=1}^{k} \Delta(r)_i \right) \setminus \left( \bigcup_{i=1}^{k-1} \Delta(r)_i \right) = \Delta(r)_k.
\]

Hence \( \Delta(t) = \Delta(r) \) and so \( r \) and \( t \) are row equivalent. As seen above \( s \) and \( r \) are column-equivalent and so (b) holds. \( \square \)

2.3 The Specht Module

**Hypothesis 2.3.1.** In the section \( n \) is a positive integer, \( F \) is a non-zero commutative ring and \( \lambda \) a partition of \( n \).

**Definition 2.3.2.** Let \( G \) be a group, \( R \) a ring and \( f \in R[G] \).

(a) \( f \) is called multiplicative, if \( f_{ab} = f_a f_b \) for all \( a, b \in G \).

(b) \( f \) is called a class function if \( f_a = f_b \) for any conjugate elements \( a, b \in G \).

(c) Let \( H \subseteq G \). Then \( f_H := \sum_{h \in H} f_h \).

**Example 2.3.3.** Define

\[
\text{sgn} := \text{sgn}^n_F : \text{Sym}(n) \to F, \quad \pi \mapsto \begin{cases} 1_F & \text{if } \pi \text{ is even} \\ -1_F & \text{if } \pi \text{ is odd} \end{cases}
\]

Then

\[
\text{sgn} = \sum_{\pi \in \text{Sym}(n)} \text{sgn}(\pi) \pi \in F[\text{Sym}(n)]
\]

and for \( A \subseteq \text{Sym}(n) \)

\[
\text{sgn}_A = \sum_{\pi \in A} \text{sgn}(\pi) \pi \in F[\text{Sym}(n)].
\]

**Lemma 2.3.4.** Let \( G \) be a group, \( R \) a ring and \( f \in R[G] \). Suppose that \( f \) is multiplicative.

(a) Let \( A, B \subseteq G \) such that function \( A \times B \to G, (a, b) \mapsto ab \) is 1-1, then \( f_{AB} = f_A f_B \).

(b) Let \( A \leq B \leq G \) and \( T \) a left-transversal to \( A \) in \( B \). Then \( f_B = f_T f_A \).

(c) Let \( A_1, A_2, \ldots, A_n \subseteq G \) and put \( A := \langle A_i \mid 1 \leq i \leq n \rangle \). If \( A = \times_{i=1}^{n} A_i \), then \( f_A = f_{A_1} f_{A_2} \cdots f_{A_n} \).
Proof. (a) Since the function \((a, b) \to ab\) is \(1 - 1\), every element in \(AB\) can be uniquely written as \(ab\) with \(a \in A\) and \(b \in B\). Thus
\[
f_{A}f_{B} = \sum_{a \in A} f_{a}a \cdot \sum_{b \in B} f_{b}b = \sum_{a \in A, b \in B} f_{a}f_{b}ab = \sum_{c \in AB} f_{a}c = f_{AB}
\]

(b) Observe that \(T \times A \to G, (t, a) \to ta\) is a bijection and \(TA = B\). So (b) is a special case of (a).

(c): For \(n = 2\) this is a special case of (a). The general case follows by induction on \(n\). \(\square\)

Lemma 2.3.5. Let \(G\) be a group and \(R\) a ring. Then
\[
gf_{H}g^{-1} = f_{gHg^{-1}}
\]
for all class functions \(f \in R[G]\), all \(H \subseteq G\) and all \(g \in G\).

Proof.
\[
gf_{H}g^{-1} = g \left( \sum_{h \in H} f_{h}h \right) g^{-1} = \sum_{h \in H} f_{h}ghg^{-1} = \sum_{h \in H} f_{ghg^{-1}}ghg^{-1} = \sum_{k \in gHg^{-1}} f_{k}k = f_{gHg^{-1}}
\]

Remark 2.3.6. Let \(\lambda\) be a partition of \(n\). Observe that the function
\[
\tilde{t} \mapsto \Delta(t)
\]
is a well-defined bijection between the \(\lambda\)-tabloids and the \(\lambda\)-partitions of \(I_{n}\). We often identify \(\tilde{t}\) with \(\Delta(t)\). In particular, \(\tilde{t} \in M^{\lambda}\).

Definition 2.3.7. Let \(t\) be \(\lambda\)-tableau.

(a) \(k_{t} := \text{sgn}_{C_{t}} \equiv \sum_{\pi \in C_{t}} \text{sgn}(\pi)\pi \in F[\text{Sym}(n)]\).

(b) \(e_{t} := k_{\tilde{t}} = \sum_{\pi \in C_{t}} \text{sgn}(\pi)\tilde{\pi} \in M^{\lambda}\). \(e_{t}\) is called the \(\lambda\)-polytabloid associated to \(t\).

Example 2.3.8. Let \(t = \begin{array}{ccc}
3 & 2 & 5 \\
1 & 4 & \end{array}\).

Then
\[
C_{t} = \text{Sym}\{1, 3\} \times \text{Sym}\{2, 4\}
\]
\[
k_{t} = \left(1 - (13)\right) \cdot \left(1 - (24)\right) = 1 - (13) - (24) + (13)(24).
\]

and
\[
e_{t} = \begin{array}{ccc}
\frac{3}{1} & \frac{2}{4} & \frac{5}{3} \\
\frac{1}{4} & \frac{5}{3} & \frac{2}{1} \end{array} - \begin{array}{ccc}
\frac{1}{3} & \frac{2}{4} & \frac{5}{2} \\
\frac{3}{1} & \frac{2}{4} & \frac{5}{2} \end{array} + \begin{array}{ccc}
\frac{1}{3} & \frac{2}{4} & \frac{5}{2} \\
\frac{3}{1} & \frac{2}{4} & \frac{5}{2} \end{array}.
\]

Remark 2.3.9. Let \(t\) and \(s\) be \(\lambda\)-tableau.

(a) Suppose \(t\) and \(s\) are row- and column-equivalent. Then \(t = s\).
(b) Let \( t \) be a \( \lambda \)-tableau and \( \alpha, \beta \in C \), with \( \alpha \neq \beta \). Then \( \overline{\alpha t} \neq \overline{\beta t} \).

Proof. (a): For all \( a \in I_n \), \( a \) lies in the same row and column of \( t \) as of \( s \). So \( t = s \).

(b): Suppose \( \overline{\alpha t} = \overline{\beta t} \). Then \( \alpha t \) and \( \beta t \) are row-equivalent. Since \( \alpha, \beta \in C \), \( \alpha t \) and \( \beta t \) are also column-equivalent. So (a) shows that \( \alpha t = \beta t \). Since \( t \) is a bijection this gives \( \alpha = \beta \). \( \square \)

**Lemma 2.3.10.** Let \( s \) and \( t \) be \( \lambda \)-tableaux. Then the following statements are equivalent.

(a) \( |R \cap C| \leq 1 \) for all rows \( R \) of \( t \) and all columns \( C \) of \( s \).

(b) There exists a \( \lambda \)-tableau which is row-equivalent to \( t \) and column-equivalent to \( s \).

(c) \( \bar{I} \cap |s| \neq \emptyset \).

(d) \( |\bar{I} \cap |s| | = 1 \).

(e) \( \bar{I} = \overline{\pi s} \) for some \( \pi \in C_x \).

(f) \( (\bar{I} | e_s) = \pm 1 \).

(g) \( (\bar{I} | e_s) \neq 0 \).

Proof. (a) \( \implies \) (b): See 2.2.25(b).

(b) \( \implies \) (a): Suppose \( \pi \) is a \( \lambda \)-partition which is row-equivalent to \( t \) and column-equivalent to \( s \). Let \( R \) be a row of \( t \) and \( C \) a column of \( s \). Then \( R \) is a row of \( r \) and \( C \) is a column of \( r \). Hence \( |R \cap C| \leq 1 \).

(b) \( \iff \) (c): Just recall the \( \bar{I} \) is the set of \( \lambda \)-tableaux which are row-equivalent to \( t \) and that \( |s| \) is the set of \( \lambda \)-tableaux which are column-equivalent to \( s \).

(c) \( \iff \) (d): If \( u, r \in \bar{I} \cap |s| \), then \( u \) is row- and column-equivalent to \( r \) and so \( u = r \).

(b) \( \iff \) (d): Let \( r \) be a \( \lambda \)-tableau and \( \pi \in \text{Sym}(n) \) with \( r = \pi s \). Then \( r \) is row equivalent to \( t \) if and only if \( \bar{I} = \bar{r} \) and if and only if \( \bar{I} = \overline{\pi s} \). Also \( r \) is column-equivalent to \( s \) if and only if \( \pi \in C_x \).

(c) \( \iff \) (e) \( \iff \) (f): Recall that \( e_s = \sum_{\pi \in C_x} \text{sgn}(\pi)\overline{\pi s} \) and that the \( \overline{\pi s} \), \( \pi \in C_x \), are pairwise distinct. Also \( \{\bar{I} | r \text{ a \( \lambda \)-tableau}\} \) is an orthonormal basis for \( M^\lambda \). If \( \bar{I} = \overline{\pi s} \) for some \( \pi \in C_x \) we conclude that \( (\bar{I} | e_s) = \text{sgn}(\pi) = \pm 1 \neq 0 \). If \( \bar{I} \neq \overline{\pi s} \) for all \( \pi \in C_x \), then \( (\bar{I} | e_s) = 0 \neq \pm 1 \). \( \square \)

**Definition 2.3.11.** (a) \( S^\lambda \) is the \( F \)-submodule of \( M^\lambda \) spanned by the \( \lambda \)-polytabloids. \( S^\lambda \) is called the Specht module associated to \( \lambda \).

(b) \( F^\lambda \) is the left ideal in \( F[\text{Sym}(n)] \) generated by the \( k_t \), \( t \) a \( \lambda \)-tableau.

(c) For \( W \subseteq M^\lambda \) define \( W^\perp := \{m \in M^\lambda \ | \ (w | m) = 0 \text{ for all } w \in W\} \).

(d) \( D^\lambda := S^\lambda / (S^\lambda \cap S^{\lambda \perp}) \).

**Example 2.3.12.** Suppose \( n \geq 2 \). Let \( \lambda = (n - 1, 1) \) and let \( t \) be \( \lambda \)-tableau of shape \( t = \begin{array}{cccc} i & \cdots & \cdots & j \\ \vdots & \ddots & \ddots & \vdots \\ j & \cdots & \cdots & i \end{array} \).

Then \( C_t = \text{Sym}(i, j) = \{1, (i, j)\} \), \( k_t = 1 - (i, j) \) and

\[
e_t = \begin{array}{cccc} i & \cdots & \cdots & j \\ \dfrac{j}{i} & \ddots & \ddots & \dfrac{i}{j} \\ \dfrac{j}{i} & \cdots & \cdots & \dfrac{i}{j} \end{array}
\]
For \( i \in I_n \) let \( x_i \) be the \( \lambda \)-partition

\[
x_i := (I_n \setminus \{i\}, \{i\}) = \frac{1}{i} \begin{array}{cccc}
1 & 2 & \ldots & i - 1 & i + 1 & \ldots & n \\
\end{array}
\]

Then \( M^{(n-1,1)} \) is the free \( F \)-module with basis \( (x_i)_{i=1}^n \). Moreover, \( e_i = x_j - x_i \). Thus

\[
S := S^{(n-1,1)} = \langle x_j - x_i \mid i \neq j \in I_n \rangle_F = \left\{ \sum_{i=1}^n f_i x_i \mid f_i \in F, \sum_{i=1}^n f_i = 0 \right\} = (x_1 + x_2 + \ldots + x_n)^\perp = x^\perp,
\]

where \( x = x + 1 + x_2 + \ldots + x_n \).

Note that \( (\sum_{i=1}^n x_i x_i) = f_i - f_j \) so

\[
S^\perp = \{ \sum_{i=1}^n f_i x_i \mid f_i = f_j \text{ for all } i \neq j \in I_n \} = Fx.
\]

Hence

\[
S \cap S^\perp = \{ fx \mid f \in F, nf = 0 \}.
\]

Assume that \( F \) is an integral domain and put \( p := \text{char } F \). If \( p \nmid n \), then \( S \cap S^\perp = 0 \) and so \( D^4 = S^\perp \). If \( p \mid n \), then \( S^\perp \leq S \) and so \( D^4 = S/S^\perp \).

Suppose that \( n = 2 \) and \( p = 2 \). Then \( \lambda = (1,1) \) and \( \lambda \) is 2-singular. Also \( S = \langle x_1 - x_2 \rangle_F = \langle x_1 + x_2 \rangle_F = S^\perp \) and so \( D^4 = 0 \).

Suppose \( (n,p) \neq (2,2) \). We claim that \( \lambda \) is \( p \)-regular. Assume that \( n > 2 \) then \( n - 1 \neq 1 \) and so \( \lambda \) is \( p \)-regular. Assume that \( n = 2 \), then \( \lambda = (1,1) \). Note that \( p = 0 \) or \( p > 2 \) and so again \( \lambda \) is \( p \)-regular.

If \( (n,p) \neq (2,2) \) and \( F \) is a field, then \[1.1.17\] shows that \( D^4 \) is a simple \( F[\text{Sym}(n)] \)-module.

Lemma 2.3.13. Let \( \pi \in \text{Sym}(n) \) and \( t \) a tableau.

(a) \( \pi k \pi^{-1} = k_t \)
(b) \( \pi e_t = e_{\pi t} \).
(c) \( \text{Sym}(n) \) acts transitively on the set of \( \lambda \)-polytabloids.
(d) \( S^\perp \) is a \( F[\text{Sym}(n)] \)-submodule of \( M^4 \).
(e) If \( \pi \in C_t \), then \( k_\pi = k_t, \pi k_t = k_\pi, \text{sgn}(\pi)k_t = \text{sgn}(\pi)k_\pi \) and \( \pi e_t = e_{\pi t} = \text{sgn}(\pi)e_t \).
(f) Let \( s \) and \( t \) be column-equivalent \( \lambda \)-tableaux. Then \( e_s = \pm e_t \).

Proof. (a): We have \( C_t = \pi C_\pi \pi^{-1} \) and so by \[2.3.3\] applied to the class function \( \text{sgn} \) on \( \text{Sym}(n) \),

\[
k_\pi = \text{sgn}_{C_\pi} = \text{sgn}_{\pi C_\pi \pi^{-1}} = \pi C_\pi \pi^{-1} = \pi k_\pi \pi^{-1}
\]

(b): Using (a)

\[
e_{\pi t} = k_{\pi t} e_t = \pi k_t \pi^{-1} e_t = \pi k_\pi = e_t
\]

(c): By \[2.2.19\] \( \text{Sym}(n) \) act transitively on the set of \( \lambda \)-tableaux. Hence (b) implies that \( \text{Sym}(n) \) also transitively on the set of \( \lambda \)-polytabloids.
CHAPTER 2. REPRESENTATIONS OF THE SYMMETRIC GROUPS

(F) Follows from (E).

Since \( \pi \in C_r \), \( C_{\pi r} = \pi C_r \pi^{-1} = C_r \). Thus

\[
k_i = C_i^{-} = C_{\pi r}^{-} = \pi k_i \pi^{-1}.
\]

In particular, \( \pi k_i = k_i \pi \).

Moreover, \( C_i \to C_{\beta} \), \( \beta \mapsto \beta \pi \) is a bijection and so

\[
k_i = \sum_{\alpha \in C_i} \text{sgn}(\alpha)\alpha = \sum_{\beta \in C_i} \text{sgn}(\beta \pi)\beta \pi = \sum_{\beta \in C_i} \text{sgn}(\beta)\beta \pi = \text{sgn}(\pi)k_i \pi
\]

Multiplying with \( \text{sgn}(\pi) \) from the left gives

\[
\text{sgn}(\pi)k_i = k_i \pi.
\]

Hence

\[
\pi e_r = e_{\pi r} = k_{\pi r} \pi = k_i \pi = \text{sgn}(\pi)k_i \pi = e_r.
\]

(E) By the column version of 2.2.23 there exists \( \pi \in C_r \) with \( s = \pi t \). Hence by \( e_r = e_{\pi r} = \text{sgn}(\pi) e_r \).

\[\Box\]

Lemma 2.3.14. Let \( \lambda \) and \( \mu \) be partitions of \( n \).

(a) Suppose \( s \) is a \( \mu \)-tableau and \( t \) is a \( \lambda \)-tableau with \( k_s \pi \neq 0 \). Then \( |R \cap C| \leq 1 \) for any row \( R \) of \( t \) and any column \( C \) of \( s \). In particular, \( \lambda \leq \mu \).

(b) Suppose that \( F^\mu M^\lambda \neq 0 \). Then \( \lambda \leq \mu \).

(c) Suppose \( r, s, t \) are \( \lambda \)-tableaux such that \( r \) is row-equivalent to \( t \) and column-equivalent to \( s \). Then \( k_s \pi = e_r = \pm e_s \). In particular, \( k_i e_r = e_r \).

(d) Suppose \( t \) and \( s \) are \( \lambda \)-tableau with \( k_s \pi \neq 0 \). Then \( k_t \pi = \pm e_s \).

Proof. (a) Otherwise there exists \( i \neq j \in I_\lambda \) such that \( i \) and \( j \) are in the same row of \( t \) and in the same column of \( s \). Put \( H := \text{Sym}\{i, j\} = \{1, (i, j)\} \). Since \( i \) and \( j \) are the same row of \( t \) we have \( (i, j) \pi \pi = \pi = \pi \) and so

\[
H^{-1} \pi = \pi + \text{sgn}(i, j)(i, j) \pi \pi = \pi - \pi = 0.
\]

Since \( i, j \) are in the same column of \( s \) we have \( H \leq C_s \). Choose transversal \( T \) to \( H \) in \( C_s \). By 2.3.4(b) we get

\[
k_i = C_i^{-} = T^{-} H^{-}
\]

and so

\[
k_s \pi = (\text{sgn}_T \text{sgn}_H) \pi = \text{sgn}_T(\text{sgn}_H \pi) = 0
\]

contrary to our assumption. The second statement now follows from 2.2.25.

(F) Follows from (E).

(F) : Since \( r \) and \( s \) are column equivalent we know that \( C_r = C_s \) and \( e_s = \pm e_r \). In particular, \( k_r = k_s \).

Since \( r \) and \( t \) are row equivalent we have \( \pi = \pi \). Thus

\[
k_s \pi = k_t \pi = e_s = \pm e_r.
\]

(F) Apply (F) with \( \lambda = \mu \). Then 2.2.25 shows that there exists a \( \lambda \)-tableau \( r \) which is row equivalent to \( t \) and columns equivalent to \( s \). Thus (F) follows from (E).

\[\Box\]
Lemma 2.3.15. Let $V$ and $W$ be $F$-modules and $s : V \times W \to F$ an $F$-bilinear function. Let $\alpha : V \to V$ and $\beta : W \to W$ be $F$-linear and suppose that
\[ s(\alpha v, w) = s(v, \beta w) \]
for all $v \in V$ and $w \in W$. Then
\[ \ker \alpha \subseteq (\Im \beta) \]
with equality if $W \perp = 0$.

Proof. Let $v \in V$. Then
\[ v \in \ker \alpha \quad \iff \quad \alpha v = 0 \]
\[ \implies \quad s(\alpha v, w) = 0 \text{ for all } w \in W \]
\[ \iff \quad s(v, \beta w) = 0 \text{ for all } w \in W \]
\[ \iff \quad v \in (\Im \beta) \]
Moreover, if $W \perp = 0$, the $\implies$ becomes an $\iff$.

Definition 2.3.16. Let $R$ be ring and $G$ a group. For $a \in R \rtimes G$ define $a^\circ := \sum_{g \in G} a g g^{-1}$. For $A \subseteq R[G]$ define $A^\circ = \{ a^\circ \mid a \in A \}$.

Lemma 2.3.17. Let $G$ be a group, let $V$ and $W$ be $R \rtimes G$-modules and let $s : V \times W \to F$ be an $G$-invariant $F$-bilinear function. Then
\[ (a) \quad (av \mid w) = (v \mid a^\circ w) \text{ for all } a \in F[G], v \in V \text{ and } w \in W. \]
\[ (b) \quad \text{Let } B \subseteq F[G]. \quad \text{Then } A_V(B) \subseteq (B \circ W) \]
\[ \text{with equality if } W \perp = 0. \]

Proof. (a): Let $g \in G$, $v \in V$ and $w \in W$. Then
\[ s(gv, w) = s(g^{-1}g, g^{-1}w) = s(v, g^{-1}w) = s(v, g^\circ w) \]
As $s$ is $F$-bilinear, this gives (a).

(b): Let $a \in F[G]$. By (a) $s(av, w) = s(v a^\circ, w)$ and so 2.3.15 shows that $A_V(a) \subseteq (a^\circ W) \perp$ with equality of $W \perp = 0$. Hence
\[ A_V(B) = \bigcap_{b \in B} A_V(b) \subseteq \bigcap_{b \in B} (b^\circ W) \perp = \left( \bigcup_{b \in B} b^\circ W \right) \perp = (B^\circ W) \perp \]
with equality if $W \perp = 0$.

Lemma 2.3.18. Let $\mathbb{F} \leq \mathbb{K}$ be an extension of rings and $V$ an $\mathbb{F}$-space. For an $\mathbb{F}$-subspace $U$ put
\[ \overline{U} := \langle k \otimes u \mid k \in \mathbb{K}, u \in U \rangle \leq \mathbb{K} \otimes \mathbb{F} V \]
\[ (a) \quad \text{Let } U \text{ be an } \mathbb{F}-\text{subspace of } V. \quad \text{Then } \mathbb{K} \times U \to \overline{U}, (k, u) \to k \otimes u \text{ is a tensor product of } \mathbb{K} \text{ and } U \text{ over } \mathbb{F}. \]
(b) Let $U$ be an $F$-subspace of $V$. Then $\mathbb{K} \times V/U \rightarrow \overline{V/U}, (k, u + U) \mapsto (k \otimes u) + \overline{U}$ is a tensor product of $\mathbb{K}$ and $V/U$ over $F$.

(c) Let $U$ be a set of $F$-subspaces of $V$. Then

$$\bigcap_{\text{set } U} U = \bigcap_{\text{set } U} \overline{U}$$

(d) Let $s : V \otimes W \rightarrow F$ be an $F$-bilinear form and extend $s$ to a bilinear form

$$\tilde{s} : \mathbb{K} \otimes_F V \times \mathbb{K} \otimes_F W \rightarrow \mathbb{K} \text{ with } (k \otimes v, l \otimes w) \mapsto kl s(v, w).$$

Let $X$ an $F$-subspace of $V$. Then $X^\perp = \overline{X}$. 

Proof. (a) and (b): Let $V = U \oplus W$ for some $F$-subspace $W$ of $V$. Then $K \otimes F V = \mathbb{K} \otimes F U \oplus \mathbb{K} \otimes F W$. Note that $\overline{U}$ is the direct summand $\mathbb{K} \otimes F U$ and $\mathbb{K} \otimes F V/\overline{U} \cong \mathbb{K} \otimes F W$. This gives (a) and (b).

(c): Suppose first that $U = \{U_1, U_2\}$. Then there exists $F$-subspaces $X_i$ of $U_i$ with $U_i = X_i \oplus (U_1 \cap U_2)$. Observe that $U_1 + U_2 = (U_1 \cap U_2) \oplus X_1 \oplus X_2$. Then $U_1 = U_1 \cap U_2 \oplus X_1 \cap X_2$ and $U_1 + U_2 = U_1 \cap U_2 \oplus X_1 \oplus X_2$ and so $U_1 \cap U_2 = U_1 \cap U_2$. So (c) holds if $|U| = 2$. By induction it holds if $U$ is finite.

Let $W$ be finite dimensional $F$-subspace of $V$. The also $\overline{W}$ is finite dimensional over $\mathbb{K}$. So we can choose finite subset $U_X$ of $U$ such that

$$\overline{W} \cap \bigcap_{\text{set } U} U = \overline{W} \cap \bigcap_{\text{set } U} \overline{U} \quad \text{and} \quad W \cap \bigcap_{\text{set } U} U = W \cap \bigcap_{\text{set } U} \overline{U}.$$ 

Since $\{W\} \cup U_w$ is finite, the proven finite case of (c) shows that

$$\overline{W} \cap \bigcap_{\text{set } U} U = W \cap \bigcap_{\text{set } U} \overline{U}$$

and so

$$\overline{W} \cap \bigcap_{\text{set } U} \overline{U} = W \cap \bigcap_{\text{set } U} U.$$ 

Let $W$ be the set of finite dimension $F$-subspaces of $V$. Observe that $V = \bigcup_{W \in W} W$ and $\overline{V} = \bigcup_{W \in W} \overline{W}$. Hence

$$\bigcap_{\text{set } U} U = \bigcup_{W \in W} \left( \overline{W} \cap \bigcap_{\text{set } U} \overline{U} \right) = \bigcup_{W \in W} \left( W \cap \bigcap_{\text{set } U} \overline{U} \right) = \bigcap_{\text{set } U} \overline{U}$$

(c) Let $U$ be the set of 1-dimensional $F$-subspaces of $X$. Let $U \in U$. Note that $\tilde{s}(k \otimes u, lw) = 0$ for all $k, l \in \mathbb{K}, u \in U$ and $w \in U^\perp$. Since $\tilde{s}$ is $\mathbb{Z}$-bilinear this gives $\tilde{s}(\overline{u}, \overline{w}) = 0$ for all $\overline{u} \in \overline{U}$ and $\overline{w} \in \overline{U^\perp}$. Thus $U^\perp \leq \overline{U}$. If $U^\perp = V$ this gives $U^\perp = \overline{V}$. Suppose $U^\perp \neq \overline{U}$. Let $u \in U$ and $w \in W$ with $s(u, w) \neq 0$. Then also $s(1 \otimes u, 1 \otimes w) \neq 0$ and so $\overline{U} \neq \overline{W}$. Since $U$ is 1-dimensional over $F$ we have $U = Fu$ and so $U^\perp = u^\perp$. It follows that $U^\perp$ is the kernel of the $F$-linear function $W \rightarrow F, w \mapsto s(u, w)$ and so $W/\overline{U}^\perp$ is 1-dimensional. By (d) $W/\overline{U}^\perp = \mathbb{K} \otimes_F W/\overline{U}^\perp$ and so $\overline{W}/\overline{U}^\perp$ is 1-dimensional over $\mathbb{K}$. As $\overline{U}^\perp \leq \overline{U} \leq \overline{W}$ this shows that $\overline{U}^\perp = \overline{U}$. Observe that $X = \langle U \in U \rangle_F$ and $\overline{X} = \langle \overline{U} \mid U \in U \rangle_F$. Thus
Lemma 2.3.19. Let $\lambda$ and $\mu$ be partitions of $n$ and $t$ an $\lambda$-tableau. Then

(a) $k_i = k_i^2$

(b) $(k_i M^\mu)^\perp = A_{M^\mu}(k_i)$.

(c) $k_i M^4 = Fe_t$ and $A_{M^\mu}(k_i) = e_t^\perp$.

(d) $(k_i v \mid w) = (v \mid k_i w)$ for all $v, w \in M^\mu$.

(e) $k_i v = (v \mid e_i) e_i$ for all $v \in M^4$.

Proof. (a) $k_i^2 = \sum_{\pi \in C_i} \text{sgn}(\pi) \pi^{-1} = \sum_{\pi \in C_i} \text{sgn}(\pi-1) \pi^{-1} = \sum_{\pi \in C_i} \text{sgn}(\pi) \pi = k_i$.

(b) Recall that $(\cdot \mid \cdot)$ is a Sym(n)-invariant non-degenerate $F$-bilinear form on $M^\mu$. Hence 2.3.17(b) shows that $A_{M^\mu}(k_i) = (k_i M^\mu)^\perp$. By (a) $k_i = k_i^2$ and so (b) holds.

(c) Let $s$ be a $\lambda$-tableau. By 2.3.14 $k_i e_s = 0$ or $k_i e_s = \pm e_s$. Also $k_i e_t = e_t$. Since $S^4$ is spanned by $e_t$, this gives $k_i M^4 = Fe_t$. Hence by (b) $A_{M^\mu}(k_s) = Fe_t^\perp = e_t^\perp$.

(d) By 2.3.17(a) we have $(k_i v \mid w) = (v \mid k_i^2 w)$ and by (a) $k_i = k_i^2$.

(e) By (c) $k_i v = fe_t$ for some $f \in F$. Note also that $(e_t \mid \bar{t}) = 1$. So

$$(v \mid e_t) = (v \mid k_i \bar{t}) = (k_i v \mid \bar{t}) = (fe_t \mid \bar{t}) = f (e_t \mid \bar{t}) = f.$$

Lemma 2.3.20. $F^4 M^4 = S^4$ and $A_{M^4}(F^4) = S^4\perp$.

Proof. This follows immediately from 2.3.19(a) and 2.3.19(c). □

Lemma 2.3.21. Suppose $F$ is a field and let $V$ be an $F[\text{Sym}(n)]$-submodule of $M^4$. Then either $(F^4 V = S^4$ and $S^4 \leq V)$ or $(F^4 V = 0$ and $S^4 \leq V)$.

Proof. If $F^4 V = 0$, then by 2.3.20 $V \leq A_{M^4}(F^4) = S^4\perp$.

So suppose $F^4 V \neq 0$. Let $T^4$ be the set of $\lambda$-tableaux. As $F^4$ is generated by the $k_i, t \in T^4$ we conclude that $k_i V \neq 0$ for some $\lambda$-tableau $t$. By 2.3.19 $k_i M^4 = Fe_t$. Hence $0 \neq k_i V \leq Fe_t$. Since $F$ is a field and $kS$ is an $F$-submodule of $V$, we conclude that $k_i V = Fe_t$. Let $\pi \in \text{Sym}(n)$. Since $V$ is $\text{Sym}(n)$-invariant we have $\pi^{-1} V = V$. Then

$$k_{\pi} V = \pi k_i \pi^{-1} V = \pi k_i V = \pi Fe_t = F\pi e_t = F e_\pi.$$ 

By 2.3.13(a) $\text{Sym}(n)$ acts transitively $T^4$, so $k_i V = Fe_t$ for all $t \in T^4$. Thus

$$F^4 V = F[\text{Sym}(n)] \{ k_i \mid t \in T^4 \} V = F[\text{Sym}(n)] \{ e_t \mid t \in T^4 \} = S^4.$$ 

Note that $F^4 V \leq V$, so $S^4 \leq V$. □
Proposition 2.3.22. Suppose that $F$ is a field and $D^3 \neq 0$. Then $D^3$ is a simple $F[\text{Sym}(n)]$-module.

Proof. Let $W$ be an $F[\text{Sym}(n)]$-submodule of $D^3$. Then $W = \langle S^3 \rangle$ for some $F[\text{Sym}(n)]$-submodule $V$ of $S^3$. By 2.3.21 $S^3 \leq V$ or $V \leq S^3$. In the first case $V = S^3$ and in the second case $V = S^3 \cap S^3$. Hence $W = D^3$ or $W = 0$. As $D^3 \neq 0$, this shows that $D^3$ is a simple $F[\text{Sym}(n)]$-module. \hfill $\square$

Lemma 2.3.23. Suppose $F$ is a field, that $D^3 \neq 0$ and that either $\text{char} F \neq 0$ or $\text{char} F > n$. Then $S^3 \cap S^3 = 0$. In particular, $S^3 \cong D^3$ as an $F[\text{Sym}(n)]$-module.

Proof. Note that $\text{char} F \neq n! = |\text{Sym}(n)|$ and so Maschke’s Theorem [1.3.1] shows that $S^3 = U \oplus (S^3 \cap S^3)$ for some $F[\text{Sym}(n)]$-submodule $U$ of $S^3$. Since $D^3 \neq 0$ we have $U \neq 0$ and so $U \leq S^3$. Hence 2.3.21 shows that $S^3 \leq U$. Thus $S^3 \cap S^3 \leq U \cap S^3 = 0$. \hfill $\square$

Lemma 2.3.24. Suppose $F$ is a field. Then $F^3 D^3 = D^3$.

Proof. By 2.3.21 either $F^3 S^3 = S^3$ or $S^3 \leq S^3$. In the first case $F^3 D^3 = D^3$ and in the second $D^3 = 0$ and again $F^3 D^3 = D^3$. \hfill $\square$

Proposition 2.3.25. Let $\lambda$ and $\mu$ be partitions of $n$. Suppose that $F$ is a field and $D^3 \neq 0$.

(a) Suppose that $D^3$ is isomorphic to an $F[\text{Sym}(n)]$-section of $M^\mu$. Then $\lambda \leq \mu$.

(b) If $D^3$ and $D^\mu$ are isomorphic $F[\text{Sym}(n)]$-modules, then $\lambda = \mu$.

Proof. (a) By 2.3.24 $F^3 D^3 = D^3 \neq 0$. Since $D^3$ isomorphic to an $F[\text{Sym}(n)]$ section of $M^\mu$ this gives $F^3 M^\mu \neq 0$. So 2.3.14(b) shows that $\lambda \leq \mu$.

(b) Suppose that $D^3 \cong D^\mu$ as $F[\text{Sym}(n)]$-modules. Then $D^\mu$ is isomorphic to $F[\text{Sym}(n)]$-section of $M^\lambda$ and $D^3$ is isomorphic to $F[\text{Sym}(n)]$-section of $M^\lambda$. Thus (a) shows that $\lambda \leq \mu$ and $\mu \leq \lambda$. Hence $\lambda = \mu$. \hfill $\square$

Lemma 2.3.26. Let $\mathbb{K} \subseteq \mathbb{K}$ be a field extension.

(a) $M^3_\mathbb{K}$ is an $\mathbb{F}[\text{Sym}(n)]$-submodule of $M^3_{\mathbb{K}}$.

(b) $M^3_\mathbb{K} = \mathbb{K} M^3_\mathbb{F} \cong \mathbb{K} \otimes_{\mathbb{F}} M^3_\mathbb{F}$.

(c) $S^3_\mathbb{K} = \mathbb{K} S^3_\mathbb{F} \cong \mathbb{K} \otimes_{\mathbb{F}} S^3_\mathbb{F}$.

(d) $S^3_\mathbb{K} = \mathbb{K} S^3_\mathbb{F} \cong \mathbb{K} \otimes_{\mathbb{F}} S^3_\mathbb{F}$.

(e) $S^3_\mathbb{K} \cap S^3_\mathbb{F} = \mathbb{K} (S^3_\mathbb{F} \cap S^3_\mathbb{F}) = \mathbb{K} \otimes_{\mathbb{F}} (S^3_\mathbb{F} \cap S^3_\mathbb{F})$.

(f) $D^3_\mathbb{K} \cong \mathbb{K} \otimes_{\mathbb{F}} D^3_\mathbb{F}$.

Proof. (a) $M^3_\mathbb{K}$ is an $\mathbb{F}[\text{Sym}(n)]$-submodule of $M^3_{\mathbb{K}}$.

(b) Recall that $M^3_\mathbb{F}$ is a $\mathbb{F}$-basis for $M^3_\mathbb{F}$ and a $\mathbb{K}$-basis for $M^3_{\mathbb{K}}$. Thus (b) holds.

For an $\mathbb{F}$ subspace $U$ of $M^3_\mathbb{F}$ let $\overline{U} = \langle k \otimes u \mid k \in \mathbb{K}, u \in U \rangle_{\mathbb{F}} \subseteq \mathbb{K} \otimes_{\mathbb{F}} M^3_\mathbb{F}$. Note that tensor products are only defined up to isomorphism. So we may write $M^3_{\mathbb{K}} = \mathbb{K} \otimes_{\mathbb{F}} M^3_\mathbb{F}$. Then $\overline{U} = \mathbb{K} U$. By 2.3.18(a) we have $\overline{U} \cong \mathbb{K} \otimes_{\mathbb{F}} U$. Thus

\[ \mathbb{K} U = \overline{U} \cong \mathbb{K} \otimes_{\mathbb{F}} U. \]
2.4. STANDARD BASIS FOR THE SPECHT MODULE

Let $\mathcal{P}^\lambda$ be the set of $\lambda$-polytabloids. Then $S^\lambda_{\mathbb{K}} = \langle \mathcal{P}^\lambda \rangle_{\mathbb{K}}$ and $S^\lambda_{\mathbb{F}} = \langle \mathcal{P}^\lambda \rangle_{\mathbb{F}}$. As $\mathbb{F} \subseteq \mathbb{K}$ this gives $S^\lambda_{\mathbb{K}} = \mathbb{K}S^\lambda_{\mathbb{F}}$. Hence

\[ S^\lambda_{\mathbb{K}} = \mathbb{K}S^\lambda_{\mathbb{F}} \quad \text{(**) \quad \text{So (c) holds.}} \]

\[ S^\lambda_{\mathbb{F}} \cong S^\lambda_{\mathbb{F}} \quad \text{(*) \quad \text{by 2.3.26(f)}} \]

\[ S^\lambda_{\mathbb{K}} \cong \mathbb{K}S^\lambda_{\mathbb{F}} \quad \text{(*) \quad \text{by 2.3.26(f)}} \]

\[ D^\lambda_{\mathbb{F}} = S^\lambda_{\mathbb{F}} / \langle S^\lambda_{\mathbb{K}} \rangle \quad \text{(**) \quad \text{by 2.3.26(f)}} \]

**Lemma 2.3.27.** Let $\mathbb{F}$ be a field. Then $D^\lambda_{\mathbb{F}}$ is an absolutely simple $\mathbb{F}[\text{Sym}(n)]$-module.

**Proof.** Let $\mathbb{F} \subseteq \mathbb{K}$ be a field extension. By 2.3.26(f), $\mathbb{K} \otimes_{\mathbb{F}} D^\lambda_{\mathbb{F}} = D^\lambda_{\mathbb{K}}$ and by 2.3.22 $D^\lambda_{\mathbb{K}}$ is a simple $\mathbb{K}[\text{Sym}(n)]$-module. Note also that $\mathbb{K}[\text{Sym}(n)] = \mathbb{K} \otimes_{\mathbb{F}} \mathbb{F}[\text{Sym}(n)]$. Hence $\mathbb{K} \otimes_{\mathbb{F}} D^\lambda_{\mathbb{F}}$ is a simple $\mathbb{K} \otimes_{\mathbb{F}} \mathbb{F}[\text{Sym}(n)]$-module. Thus, by definition, $D^\lambda_{\mathbb{F}}$ is absolutely simple $\mathbb{F}[\text{Sym}(n)]$-module.

\[ \square \]

**2.4 Standard basis for the Specht module**

We continue to assume:

**Hypothesis 2.4.1.** In the section $n$ is a positive integer, $F$ is a non-zero commutative ring and $\lambda$ a partition of $n$.

**Lemma 2.4.2.** Let $G$ be a groups and $A$ and $B$ subgroups of $B$. Let $\mathcal{R}$ be transversal to $A \cap B$ to $A$. Then $\mathcal{R}$ is also as transversal to $B$ in $AB$. In particular, the function $\mathcal{R} \times B \to AB, (r, b) \mapsto rb$ is a bijection.

**Proof.** Recall from the second isomorphism theorem that the function $A/A \cap B \to AB/B, a(A \cap B) \mapsto AB/B$ is a bijection. As $\mathcal{R}$ is a transversal to $\lambda \cap B$ in $A$, we conclude that $\mathcal{R}$ is also a transversal to $B$ in $AB$. \[ \square \]

**Proposition 2.4.3.**

(a) Let $\alpha, \beta \in \text{Sym}(n)$ with $\alpha C_\gamma = \beta C_\gamma$. Then $\text{sgn}(\alpha)\alpha e_\gamma = \text{sgn}(\beta)\beta e_\gamma$.

(b) Let $H \subseteq \text{Sym}(n)$ and $\mathcal{R}$ a transversal to $H \cap C_\gamma$ in $H$. Then

\[ \text{sgn}_{H \cap C_\gamma} = \text{sgn}_{\mathcal{R}e_\gamma}. \]

(c) Let $Z \subseteq I_n$ such that $|Z| > |C|$ for all columns $C$ of $n$ with $|Z \cap C| \neq \emptyset$. Then $\text{sgn}_{\text{Sym}(Z)C_\gamma} = 0$. 

\[ \text{sgn}_{\text{Sym}(Z)C_\gamma} = 0. \]
**Proof.**

(a) Pick \( p \in C_i \) with \( \alpha = \beta p \). Then
\[
\sgn(\alpha) \alpha e_i = \sgn(\alpha p) \alpha p e_i = (\sgn(\alpha) \alpha)(\sgn(\rho) p e_i) = \sgn(\alpha) \alpha e_i,
\]
and so (a) holds.

(b) By 2.4.2 \( R \times C_i \to HC_i, (\alpha, \beta) \mapsto \alpha \beta \) is a bijection. Thus using 2.3.4(a)
\[
\sgn_{HC_i}^\lambda = \sgn_R \sgn_{C_i}^\lambda = \sgn_R e_i.
\]

(c) Set \( J = \{ j \in I_n \mid Z \cap \Delta'(t)_j \neq \emptyset \} \). Put \( k := \max \{ \lambda'_j \mid j \in J \} \) and \( D := \bigcup_{j \in J} [\lambda'_j] \). Observe that \( t(D) = \bigcup_{j \in J} \Delta'(t)_j \), so \( Z \subseteq t(D) \). Thus \( t(D) \) is invariant under \( \text{Sym}(Z) \). As \( C_i \) fixes all columns of \( t \), \( t(D) \) is also invariant under \( C_i \). Let \( \alpha \in \text{Sym}(Z)/C_i \). Then \( (\alpha t)(D) = \alpha(t(D)) = t(D) \). Hence
\[
Z \subseteq t(D) = (\alpha t)(D) = \bigcup_{j \in J} \Delta'(\alpha t)_j.
\]

As \( \alpha t \) is \( \lambda \)-tableau and \( k \geq \lambda'_j \) for all \( j \in J \), this shows that \( Z \) is contained in the first \( k \)-rows of \( \alpha t \). As \( |Z| > k \) there exist distinct \( y, z \in Z \) which lie in the same row of \( \alpha t \). Choose such \( y \) and \( z \) and then \( z \) minimal and define \( \pi_y := (y, z) \in \text{Sym}(Z) \) and \( \alpha' := \pi_y \alpha \). Then \( \alpha' \in \text{Sym}(Z)/C_i \). Observe that \( \alpha t \) and \( \alpha' t \) have the same rows. So \( \alpha t \alpha'^{-1} = \alpha'^{-1} \alpha t \) and \( \pi_{y'} = \pi_y \). Thus \( \alpha'' = \pi_{y'} \alpha' = \pi_y \pi_{y'} \alpha = \alpha \). Note also that \( \alpha \neq \alpha' \).

Hence we can partition \( \text{Sym}(Z)/C_i \) into pairs \( \{ \alpha, \alpha' \} \). Since
\[
\sgn(\alpha) \alpha \mathbf{I} + \sgn(\alpha') \alpha' \mathbf{I} = \sgn(\alpha) \mathbf{I} + \sgn(\alpha') \alpha' \mathbf{I} = \sgn(\alpha) \mathbf{I} - \sgn(\alpha) \mathbf{I} = 0
\]
this shows that
\[
\sgn_{\text{Sym}(Z)/C_i}^\lambda \mathbf{I} = \sum_{\alpha \in \text{Sym}(Z)/C_i} \sgn(\alpha) \alpha \mathbf{I} = 0.
\]

\[\Box\]

**Definition 2.4.4.** Let \( t \) be a \( \lambda \)-tableau and \( Z \subseteq I_n \).

(a) \( \text{row}_Z := \text{row}_t \). Note here that \( \text{row}_t \) only depends on \( \mathbf{I} \), so this is well-defined.

(b) \( t \) is row-increasing on \( Z \) if \( \text{row}_t \) is increasing on \( Z \cap R \) for each row \( R \) of \( t \). We say that \( t \) is row-increasing if \( t \) is row-increasing on \( I_n \).

(c) \( t \) is columns-increasing on \( Z \) if \( \text{col}_t \) is increasing on \( Z \cap C \) for each column \( C \) of \( t \). We say that \( t \) is column-increasing if \( t \) is columns-increasing on \( I_n \).

**Remark 2.4.5.** Let \( t \) be a \( \lambda \)-tableau and \( D \subseteq [A] \). Then \( t \) is row increasing on \( t(D) \) if and only if \( t \) is increasing on \( D \cap R \) for each row \( R \) of \( [A] \).

**Definition 2.4.6.** Let \( t \) be a \( \lambda \)-tableau and \( Z \subseteq I_n \).

(a) \( \mathcal{R}_{Z \lambda} \) is the set of all \( \pi \in \text{Sym}(Z) \) such that \( \pi t \) is column-increasing on \( Z \).

(b) \( G_{Z \lambda} := \mathcal{R}_{Z \lambda} \) is called a Garnir element of \( F[\text{Sym}(n)] \).

**Lemma 2.4.7.** Let \( t \) be a \( \lambda \)-tableau and \( Z \subseteq I_n \).

(a) Let \( C \) be the set of columns of \( t \). Then \( \text{Sym}(Z) \cap C_t = \times_{C \in C} \text{Sym}(Z \cap C) \).

(b) \( \mathcal{R}_{Z \lambda} \) is a transversal to \( \text{Sym}(Z) \cap C_t \) in \( \text{Sym}(Z) \).
(c) Suppose that \(|Z| > |Z \cap C|\) for all columns \(C\) of \(t\) with \(Z \cap C \neq \emptyset\). Then
\[
G_Z t e_t = \sum_{\pi \in R_Z} \text{sgn}(\pi)e_{\pi t} = 0.
\]

Proof. (a) follows from \(C_t = \times_{C \in C} \text{Sym}(C)\).

(b) Let \(\pi \in \text{Sym}(Z)\) and let \(C\) be a column of \(\pi t\). Since the restriction of \(\text{row}_{\pi t}\) to \(Z \cap C\) is 1-1, \(2.2.24\) shows that there exists a unique element \(\beta_C \in \text{Sym}(Z \cap C)\) such that \(\text{row}_{\pi t} \circ \beta_C^{-1}\) is increasing on \(Z \cap C\).

By \(2.2.19\) \(\text{row}_{\pi t} \circ \beta_C^{-1} = \text{row}_{\beta_C \pi t}\). Define \(\beta \in \text{Sym}(Z)\) by \(\beta|_C = \beta_C\) for all columns \(C\) of \(\pi t\). Observe that \(\pi t\) and \(\beta \pi t\) have the same columns. Hence \(\beta\) is the unique element of \(\text{Sym}(Z) \cap C_t\) such that \(\beta \pi t\) is column-increasing on \(Z\).

Thus \(\pi \pi t\) is the unique element of \(R_Z\) contained in the coset \(\pi(\text{Sym}(Z) \cap C_t)\). (c) Since \(R_Z\) is a transversal to \(\text{Sym}(Z) \cap C_t\) in \(\text{Sym}(Z)\) we can apply \(2.4.3\). Thus
\[
G_Z t e_t = \text{sgn}_{R_Z} e_t \text{Sym}(Z)C_t = 0.
\]

\(\square\)

Example 2.4.8. Consider \(n = 4, \lambda = (2^2), t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}, Z = \{1, 2, 3, 4\}\). For a tableau \(s\) write \(e(s)\) for \(e_t\). Then \(G_Z t e_t = 0\) gives
\[
e(\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}) - e(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}) + e(\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}) + e(\begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}) - e(\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}) + e(\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}) = 0\]

To confirm

\[
\begin{array}{cccccccc}
+ & 1 & 3 & - & 1 & 2 & + & 1 & 2 \\
2 & 4 & & & & & & \\
- & 2 & 3 & & 3 & 2 & - & 4 & 2 \\
& 1 & 4 & & 1 & 4 & & 1 & 3 \\
& & & & & & & & \\
- & 1 & 4 & & 1 & 4 & & 1 & 3 \\
& 2 & 4 & & 2 & 3 & & 3 & 2 \\
& & & & & & & & \\
& 2 & 3 & & 3 & 2 & & 4 & 2 \\
& & & & & & & & \\
+ & 1 & 3 & & 1 & 2 & + & 1 & 2 \\
& 2 & 1 & & 2 & 1 & & 3 & 1 \\
& & & & & & & & \\
\end{array}
\]
\[
= 0
\]

Lemma 2.4.9. Let \(\lambda\) be a partition of \(n\) and \(t\) a \(\lambda\)-tableau.

(a) \(\tilde{t}\) contains a unique row-increasing tableau.

(b) \(|t|\) contains a unique column-increasing tableau.
Definition 2.4.10.  
(a) A standard tableau is a row- and column-increasing tableau.  

(b) A tabloid is standard if it contains a standard tableau.  

(c) If \( t \) is a standard tableau, then \( e_t \) is called a standard polytabloid.

Remark 2.4.11.  
A standard tabloid contains a unique standard tableau.

Proof.  Let \( t \) be a standard tableau. By Lemma 2.4.9(a) \( t \) is the unique row-increasing tableau in \( \tilde{t} \), and so also the unique standard tableau in \( \tilde{t} \).

Our goal now is to show that \( S^I \) is a free \( F \)-module with basis the standard \( \lambda \)-polytabloids.

For this we introduce a total order on the tabloids

Definition 2.4.12.  
(a) Let \( \tilde{I} \) and \( \tilde{J} \) be the distinct \( \lambda \)-tabloids. Let \( i \in I_n \) be maximal with \( \text{row}_{\tilde{I}}^\lambda(i) \neq \text{row}_{\tilde{J}}^\lambda(i) \). Define \( \tilde{I} < \tilde{J} \) if \( \text{row}_{\tilde{I}}^\lambda(i) < \text{row}_{\tilde{J}}^\lambda(i) \).

(b) Let \( |I| \) and \( |J| \) be distinct column equivalence classes of \( \lambda \)-tabloids. Let \( i \in I_n \) be maximal with \( \text{col}_{\tilde{I}}^\lambda(i) \neq \text{col}_{\tilde{J}}^\lambda(i) \). Define \( \tilde{I} < \tilde{J} \) provided that \( \text{col}_{\tilde{I}}^\lambda(i) < \text{col}_{\tilde{J}}^\lambda(i) \).

Lemma 2.4.13.  \( < \) is a total ordering on the set of \( \lambda \)-tabloids.

Proof.  Any tabloid \( \tilde{t} \) is uniquely determined by the tuple \( (\text{row}_{\tilde{I}}^\lambda(i))_{i=1}^n \). Moreover the ordering is just a lexicographic ordering in terms of its associated tuple.

Lemma 2.4.14.  Let \( A \) and \( B \) be totally ordered sets and let \( f : A \to B \) be a function. Suppose that \( A \) is finite and that \( \pi \in \text{Sym}(A) \) with \( f \neq f \circ \pi \). Let \( a \in A \) be maximal such that \( f(a) \neq f(\pi a) \). If \( f \) is non-decreasing then \( f(a) \geq f(\pi a) \) and if \( f \) is non-increasing then \( f(a) < f(\pi a) \).

Proof.  The second assertion follows from the first by reversing the ordering on \( B \). So suppose that \( f \) is non-decreasing. Let \( J := \{ j \in A \mid f(j) > f(a) \} \) and let \( j \in J \). If \( j \leq a \), then also \( f(j) \leq f(a) \), since \( f \) is non-decreasing. Thus \( j > a \) and the maximal choice of \( a \) implies that \( f(\pi j) = f(j) \). As \( f(j) > f(a) \) this gives \( f(\pi j) > f(a) \). Hence \( \pi j \in J \), so \( \pi(j) \subseteq J \).

Since \( J \) is finite and \( \pi \) is 1-1 this implies \( \pi(J) = J \). As \( \pi \) is a bijection we conclude that \( \pi(I \setminus J) = I \setminus J \).

Note that \( a \notin J \). So also \( \pi a \notin J \). Thus \( f(\pi a) \leq f(a) \). By choice of \( a \) we have \( f(\pi a) \neq f(a) \), thus \( f(\pi a) < f(a) \).

Remark 2.4.15.  The above lemma is false if \( I \) is not finite (even if there exists a maximal \( a \)). Define

\[
\begin{align*}
&f : \mathbb{Z} \to \{0,1\}, \quad i \mapsto \begin{cases} 0 & \text{if } i \leq 0 \\ 1 & \text{if } i \neq 0 \end{cases} \quad \text{and} \quad \\
&\pi : \mathbb{Z} \to \mathbb{Z}, \quad i \mapsto i + 1.
\end{align*}
\]

Then \( f \) is non-decreasing and \( a = 0 \) is the unique element in \( \mathbb{Z} \) with \( f(a) \neq f(\pi a) \). But \( f(a) = f(0) = 0 < 1 = f(1) = f(\pi a) \).

Definition 2.4.16.  Let \( F \) be a commutative ring, \( G \) a group and \( V \) an \( F[G] \)-module.

(a) The dual \( V^* \) of \( V \) is the \( F[G] \)-module defined by \( V^* := \text{Hom}_F(V,F) \) as an \( F \) module and

\[
(g \phi)(v) = \phi(g^{-1}v)
\]

for all \( g \in G, \phi \in V^* \) and \( v \in V \).
(b) \( V \) is called self-dual if \( V \cong V^* \) as an \( F[G] \)-module.

**Lemma 2.4.17.** Let \( F \) be a commutative ring, \( G \) a group, \( V \) and \( F[G] \)-module and \( s : V \times V \to F \) a \( G \)-invariant, \( F \)-bilinear, symmetric form on \( V \).

(a) is readily verified.

Let \( g \) and so \( g \).

Lemma 2.4.17.

Let \( F \) be a commutative ring, \( G \) a group, \( V \) and \( F[G] \)-module and \( s : V \times V \to F \) a \( G \)-invariant, \( F \)-bilinear, symmetric form on \( V \).

(a) Let \( W \) be an \( F[G] \)-submodule of \( V \).

Then \[
\overline{s}_W : W/W \cap W^\perp \times W/W \cap W^\perp \to F, \quad (v + (W \cap W^\perp), w + (W \cap W^\perp)) \mapsto s(v, w)
\]

is a well-defined \( G \)-invariant, \( F \)-bilinear, non-degenerate, symmetric form of \( W/W \cap W^\perp \).

(b) Suppose \( s \) is non-degenerate. Then the function \[
\Phi : V \to V^*, \quad v \mapsto (w \mapsto s(v, w))
\]

is a well-defined, \( R[G] \)-monomorphism. If, in addition, \( F \) is a field and \( \dim_F V \) is finite, then \( \Phi \) is an \( R[G] \)-isomorphism and \( V \) is self-dual.

**Proof.** \( \square \) It is easy to verify that \( \Phi \) is well-defined and \( F \)-linear. Note that \( \ker \Phi = V^\perp = 0 \) and so \( \Phi \) is injective. Let \( g \in G \) and \( v, w \in V \). Then

\[(g \Phi(v))(w) = \Phi(v)(g^{-1}w) = s(g^{-1}v, w) = s(gv, w) = \Phi(gv)(w)\]

and so \( g \Phi = \Phi(gv) \). Thus \( \Phi \) is \( F[G] \)-linear.

Suppose now that \( F \) is a field and \( \dim_F V \) is finite. Then \( \dim_F V = \dim_F V^* \) and since \( \Phi \) is \( F \)-monomorphism, we conclude that \( \Phi \) is an \( F \)-isomorphism and so also an \( F[G] \)-isomorphism. \( \square \)

**Lemma 2.4.18.** Let \( V \) be a free \( R \)-module with totally ordered basis \( \mathcal{B} \) and let \( \mathcal{L} \subseteq V \setminus \{0\} \). For \( v \in V \) define \( (v_b)_{b \in \mathcal{B}} \in F_{\mathcal{B}} \) by \( v = \sum_{b \in \mathcal{B}} v_b b \). Let \( b \in \mathcal{B} \) and \( v \in V \). We say that \( b \) is involved in \( v \) with respect to \( \mathcal{B} \) if \( v_b \neq 0 \). If \( v \neq 0 \) let \( b_v \) be maximal element of \( \mathcal{B} \) involved in \( v \).

Put \( C := \{ b_l \mid l \in \mathcal{L} \} \) and \( \mathcal{D} := \mathcal{B} \setminus C \). Let \( (\cdot, \cdot) \) be the unique \( F \)-bilinear form on \( V \) with orthonormal basis \( \mathcal{B} \). Suppose that

(i) \( \mathcal{L} \) is finite.

(ii) the \( b_l, l \in \mathcal{L} \), are pairwise distinct, and

(iii) \( v_b \) is a unit in \( F \) for all \( l \in \mathcal{L} \).

Then

(a) \( \mathcal{L} \cup \mathcal{D} \) is an \( F \)-basis for \( V \). In particular, \( V = F \mathcal{L} \oplus F \mathcal{D} \).

(b) \( V = FC \oplus \mathcal{L}^\perp \), that is for each \( v \in V \) there exists a unique \( e_v \in v + FC \) such that \( e_v \in \mathcal{L}^\perp \).

(c) \( (e_d)_{d \in \mathcal{D}} \) is an \( F \)-basis for \( \mathcal{L}^\perp \).

(d) Let \( v \in F \mathcal{D} \). Then \( v = \sum_{d \in \mathcal{D}} (e_v, d) d \). In particular, \( F \mathcal{D} \cap \mathcal{L}^\perp = 0 \).

(e) \( \mathcal{L}^{\perp \perp} = F \mathcal{L} \).

(f) Suppose \( \mathcal{L} \) is finite. Then

\[
\Phi : \quad V/\mathcal{L}^\perp \to (F \mathcal{L})^*, \quad v + \mathcal{L}^\perp \mapsto (w \mapsto (v, w))
\]

is an \( F \)-isomorphism.
Proof. Replacing $l \in \mathcal{L}$ by $v^{-1}l$ we may assume that $v_b = 1$ for all $l \in \mathcal{L}$.

(a) Note that $\mathcal{D}$ is an $F$-basis for $F^{\mathcal{D}}$ and

$$F^{\mathcal{D}} = \{ v \in V \mid v_b = 0 \text{ for all } l \in \mathcal{L} \}$$

Let $0 \neq (f_l)_{l \in \mathcal{L}} \in F^{\mathcal{L}}$ and put $v := \sum_{l \in \mathcal{L}} f_l l$. We will show that $v \notin F^{\mathcal{D}}$. Note that this implies that $\mathcal{L} \cup \mathcal{D}$ is linearly independent over $F$.

Choose $k \in \mathcal{L}$ such that $b_l$ is maximal (in the totally ordered set $\mathcal{B}$) with respect to $f_k \neq 0$. Let $l \in \mathcal{L}$ with $l \neq k$ and $f_l \neq 0$. It follows that $b_k > b_l$ and so $l_b = 0$. Hence $v_{b_l} = f_k v_{b_k} = f_k \neq 0$. Hence $v \notin F^{\mathcal{D}}$ and $\mathcal{L} \cup \mathcal{D}$ is linearly independent.

Next let $m \in V$. We will show that $m \in F \mathcal{L} + F^{\mathcal{D}}$. Note that this implies that $\mathcal{L} \cup \mathcal{D}$ spans $V$ and so $\mathcal{L} \cup \mathcal{D}$ is a basis for $V$.

If $m_{b_l} = 0$ for all $l \in \mathcal{L}$, then $m \in F^{\mathcal{D}}$ and we are done. Otherwise pick $k \in \mathcal{L}$ such that $b_k$ is maximal with $m_{b_k} \neq 0$. Put $w := m - m_{b_k} k$. Then $w_{b_k} = 0$ for all $l \in \mathcal{L}$ with $b_l \geq b_k$. Induction on $b_k$ shows that $w \in F \mathcal{L} + F^{\mathcal{D}}$. So also $m \in F \mathcal{L} + F^{\mathcal{D}}$ and (a) is proved.

(b) Let $k, l \in \mathcal{L}$ with $b_k < b_l$. Then $b_l$ is not involved in $k$. Thus

$$(b_l | k) = 0.$$

Also the coefficient of $b_k$ in $k$ is equal to 1, so

$$(b_k | k) = 1.$$

We will now show that

$$FC \cap \mathcal{L}^\perp = 0.$$

For this let $m \in FC$ with $m \neq 0$. Then $m = \sum_{l \in \mathcal{L}} m_l b_l$ with $m_l \in F$. Choose $k \in \mathcal{L}$ such that $b_k$ minimal subject to $m_k \neq 0$. Let $l \in \mathcal{L}$. If $b_l < b_k$, the minimality of $b_k$ gives $m_l = 0$ and so $(m b_l | k) = 0$. If $b_l = b_k$, then $k = l$ and so $(m b_l | k) = m_k (b_l | k) = m_k$. If $b_l > b_k$, then $(b_l | k) = 0$ and so $(m b_l | k) = 0$. Thus $(m | k) = m_k \neq 0$, so $m \notin \mathcal{L}^\perp$.

Let $\mathcal{L} = \{ l_1, l_2, \ldots, l_n \}$ and $b_i := b_{l_i}$ such that $b_1 < b_2 < \ldots < b_n$. Let $0 \leq i \leq n$. We will inductively define $e_i \in V$ such that

$$e_i \in v + F\{ b_1, \ldots, b_i \} \quad \text{and} \quad e_i \in \{ l_1, \ldots, l_i \}^\perp$$

For $i = 0$ just choose $e_0 := b$. Suppose $i < n$ and that (**) holds. Define

$$e_{i+1} := e_i - (e_i | l_{i+1}) b_{i+1}.$$  

Let $1 \leq k \leq i$. Then $(b_{i+1} | l_k) = 0$ and $(e_i | l_k) = 0$, so $(e_{i+1} | l_k) = 0$. Since $(b_{i+1} | l_{i+1}) = 1$ we also have $(e_{i+1} | l_{i+1}) = 0$. Thus (**) also holds for $i + 1$.

Put $e_n := e_n$. Then $e_n \in v + FC$ and $e_n \in \mathcal{L}^\perp$. By (**), $e_n$ is unique.

(c) Since $\mathcal{B}$ is linearly independent over $F$ also $(d + FC)_{d \in \mathcal{D}}$ is linearly independent. Thus also $(e_d + FC)_{d \in \mathcal{D}}$ is linearly independent and so $(e_d)_{d \in \mathcal{D}}$ is linearly independent.

Let $m = \sum_{d \in \mathcal{D}} m_d b_d \in \mathcal{L}^\perp$. Let $d \in \mathcal{D}$. Since $e_d \in d + FC$ we have $e_d + FC = d + FC$. Also $e_d + FC = 0 + FC$ for all $c \in C$. Hence

$$m + FC = \sum_{d \in \mathcal{D}} m_d d + \sum_{c \in C} m_c c + FC = \sum_{d \in \mathcal{D}} m_d e_d + FC.$$
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and so

\[ m - \sum_{d \in D} m_de_d \in FC \cap \mathcal{L}^\perp \quad \square \]

Thus \( m = \sum_{d \in D} m_de_d \) and so \((e_d)_{d \in D}\) is an \( F \)-basis for \( \mathcal{L}^\perp \).

Let \( d \in D \) and \( v \in FD \). Note that \( FC \leq FD^\perp \). As \( v \in e_d + FC \) this gives \( (e_d \mid d) = (v \mid d) \). Since \( \mathcal{B} \) is an orthonormal basis for \( V \), \( (v \mid d) = v_d \) and so \( v = \sum_{d \in D} v_d = \sum_{d \in D} (e_d \mid d) \).

Recall that \( e_d \in \mathcal{L}^\perp \). If \( v \in \mathcal{L}^\perp \) we get \( (e_d \mid d) = 0 \) for all \( d \in D \) and so \( v = 0 \).

By (a) \( V = FL + FD \). Note that \( FL \leq \mathcal{L}^\perp \) and the modular law implies that

\[ \mathcal{L}^\perp = FL + (\mathcal{L}^\perp \cap FD) \quad \square \quad FL + 0 = FL. \]

By (2.4.17) we know that \( \Phi \) is a well-defined, \( F \)-linear and 1-1. So we just need to show that \( \Phi \) is onto.

For \( 0 \neq b_\alpha \in (FL)^* \) define \( b_\alpha := \min \{ b \mid I, \alpha(l) \neq 0 \} \) and let \( k \in L \) with \( \alpha(k) \neq 0 \) and \( b_\alpha = b_k \). Define \( \beta := \alpha - \alpha(k)\Phi(b_k) \). Let \( l \in L \).

If \( b_k < b_\alpha \), then \( \alpha(l) = 0 \) and \( \Phi(b_k)(l) = (b_k \mid l) = 0 \). So \( \beta(l) = 0 \). Also

\[ \beta(k) = (\alpha(k) - \alpha(k)(b_k \mid k) = (\alpha(k) - \alpha(k)1 = 0 \]

Thus either \( \beta = 0 \) or \( b_\beta > b_\alpha \). By downwards induction on \( b_\alpha \) we may assume that \( \beta \in \text{Im } \Phi \). Hence also \( \alpha \in \text{Im } \Phi \). Thus \( \Phi \) is onto. \( \square \)

**Lemma 2.4.19.** Let \( t \) be a \( \lambda \)-tableau and \( \pi \in \text{Sym}(n) \).

(a) Suppose that \( \text{row}_i \) is non-decreasing on each (non-trivial) orbit of \( \pi \) on \( I_\pi \). Then \( \pi \alpha \leq \pi \).

(b) Suppose that \( \text{row}_i \) is non-increasing on each (non-trivial) orbit of \( \pi \) on \( I_\pi \). Then \( \pi \alpha \geq \pi \).

**Proof.** We will prove (a) and (b) simultaneously. Without loss \( \pi \alpha \neq \pi \).

Let \( i \) be maximal in \( I_\pi \) with \( \text{row}_i(i) \neq \text{row}_\pi(i) \). Let \( X \) be the orbit of \( \pi \) on \( I_\pi \) containing \( i \). Under the hypothesis of (a), \( \text{row}_i \) is non-decreasing on \( X \), and under the hypothesis of (b), \( \text{row}_i \) is non-increasing on \( X \). Recall from (2.4.17) that \( \text{row}_\pi = \text{row}_i \circ \pi^{-1} \). Hence we can apply 2.4.14 with \( (\text{row}_i \mid X, \pi^{-1} \mid X) \) in place of \((f, \pi)\). If \( \text{row}_i \) is non-decreasing on \( X \) we get \( \text{row}_\pi(i) = \text{row}_i(\pi^{-1}i) < \pi^{-1}(i) \) and if \( \text{row}_i \) is non-decreasing on \( X \), we conclude that \( \text{row}_\pi(i) = \text{row}_i(\pi^{-1}i) > \pi^{-1}(i) \).

Thus \( \pi \alpha \leq \pi \) and \( \pi \alpha \geq \pi \), respectively. \( \square \)

**Lemma 2.4.20.** Let \( t \) be column-increasing \( \lambda \) tableau. Then \( \pi \) is the maximal tableau involved in \( e_t \).

**Proof.** Let \( \overline{x} \) be a tableau involved in \( e_t \). Then \( \overline{x} = \pi \alpha \) for some \( \pi \in C_t \). Note that each orbit of \( \pi \) on \( I_\pi \) is contained in a column of \( t \). As \( t \) is column-increasing we conclude that \( \text{row}_i \) is increasing (and so also non-decreasing) on each orbit of \( \pi \). Thus 2.4.19 shows that \( \overline{x} = \pi \alpha \leq \pi \). \( \square \)

**Theorem 2.4.21.** (a) The set standard \( \lambda \)-polytabloids is \( F \)-basis of \( S^\lambda \). In particular, \( S^\lambda \) is a free \( F \)-module of rank the equal to number of the nonstandard \( \lambda \)-polytabloids.

(b) \( S^{\lambda \perp} = S^\lambda \).

(c) \( M^\lambda /S^{\perp} \cong (S^\lambda)^* \) as an \( F[\text{Sym}(n)] \)-module.
Proof. Let \( \mathcal{A} \) be the set of standard \( \lambda \)-tableaux and let \( \mathcal{L} := \{ e_t \mid t \in \mathcal{A} \} \) be the set standard polytableaux.

Let \( a \in \mathcal{A} \). Since \( a \) is column increasing 2.4.19[a] shows that \( \overline{a} \) is the maximal tabloid involved in \( e_a \). Let \( b \in \mathcal{A} \) with \( b \neq a \). Then both \( a \) and \( b \) are row increasing. Hence \( a \) is unique row-increasing tableau in \( \overline{a} \) and \( b \notin \overline{a} \). Thus \( \overline{a} \neq \overline{b} \) Hence we can apply 2.4.18 to \((M^d, M^d, \mathcal{L})\) in place of \((V, B, \mathcal{L})\). In particular, \( \mathcal{L} \) is linearly independent, \( \mathcal{L}^{\perp \perp} = F \mathcal{L} \) and \( M^d/\mathcal{L}^{\perp \perp} \cong (S^d)^\ast \) as an \( F \)-module.

Let \( t \) be \( \lambda \)-tableau. We will show by downwards induction on \( |t| \) that \( e_t \in F \mathcal{L} \). Since \( e_t = \pm e_s \) for any \( s \in |t| \) may assume that \( t \) is column-increasing. If \( t \) is also row-increasing, then \( t \) is standard tableaux and \( e_t \in \mathcal{L} \subseteq F \mathcal{L} \). So suppose \( t \) is not row-increasing. Then there there exists \( (i, j) \in [\lambda] \) such that also \( (i, j + 1) \in [\lambda] \) and

\[ t(i, j) > t(i, j + 1). \]

Define

\[ X = \{ t(k, j) \mid i \leq k \leq j' \}, \quad Y := \{ t(k, j + 1) \mid 1 \leq k \leq j \} \quad \text{and} \quad Z := X \cup Y \]

Then

\[ |Z| = |X| + |Y| = (j' - (i - 1)) + i > j' + 1 > j' + 1 > j' + 1. \]

Hence 2.4.3 shows that

\[ 0 = G_Z e_t = \sum_{\pi \in \mathcal{R}_Z} \text{sgn}(\pi)e_{\pi t}. \]

Since row, is increasing on \( X \) and on \( Y \), \( t(i, j) \) is the minimal element of \( X \) and \( t(i, j + 1) \) is the maximal element of \( Y \). Thus \( x \geq t(i, j) > t(i, j + 1) \geq y \) for all \( x \in X \) and \( y \in Y \). Since \( \text{col}_\lambda(x) = j < j + 1 = \text{col}_\lambda(y) \) we conclude that \( \text{col}_\lambda \) is non-increasing on \( Z = X \cup Y \).

Let \( 1 \neq \pi \in \mathcal{R}_Z \). Observe that any non-trivial orbit of \( \pi \) on \( L_\pi \) is contained in \( Z \). Hence the column version of 2.4.19 shows that \( |\pi t| > |t| \). Since \( t \) is column-increasing, \( 1 \in \mathcal{R}_Z \). Since \( \mathcal{R}_Z \) is a transversal to \( \text{Sym}(Z) \cap C_\lambda \in \text{Sym}(Z) \) we have \( \pi C_\lambda \neq 1 C_\lambda \), so \( \pi \neq C_\lambda \) and \( |t| \neq |\pi t| \). Thus \( |\pi t| > |t| \). Hence, by downwards induction, \( e_{\pi t} \in F \mathcal{L} \). Thus

\[ e_t = - \sum_{1 \neq \pi \in \mathcal{R}_Z} \text{sgn}(\pi)e_{\pi t} \in F \mathcal{L}. \]

Since \( S^d \) is spanned by the \( \lambda \)-polytabloids this gives \( S^d = F \mathcal{L} \) and so \( \mathcal{L} \) is an \( F \)-basis for \( S^d \). In particular, \( S^d \) is a free \( F \)-module of rank \( |\mathcal{L}| \).

As seen above, \( \mathcal{L}^{\perp \perp} = F \mathcal{L} \) so

\[ S^d^{\perp \perp} = (F \mathcal{L})^{\perp \perp} = \mathcal{L}^{\perp \perp} = F \mathcal{L} = S^d. \]

\[ \square \]

### 2.5 \( p \)-regular partitions

**Definition 2.5.1.** Let \( \lambda \) be a partition of \( n \). Then

\[ g^d := \gcd\{(e_t, e_s) \mid t, s \lambda \text{-tableaux}\} \]

**Lemma 2.5.2.** Let \( \lambda \) be a partition of \( n \). Then \( D^d = 0 \) iff char \( F \mid g^d \).
Proof. Since $S^λ$ is spanned by the $λ$-polytabloid we have

\[ D^λ = 0 \]

\[ ⇐⇒ S^λ = S^λ \cap S^{λ⊥} \]

\[ ⇐⇒ S^λ \perp S^λ \]

\[ ⇐⇒ e_t \perp e_s \quad \text{for all $λ$-tableaux } s, t \]

\[ ⇐⇒ (e_t \mid e_s) = 0 \quad \text{for all $λ$-tableaux } s, t \]

\[ ⇐⇒ \text{char } F \mid (e_t \mid e_s) \quad \text{for all $λ$-tableaux } s, t \]

\[ ⇐⇒ \text{char } F \mid g^λ \]

Lemma 2.5.3. Recall that $\hat{λ}_j = \{|i \mid λ_i = j\}$. Then $g^λ$ divides $\prod_{j=1}^∞ (\hat{λ}_j!)$ and $\prod_{j=1}^∞ \hat{λ}_j!$ divides $g^λ$ in $\mathbb{Z}$.

Proof. Define two $λ$-tabloids $\overline{t}$ and $\overline{s}$ to be equivalent if

\[ \{Δ(t)_i \mid i \in \mathbb{Z}^+\} = \{Δ(s)_i \mid i \in \mathbb{Z}\}, \]

that is, if $\overline{t}$ and $\overline{s}$ have the rows but in possible different orders. Define

\[ Z_j := \{i \in \mathbb{Z}^+ \mid λ_i = j\} \quad \text{and} \quad Z = (Z_j)_{j=1}^∞. \]

Then $Z$ is an ordered partition of $\mathbb{Z}^+$. For $π \in \text{Sym}(\mathbb{Z}^+)$ define

\[ πZ = (π(Z_j))_{j=1}^∞ \quad \text{and} \quad \text{Sym}(Z) = \{π \in \text{Sym}(\mathbb{Z}^+) \mid πZ = Z\}. \]

Note that

\[ \text{Sym}(Z) = \bigtimes_{j=1}^∞ \text{Sym}(Z_j) \quad \text{and} \quad |\text{Sym}(Z)| = \hat{λ}! = \prod_{j=1}^∞ \hat{λ}_j! \]

Observe that $\overline{t}$ and $\overline{s}$ are equivalent if and only if there exists $π = π(\overline{t}, \overline{s}) \in \text{Sym}(\mathbb{Z}^+)$ with $Δ(t)_{πi} = Δ(s)_i$ for all $i \in \mathbb{Z}^+$. Then

\[ λ_{πi} = |Δ_{πi}| = |Δ_i(s)| = λ_i. \]

Thus $πZ = Z$ and $π \in \text{Sym}(Z)$. Conversely if $π \in \text{Sym}(Z)$ then there exists a unique tabloid $\overline{t}$ with $Δ_i(s) = Δ_{πi}(t)$ and $\overline{t}$ is equivalent to $\overline{s}$.

Hence

1°. Each equivalence class of $λ$-tabloids contains $\hat{λ}!$-tabloids.

Let $M$ be the set of $λ$-tabloids. For a $λ$-tabloid $\overline{t}$ and a $λ$-tableau $t$ let $e_t(\overline{t})$ be the coefficient of $\overline{t}$ in $e_t$. So

\[ e_t = \sum_{r \in M} e_t(\overline{r})\overline{r}. \]

Next we show:
2°. Let \( \overline{t} \) and \( \overline{s} \) be equivalent \( \lambda \)-tabloids. Then there exists \( \delta(\overline{t}, \overline{s}) \in \{\pm 1\} \) such that

\[
e_i(\overline{s}) = \delta(\overline{t}, \overline{s})e_i(\overline{t}).
\]

for any \( \lambda \)-tableaux \( t \).

Let \( \pi = \pi(\overline{t}, \overline{s}) \). Let \( \pi_j \) be the restriction of \( \pi \) to \( Z_j \) and define

\[
\delta := \prod_{j \in J} \text{sgn}(\pi_j)^i.
\]

If neither \( \overline{t} \) nor \( \overline{s} \) is involved in \( t \), then both sides in (2) are zero. So we may assume that \( \overline{t} \) is involved in \( e_i \). Then \( \overline{t} = \overline{\rho} \) for some \( \rho \in C_i \). Put \( r = \rho t \). Define \( \pi^* \in \text{Sym}(n) \) by

\[
\pi^*(r(i, j)) = r(\pi(i), j).
\]

for all \( (i, j) \in [\lambda] \). Since \( \lambda_i = \lambda_{ni} \) for all \( i \in \mathbb{Z}^+ \), this is well defined. Then

\[
\text{sgn}(\pi^*) = \delta, \quad \pi^* \in C_i, \quad \pi^* \rho \in C_i, \quad \text{and} \quad \overline{s} = \overline{\pi^* \rho} = \overline{\pi^* \overline{r}}
\]

Hence the coefficient of \( \overline{t} \) in \( e_i \) is \( \text{sgn}(\rho) \) and the coefficient of \( \overline{s} \) in \( e_i \) is \( \text{sgn}(\pi^* \rho) = \text{sgn}(\pi^*) \text{sgn}(\rho) = \delta \text{sgn}(\rho) \).

3°. \( \lambda! \) divides \( g^1 \).

Let \( t, u \) be \( \lambda \)-tableaux. We need to show that \( \lambda! \) divides \( (e_i | e_u) \). Let \( \mathcal{A} \) be the set of equivalence classes of \( \lambda \)-tabloids and for \( A \in \mathcal{A} \) pick \( \overline{\lambda} \in A \).

\[
(e_i | e_u) = \left( \sum_{\overline{t} \in \mathcal{M}} e_i(\overline{t}) \left| \sum_{\overline{t} \in \mathcal{M}} e_u(\overline{t}) \overline{t} \right) \right).
\]

Let \( \overline{\lambda} \in A \) be the set of equivalence classes of \( \lambda \)-tabloids and for \( A \in \mathcal{A} \) pick \( \overline{\lambda} \in A \).

\[
(e_i | e_u) = \sum_{\overline{t} \in \mathcal{M}} e_i(\overline{t}) e_u(\overline{t})
\]

= \sum_{A \in \mathcal{A}, A \in A} |A| e_i(\overline{\lambda}) e_u(\overline{\lambda})

Thus (3) holds.

Fix a \( \lambda \)-tableau \( t \) Define \( \sigma \in \text{Sym}(n) \) by \( \sigma(t(i, j)) = t(i, \lambda_i + 1 - j) \) and put \( \overline{t} = \sigma t \). So \( \overline{t} \) is the tableaux obtained by reversing the rows of \( t \). We will show that \( (e_i | e_t) = \prod_{i=1}^{\lambda} (\lambda_i)! \). Since \( g^1 \) divides \( (e_i | e_t) \) this will complete the proof of the lemma.
Put

\[ U_i := U_i(t) := \bigcup_{k \in \mathbb{Z}_n} \Delta(t) \]

So \( U_i \) is the union of the rows of \( t \) of size \( i \). Note that \( U_i = U_i(\tilde{t}) \). Also put

\[ U_i^j := U_i^j(t) := U_i(t) \cap \Delta^j(t) \quad \text{and} \quad U := (U_i^j)_{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+} \]

Then \( U \) is a partition of \( I_n \) refining both \( (U_i^j)_{i=1}^n \) and column partition \( \Delta^j(t) \). In particular, \( \text{Sym}(U) \leq C_t \).

Observe that

\[ U_i = U_i(\tilde{t}) \quad \text{and} \quad U_i^j := U_i^j(\tilde{t}) = U_i^{i+1-j} = \sigma(U_i^j). \]

In particular, \( U \) is also a partition of \( I_n \) refining the columns partition \( \Delta^j(\tilde{t}) \). Let \( m = \lambda_1 \). Then

\[
\begin{array}{cccccccc}
U_m^1 & U_m^2 & \cdots & U_m^j & \cdots & \cdots & U_m^{m-1} & U_m^m \\
U_{m-1}^1 & U_{m-1}^2 & \cdots & U_{m-1}^j & \cdots & \cdots & U_{m-1}^{m-1} & U_{m-1}^m \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
U_{i+1}^1 & U_{i+1}^2 & \cdots & U_{i+1}^j & \cdots & \cdots & U_{i+1}^{m-1} & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
U_i^1 & U_i^2 & \cdots & U_i^j & \cdots & \cdots & U_i^{m-1} & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
U_1^1 & U_1^2 & \cdots & U_1^j & \cdots & \cdots & U_1^{m-1} & \vdots \\
\end{array}
\]

and

\[
\begin{array}{cccccccc}
U_m^1 & U_m^2 & \cdots & U_m^j & \cdots & \cdots & U_m^{m-1} & U_m^m \\
U_{m-1}^1 & U_{m-1}^2 & \cdots & U_{m-1}^j & \cdots & \cdots & U_{m-1}^{m-1} & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
U_{i+1}^1 & U_{i+1}^2 & \cdots & U_{i+1}^j & \cdots & \cdots & U_{i+1}^{m-1} & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
U_i^1 & U_i^2 & \cdots & U_i^j & \cdots & \cdots & U_i^{m-1} & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots & \vdots \\
U_1^1 & U_1^2 & \cdots & U_1^j & \cdots & \cdots & U_1^{m-1} & \vdots \\
\end{array}
\]
Since $U$ refines the columns of $t$ and of $\bar{t}$ we have
\[
\text{Sym}(U) \subseteq C_t \quad \text{and} \quad \text{Sym}(U) \subseteq C_{\bar{t}}
\]
Note that $|U_i^j(t)| = \tilde{\lambda}_i$ if $j \leq i$ and $U_i^j(t) = \emptyset$ otherwise. Thus

4°. $|\text{Sym}(U)| = \prod_{(i,j) \in \mathbb{Z}^+ \times \mathbb{Z}^+} |U_i^j(t)|! = \prod_{i=1}^{\infty} (\tilde{\lambda}_i)!$.

Next we show:

5°. Let $\pi \in \text{Sym}(U)$. Then $\epsilon_i(\pi) = \epsilon_j(\pi) = \text{sgn}(\pi)$.

Since $\pi \in C_t$ we have $\epsilon_i(\pi) = \text{sgn}(\pi)$.

Since $\pi \in C_{\bar{t}}$ we have $\epsilon_j(\pi) = \text{sgn}(\pi)$.

Since $\sigma$ fixes the rows of $t$, $\pi \sigma \pi^{-1}$ fixes the rows of $\pi t$. Thus
\[
\pi \sigma \pi^{-1} \pi t = \pi \sigma \pi^{-1} = \pi \bar{t}
\]
and so (5°) holds.

6°. Let $\pi \in C_t$ such that $\pi t$ is involved in $e_l$. Then $\pi \in \text{Sym}(U)$.

Since $\pi t$ is involved in $e_l$ there exists $\hat{\pi} \in C_t$ with $\pi t = \pi \hat{\pi} t$. Hence row$_{\pi t} =$ row$_{\pi \hat{\pi} t}$. Since $t$ and $\bar{t}$ are row-equivalent, row$_i =$ row$_{\bar{t} i}$. So
\[
\text{row}_i \circ \pi^{-1} = \text{row}_{\pi \hat{\pi} t} = \text{row}_{\pi \hat{\bar{\pi}} t} = \text{row}_i \circ \bar{\pi}^{-1}
\]
Put $\alpha := \pi^{-1}$ and $\bar{\alpha} := \pi^{*^{-1}}$.

(\*) $\alpha \in C_t, \quad \bar{\alpha} \in C_{\bar{t}}$ and row$_i \circ \alpha =$ row$_i \circ \bar{\alpha}$

We will show that $\alpha(U_i^j) = U_i^j = \bar{\alpha}(U_i^j)$ for all $i, j$. The proof uses double induction. First on $j$ and then downwards on $i$. So let $i, j \in \mathbb{Z}^+$ and assume inductively that

(\*) $\alpha(U_i^j) = U_i^j = \bar{\alpha}(U_i^j)$

whenever $l < j$ or $l = j$ and $k > i$. Observe that
\[
\Delta'(t)_j = \bigcup_{k=1}^{\infty} U_k^j
\]
Since $\alpha \in C_t$, we have $\alpha(\Delta'(t)_j) = \Delta'(t)_j$. Since $\alpha(U_{k_j}) = \alpha(U_{k_j})$ for all $k > i$ this implies $\alpha(\bigcup_{k=1}^{i} U_i^j) = \bigcup_{k=1}^{i} U_k^j$. In particular,

(\*) $\alpha(U_i^j) \subseteq \bigcup_{k=1}^{i} U_k^j \subseteq \bigcup_{k=1}^{i} U_k$.

Put $\tilde{j} := i + 1 - j$. Then $\tilde{U}_i^j = U_i^j$ and so $U_i^j$ is contained in column $\tilde{j}$ of $\bar{t}$. We have
Recall that $\tilde{U}_k^j = U_k^{i+1-j}$. If $k < i$, then $k + 1 - j < i + 1 - (i + 1) - j = j$ and the induction assumption implies that $\alpha$ fixes $U_k^{i+1-j}$. So $\alpha \left( \tilde{U}_k^j \right) = \tilde{U}_k^j$ for all $k < i$. As $\tilde{\alpha} \in C_i$ we know that $\tilde{\alpha}$ fixes $\Delta'(t)_j$. It follows that $\tilde{\alpha} \left( \bigcup_{k=i}^{\infty} \tilde{U}_k^j \right) = \bigcup_{k=i}^{\infty} \tilde{U}_k^j$.

In particular,

$$\tilde{\alpha} \left( U_i^j \right) = \tilde{\alpha} \left( \tilde{U}_i^j \right) \subseteq \bigcup_{k=i}^{\infty} \tilde{U}_k^j \subseteq U_k$$

As row$_i \circ \alpha = \text{row}_i \circ \tilde{\alpha}$ this implies

$$(+) \quad \alpha(U_i^j) \subseteq \bigcup_{k=i}^{\infty} U_k.$$

From $**$ and $(+)$ we that $\alpha(U_i^j) \subseteq U_i$. As $\text{row}_i \circ \alpha = \text{row}_i \circ \tilde{\alpha}$ this gives $\tilde{\alpha}(U_i^j) \subseteq U_i$. Since $\alpha \in C_i$ and $\tilde{\alpha} \in C_i$ this implies $\alpha(U_i^j) = U_i^j$ and $\tilde{\alpha}(U_i^j) = U_i^j$.

We proved that $\alpha(U_i^j) = U_i^j$ for all $i, j \in \mathbb{Z}^+$. So $\alpha \in \text{Sym}(U)$ and thus also $\pi = \alpha^{-1} \in \text{Sym}(U)$. This completes the proof of (6).

Let $\mathcal{N}$ be the set of $\lambda$-tabloids involved in $e_i$ and $e_i$. Then

$$\langle e_i | e_i \rangle = \sum_{T \in \mathcal{N}} e_i(T)e_i(T)$$

The $\lambda$-tabloids involved in $e_i$ are $\overline{T}, \pi \in C_i$. By (6), $\overline{T}$ is involved in $e_i$ if and only if $\pi \in \text{Sym}(U)$, so $\mathcal{N} = \{ \overline{T} | \pi \in \text{Sym}(U) \}$. Thus

$$\langle e_i | e_i \rangle = \sum_{\pi \in \text{Sym}(U)} e_i(\overline{T})e_i(\overline{T}) \otimes \sum_{\pi \in \text{Sym}(U)} \sgn(\pi)\sgn(\pi) = \sum_{\pi \in \text{Sym}(U)} 1 = |\text{Sym}(U)| \prod_{i=1}^{\infty} (\hat{\lambda}_i!)^i.$$

As $g^d$ divides $\langle e_i | e_i \rangle$ this shows that $g^d$ divides $\prod_{i=1}^{\infty} (\hat{\lambda}_i!)^i$. \hfill $\square$

**Proposition 2.5.4.** Suppose $F$ is an integral domain and put $p := \text{char} F$. Then $D^4 \neq 0$ iff $\lambda$ is p-regular.

**Proof.** By [2.5.2]

$$D^4 = 0 \iff p|g^4$$

By [2.5.3] we know that

$$\prod_{i=1}^{\infty} \hat{\lambda}_i! | g^4 \text{ and } g^4 | \prod_{i=1}^{\infty} (\hat{\lambda}_i!)^i.$$ 

Since $F$ is an integral domain, $p = 0$ or $p$ is a prime. It follows that
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\[ p|g^\lambda \iff p|\lambda_i \text{ for some } i \in \mathbb{Z}^+ \]

Observe that

\[ p|\lambda_i! \iff p \neq 0 \text{ and } p \leq \lambda_i \]

Thus

\[ D^i = 0 \iff p \neq 0 \text{ and } p \leq \lambda_i \text{ for some } i \in \mathbb{Z}^+. \]

Hence \( D^i = 0 \) if and only if \( p \neq 0 \) and the multiplicative partition \( \lambda \) is \( p \)-singular. Thus \( D^i \neq 0 \) if and only if \( \lambda \) is \( p \)-regular. By 2.1.5 \( \lambda \) is \( p \)-regular if and only if \( \lambda \) is \( p \)-regular. □

**Theorem 2.5.5.** Let \( F \) be a field and \( n \) a positive integer. Put \( p := \text{char } F \).

(a) Let \( \lambda \) be a \( p \)-regular partition of \( n \). Then \( D^\lambda \) is an absolutely simple, self-dual \( F \text{Sym}_p n \)-module.

(b) Let \( I \) be a simple \( F \text{Sym}_p n \)-module. Then there exists a unique \( p \)-regular partition \( \lambda \) of \( n \) with \( I \cong D^\lambda \).

**Proof.**

(a) By 2.5.4 \( D^\lambda \neq 0 \). As \( D^\lambda = S^\lambda/(S^\lambda \cap S^\lambda) \), 2.4.17(a) shows that \( s \) induces a symmetric, non-degenerate \( G \)-invariant \( F \)-bilinear form on \( D^\lambda \). Hence 2.4.17(b) implies that \( D^\lambda \) is self-dual. By 2.3.27 \( D^\lambda \) is absolutely simple.

(b) If \( \lambda \) and \( \mu \) are distinct \( p \)-regular partitions then by 2.3.25 and (a), \( D^\lambda \) and \( D^\mu \) are non-isomorphic simple \( FSym(n) \)-modules. Let \( \bar{F} \) be an algebraic closure of \( F \). By 1.8.3(d) the number of isomorphism classes of simple \( FSym(n) \)-modules is less or equal to the number isomorphism classes of simple simple \( \bar{F} \text{Sym}_p n \)-modules. The latter number is by 1.10.8 equal to the number of \( p \)-conjugacy classes and so by 2.1.6 equal to the number of \( p \)-regular partitions of \( n \). So (b) holds. □

### 2.6 Series of \( R \)-modules

**Definition 2.6.1.** Let \( R \) be a ring and \( M \) and \( R \)-module. Let \( S \) be a set of \( R \)-submodules of \( M \). Then \( S \) is called an \( R \)-series on \( M \) provided that:

(i) \( 0 \in S \) and \( M \in S \).

(ii) \( S \) is totally ordered with respect to inclusion.

(iii) For all \( \emptyset \neq T \subseteq S \), \( \bigcap T \in S \) and \( \bigcup T \in S \).

**Definition 2.6.2.** Let \( R \) be a ring, \( M \) an \( R \)-module and \( S \) an \( R \)-series on \( M \).

(a) For \( U \subseteq M \) define

\[ U^- := \bigcup\{D \in S \mid D \subseteq U\} \quad \text{and} \quad U^+ := \bigcap\{D \in S \mid U \subseteq D\}. \]

(b) If \( A \in S \) with \( A \neq A^- \), then \( (A^-, A) \) is called a jump of \( S \) and \( A/A^- \) a factor of \( S \).

(c) \( S \) is called a composition series for \( R \) on \( S \) provided that all factors of \( S \) are simple \( R \)-modules.
Example 2.6.3. Let $(p_i)_{i=1}^{\infty}$ be any sequence of prime integers. Then
\[ \mathbb{Z} > p_1 \mathbb{Z} > p_1 p_2 \mathbb{Z} > \ldots > p_1 p_2 \ldots p_{i-1} \mathbb{Z} > p_1 p_2 \ldots p_i \mathbb{Z} > \ldots > 0 \]
is composition series for $\mathbb{Z}$ on $\mathbb{Z}$ with factors isomorphic to $\mathbb{Z}/p_i \mathbb{Z}$, $i \in \mathbb{Z}^+$.
If $p_i = p$ for a fixed prime $p$, then all factors are isomorphic to $\mathbb{Z}/p \mathbb{Z}$.
If $p_i$ is the $i$-th positive prime, then all $\mathbb{Z}/p_i \mathbb{Z}$, $p$ a prime, occur as a composition factors.

Lemma 2.6.4. Let $R$ be a ring, $M$ an $R$-module, $S$ an $R$-series on $M$.

(a) Let $A, B \in S$ with $B \subseteq A$. Then $(B, A)$ is a jump iff $A = C$ or $B = C$ for all $C \in S$ with $B \subseteq C \subseteq A$.

(b) Let $U \subseteq M$. Then there exists a unique $A \in \mathcal{U}$ minimal with $U \subseteq A$. If $U$ is finite and contains a non-zero element then $A^- \neq A$ and $A \cup U \subseteq A^-$.

(c) Let $0 \neq m \in M$. Then there exists a unique jump $(B, A)$ if $S$ with $v \in A$ and $v \notin B$.

Proof. (a) Suppose first that $(B, A)$ is a jump. Then $B = A^-$. Let $C \in S$ with $B \subseteq C \subseteq A$ Suppose $C \subseteq A$. Then $C \subseteq A^- = B$ and $C = B$.

Suppose next that $A = C$ or $B = C$ for all $C \in S$ with $B \subseteq C \subseteq A$. Since $B \subseteq A$, $B \subseteq A^-$. Let $C \in S$ with $C \subseteq A$. Since $S$ is totally ordered, $C \subseteq B$ or $B \subseteq C$. In the latter case, $B \subseteq C \subseteq A$ and so by assumption $B = C$. So in any case $C \subseteq B$ and thus $A^- \subseteq B$. We conclude that $B = A^-$ and so $(B, A)$ is a jump.

(b) Put $A = \bigcup \{ S \in S \mid U \subseteq S \}$. By $A \in S$ and so clearly is minimal with respect to $U \subseteq A$ and is unique with respect to this property. Suppose now that $U$ is finite and contains a non-zero element. Then $A \neq 0$. Suppose that $A = A^-$. Then for each $u \in U$ we can choose $B_u \in S$ with $u \in B_u$ and $B_u \subseteq A$. Since $U$ is finite \{B_u, u \in U\} has a maximal element $B$. Then $U \subseteq B \subseteq A$, contradicting the minimality of $A$.

Thus $A \neq A^-$ and by minimality of $A$, $U \subseteq A$.

(c) Follows from (b) applied to $U = \{ m \}$.

Lemma 2.6.5. Let $R$ be a ring, $M$ a free $R$-module with basis $\mathcal{B}$ and $S$ be an $R$-series on $M$. Then the following four statements are equivalent. One of the following holds:

(a) For each $A \in S$, $A \cap \mathcal{B}$ spans $A$ over $R$.

(b) For each $B \in S$, $(a + B \mid a \in \mathcal{B} \setminus B)$ is $R$-linear independent in $V/B$. Then

(c) For each jump $(B, A)$ of $S$, $(a + B \mid a \in \mathcal{B} \cap A \setminus B)$ is $R$-linear independent in $A/B$.

(d) For all $A, B \in S$ with $B \subseteq A$, $(a + B \mid a \in \mathcal{B} \cap A \setminus B)$ is an $R$-basis for $A/B$.

Proof. (a) $\Rightarrow$ (b): $(r_a) \in \bigoplus_{a \in \mathcal{B} \setminus A} R$ with $\sum_{a \in \mathcal{B} \cap A} r_a a \in B$. Then by (a) applied to $B$ there exists $(r_a) \in \bigoplus_{a \in \mathcal{B} \cap A}$ with $\sum_{a \in \mathcal{B} \cap A} r_a a = \sum_{a \in \mathcal{B} \cap A} r_a a$.

Since $\mathcal{B}$ is linearly independent over $R$ this implies $r_a = 0$ for all $a \in \mathcal{B}$ and so (b) holds.

(b) $\Rightarrow$ (c): Obvious.

(c) $\Rightarrow$ (a): Let $a \in A$. Since $\mathcal{B}$ spans $M$ over $R$ there exists a finite subset $C$ of $\mathcal{B}$ and $(r_c) \in \bigoplus_{c \in C} R^d$ with $a = \sum_{c \in C} r_c c$. Let $D \in S$ be minimal with $C \subseteq D$. Then $(D^-, D)$ is a jump and $C \setminus D^- \neq \emptyset$. Suppose that $D \subseteq A$. Since $S$ is totally ordered, $A \subseteq D^-$. Thus

$$0_{D/D^-} = a + D^- = \sum_{c \in C} r_c c + D^- = \sum_{c \in C \setminus D^-} r_c c + D^-$$
a contradiction to \( (c) \).

\[ a \implies (d): \] (a) implies that \( \{ a + B \mid a \in \mathcal{A} \} \) and so also \( \{ a + B \mid a \in \mathcal{A} \} \) spans \( A/B \). Since (a) implies (b), \( \{ a + B \mid a \in \mathcal{B} \} \) and so also \( \{ a + B \mid a \in \mathcal{B} \cap A \} \) is \( R \)-linear independent. So (d) holds.

\[ d \implies (a): \] Just apply (d) with \( B = 0 \). □

2.7 The Branching Theorem

**Definition 2.7.1.** Let \( \lambda \) be partition of \( n \)

(a) A node \( d \in [\lambda] \) is called removable if \([\lambda] \setminus \{d\}\) is a Ferrers diagram.

(b) \( d_i = (r_i, c_i), 1 \leq i \leq k \) are the the removable nodes of \([\lambda]\) ordered such that \( r_1 < r_2 < \ldots < r_k \).

(c) \( \lambda^{(i)} := \lambda ([\lambda] \setminus \{d_i\}) \) and \( \lambda \downarrow := \{ \lambda^{(i)} \mid 1 \leq i \leq k \} \)

(d) \( e \in \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) is called an exterior node of \([\lambda]\) if \([\lambda] \cup \{e\}\) is a Ferrers diagram. \( \lambda \uparrow \) is the set of partitions of \( n + 1 \) obtained by extending \([\lambda]\) by an exterior node.

**Lemma 2.7.2.** Let \( \lambda \) be a partition of \( n \) and \( (i, j) \in D \). Then the following are equivalent

(a) \( (i, j) \) is a removable node of \([\lambda]\).

(b) \( j = \lambda_i \) and \( \lambda_i > \lambda_{i+1} \).

(c) \( i = \lambda'_j \) and \( \lambda'_j > \lambda'_{j+1} \).

(d) \( j = \lambda_i \) and \( i = \lambda'_{j} \).

**Proof.** Obvious. □

**Definition 2.7.3.** Let \( t \) be a \( \lambda \)-tableau.

(a) We say that \( t \) is restrictable if \( t^{-1}(n) \) is a removable node of \([\lambda]\). If this is the case, we define

\[ t \downarrow := t|_{t^{-1}(\lambda_{n-1})}. \]

(b) \( \tilde{t} \) is called restrictable if \( \tilde{t} \) contains a restrictable tableau \( s \). In this case we define \( \tilde{t} \downarrow := s \downarrow \). Observe that this is well-defined.

**Lemma 2.7.4.** Let \( t \) be a \( \lambda \)-tableau and \( \pi \in \text{Sym}(n - 1) \)

(a) If \( t \) is restrictable then \( t \downarrow \) is a tableau.

(b) If \( t \) is standard, then \( t \) is restrictable and \( t \downarrow \) is standard.

(c) \( t \) is restrictable if \( \pi t \) is restrictable. If this is this case, then \( (\pi t) \downarrow = \pi(t \downarrow) \).

(d) \( \tilde{t} \) is restrictable if \( \tilde{\pi} \tilde{t} \) is restrictable. If this is this case, then \( (\tilde{\pi} \tilde{t}) \downarrow = \pi(\tilde{t} \downarrow) \).

**Proof.** Obvious. □
2.7. THE BRANCHING THEOREM

**Theorem 2.7.5.** Let $1 \leq i \leq k$. For a $\lambda$-tableau $t$ put $R_i(t) := \Delta(t)_{ri}$. Let $W_i$ be the $F$-submodule of $S^d$ spanned by all $e_i$ where $t$ is a restrictable $\lambda$-tableau with $n \in R_i(t)$. Define $V_0 := 0$ and inductively $V_i = V_{i-1} + W_i$. Then
\[ 0 = V_0 < V_1 \ldots < V_{k-1} < V_k = S^d \]
as a series of $F[\text{Sym}(n-1)]$-submodules of $S^d$ and
\[ V_i/V_{i-1} \cong S^{d(i)} \]
as an $F[\text{Sym}(n-1)]$-module.

**Proof.** Clearly the set of restrictable $\lambda$ tableaux $t$ with $n \in R_i(t)$ is invariant under the action of $\text{Sym}(n-1)$. Thus each $V_i$ is an $F[\text{Sym}(n-1)]$ submodule of $S^d$. Also $V_{i-1} \subseteq V_i$ and it remains to show that $V_i/V_{i-1} \cong S^{d(i)}$. For this define an $F$-linear function

\[ \theta_i : M^d \rightarrow M^{d(i)}, \quad \tilde{t} \rightarrow \begin{cases} \tilde{t} \downarrow & \text{if } n \text{ is in row } R_i(t) \\ 0 & \text{otherwise} \end{cases} \]

Observe that $\theta_i$ commutes with the action of $\text{Sym}(n-1)$ and so $\theta_i$ is $F[\text{Sym}(n-1)]$ linear. Let $1 \leq j \leq i \leq k$. Let $\lambda$ be a restrictable tableau with $n \in R_j(t)$ and $\pi \in C_j$. Then $\pi(n) \in \Delta(t)_{wi}$ for some $u \leq j$ with $u = j$ if and only if $\pi(n) = n$. If follows that $\pi(n) \in R_i(t)$ if and only if $\pi \in \text{Sym}(n-1)$ and $i = j$ and so if and only if $i = j$ and $\pi \in C_i$. Thus

\[ \theta_i(e_s) = \begin{cases} e_\tilde{s}_{\tilde{i}} & \text{if } j = i \\ 0 & \text{if } j < i \end{cases} \]

In particular, $\theta_i(V_{i-1}) = 0$ and $\theta_i(V_i) \subseteq S^{d(i)}$.

If $s$ is a $\lambda^{(i)}$-tableau, then $s = t \downarrow$ for a (unique) restrictable $\lambda$ tableau $t$ with $n \in R_i(t)$. Then $\theta_i(e_s) = e_s$ and so $\theta_i(V_i) = S^{d(i)}$.

Hence

\[ V_{i-1} \subseteq V_i \cap \ker \theta_i \quad \text{and} \quad V_i/V_i \cap \ker \theta_i \cong \text{Im} \theta_i = S^{d(i)} \]

Let $B$ be the set of standard $\lambda$-polytabloids and $B_i$ the $e_i$ with $t$ standard and $n$ in row $r_i$. Then by (1) $\theta_i(B_i)$ is the standard basis for $S^{d(i)}$ and so is linear independently. Thus also the image of $B_i$ in $V_i/V_i \ker \theta_i$ is linearly independent. Consider the series of $F$-modules

\[ 0 = V_0 \subseteq V_1 \cap \ker \theta_1 \subseteq V_1 \subseteq V_2 \cap \ker \theta_2 \subseteq V_2 \subseteq \ldots \subseteq V_{k-1} \subseteq V_k \cap \ker \theta_k \subseteq V_k \subset S^d \]

Each $e_i \in B_i$ lies in a unique $B_i$ and so in $V_i/(V_i \cap \ker \pi_i)$. Thus $B \cap V_i \cap \ker \theta_i \subseteq V_{i-1}$. So we can apply 2.6.5 to the series of $F$-modules and conclude that $V_i \cap \ker \theta_i/V_{i-1}$ is as the emptyset as an $R$-basis. Hence $V_{i-1} = V_i \cap \ker \theta_i$. For the same reason $V_k = S^d$ and theorem now follows from (3).

**Theorem 2.7.6** (Branching Theorem). Let $F$ be a field and $\lambda$ a partition of $n$. Put $p := \text{char } F$.

(a) Suppose that $p = 0$ or $p \geq n$. Then
\[ S^d \downarrow_{\text{Sym}(n-1)} = \bigoplus_{\mu \in \Delta} S^\mu \]
(b) Suppose that \( p = 0 \). Then

\[
S^d \uparrow \text{Sym}(n+1) = \bigoplus_{\mu \in \mathcal{P}} S^\mu
\]

**Proof.** By 2.7.5 there exists a series of \( F[\text{Sym}(n-1)] \)-submodules

\[
0 = V_0 < V_1 \ldots < V_{k-1} < V_k = S^d
\]
of \( S^d \) with

\[
V_i / V_{i-1} \cong S^{d(i)}
\]
as an \( F[\text{Sym}(n-1)] \)-module.

Suppose that \( \text{char } F = 0 \) or \( \text{char } F \geq n \). Then \( \text{char } F \not| (n-1)! = |\text{Sym}(n-1)| \) and so Maschke’s Theorem 1.3.1 implies for each \( 1 \leq i \leq k \) there exists an \( F[\text{Sym}(n-1)] \)-submodule \( U_i \) of \( V_i \) with \( V_i = V_{i-1} \oplus U_i \). Then

\[
S^d = \bigoplus_{i=1}^k U_i \cong \bigoplus_{i=1}^k V_i / V_{i-1} \cong \bigoplus_{\mu \in \mathcal{P}} S^{d(i)} = \bigoplus_{\mu \in \mathcal{P}} S^\mu.
\]

Put \( p := \text{char } F \) and let \( \mathcal{P} \) be the set of of \( n+1 \). Then by 2.5.5 \( D\mu, \mu \in \mathcal{P} \) is a set of representatives for the isomorphism classes of simple \( F[\text{Sym}(n)] \)-modules. By 2.3.23 we know that \( D\mu \cong S^\mu \) as an \( F[\text{Sym}(n)] \)-module. Also by Maschke’s Theorem 1.3.1 \( S^d \uparrow \text{Sym}(n+1) \) is a semisimple \( F[\text{Sym}(n+1)] \)-module. Hence

\[
(*) S^d \uparrow \text{Sym}(n+1) \cong \bigoplus_{\mu \in \mathcal{P}} (S^\mu)^{n_\mu}
\]

for some \( n_\mu \in \mathbb{N} \).

Let \( \eta, \mu \in \mathcal{P} \). By 2.3.27 \( D\eta \) is an absolutely simple \( F[\text{Sym}(n+1)] \)-module over \( F \) and so 1.8.6 shows that \( \text{End}_{F[\text{Sym}(n+1)]}(D\eta) = F[D\eta] \). As \( S^{\eta} \cong D\eta \) we conclude that

\[
(**) \dim_F \text{Hom}_{F[\text{Sym}(n+1)]}(S^\mu, S^{\eta}) = \begin{cases} 1 & \text{if } \eta = \mu \\ 0 & \text{if } \eta \neq \mu \end{cases}
\]

and \( (*) \) implies

\[
\dim_F \text{Hom}_{F[\text{Sym}(n+1)]}(S^d \uparrow \text{Sym}(n+1), S^{\eta}) = n_\eta.
\]

By Frobenius’ Reciprocity 1.7.4 we have

\[
\text{Hom}_{F[\text{Sym}(n+1)]}(S^d \uparrow \text{Sym}(n+1), S^{\eta}) \cong \text{Hom}_{F[\text{Sym}(n)]}(S^d, S^{\eta} \downarrow \text{Sym}(n)).
\]

and so

\[
(* *) \dim_F \text{Hom}_{F[\text{Sym}(n)]}(S^d, S^{\eta} \downarrow \text{Sym}(n)) = n_\eta.
\]

By \( (*) \) applied to \( \text{Sym}(n+1) \) in place of \( \text{Sym}(n) \) we get

\[
S^{\eta} \downarrow \text{Sym}(n) \cong \bigoplus_{\mu \in \mathcal{P}} S^\mu.
\]

Hence \( (**) \) applied to \( \text{Sym}(n) \) gives
2.8. The dual of a Specht module

**Definition 2.8.1.** Let $R$ be a ring, $G$ a group, $M$ an $R[G]$-module and $\epsilon: G \rightarrow \mathbb{Z}/(R)\mathbb{Z}$ a multiplicative homomorphism. Then $M_\epsilon$ is the $R[G]$-module with $M_\epsilon = M$ as an $R$-module and

$$g \cdot_\epsilon m = \epsilon(g)m$$

for all $g \in G, m \in M$.

**Proposition 2.8.2.**

$$(S^\lambda)^* \cong M^\lambda / S^\lambda_{\epsilon} \cong S^\lambda_{\epsilon}$$

as an $F[\text{Sym}(n)]$-module.

**Proof.** By [2.4.21](#) know that $(S^\lambda)^* \cong M^\lambda / S^\lambda\perp$ as an $F[\text{Sym}(n)]$-module. So we only need to prove the second statement.

Fix a $\lambda$ tableau $s$. Let $\pi \in R_s$. Note that $R_s = C_{\pi'}$, so $\pi \in C_{\pi'}$ and [2.3.13](#) shows that $\pi e_{\pi'} = \text{sgn}(\pi)e_{\pi'}$. Thus

$$\pi \cdot_{\text{sgn}} e_{\pi'} = \text{sgn}(\pi)\pi e_{\pi'} = \text{sgn}(\pi)\text{sgn}(\pi)e_{\pi'} = e_{\pi'}$$

Hence there exists a unique $F[\text{Sym}(n)]$-function

$$(*) \quad \alpha_s : M^\lambda \rightarrow M^\lambda_{\epsilon} \quad \text{with} \quad 3 \rightarrow e_{\pi'}$$

Let $t$ be any $\lambda$-tabloid. Then the exists $\pi \in \text{Sym}(n)$ with $\pi s = t$ (namely $\pi = ts^{-1}$) and so

$$\alpha_s(t) = \alpha_s(\pi t) = \alpha_s(\pi s) = \pi \cdot_{\text{sgn}} e_{\pi'} = \text{sgn}(\pi)\pi e_{\pi'} = e_{\pi'}$$

Comparing with $(*)$ yields

$$n_\eta = \begin{cases} 1 & \text{if } \lambda \in \eta \downarrow \\ 0 & \text{if } \lambda \not\in \eta \downarrow \end{cases}$$

Observe that $\lambda \in \eta \downarrow$ if and only if $\eta \in \lambda \uparrow$. So

$$n_\eta = \begin{cases} 1 & \text{if } \eta \in \lambda \uparrow \\ 0 & \text{if } \eta \not\in \lambda \uparrow \end{cases}$$

It follows that

$$S^\lambda_{\epsilon} = \bigoplus_{\mu \in \lambda} (S^\mu)^{\eta_{\mu}} = \bigoplus_{\mu \in \lambda} S^\mu$$

$\square$
that is

\[(**\ast\ast) \quad \alpha_s(\overline{t}) = \text{sgn}(ts^{-1}) e_x\]

Observe that \((**\ast\ast)\) implies

\[(**\ast\ast\ast) \quad \text{Im } \alpha_s = S^{\lambda'_x}_{\text{sgn}}\]

Since \(\lambda'' = \lambda\) we also obtain a unique \(FSym(n)\) linear function

\[(+\ast) \quad \alpha_x : M^{\lambda'}_{\text{sgn}} \rightarrow M^d, \quad \overline{t} \mapsto \text{sgn}(ts^{-1}) e_t\]

Then

\[(+\ast\ast) \quad \text{Im } \alpha_x = S^d.\]

We claim that \(\alpha_x\) is the adjoint of \(\alpha_s\), that is

\[(+\ast\ast\ast) \quad \left( \begin{array}{c|c} \alpha_s(\overline{t}) & \overline{r} \\ \hline \overline{t} & \alpha_x(\overline{r}) \end{array} \right) = \left( \begin{array}{c} \text{sgn}(ts^{-1}) e_x \\ \hline \text{sgn}(ts^{-1}) e_t \end{array} \right)\]

for all \(\lambda\)-tableaux \(t, r\).

Indeed suppose that \(\overline{r}\) is involved in \(\alpha_s(\overline{t})\). Since \(\alpha_s(\overline{t}) = \text{sgn}(ts^{-1}) e_x\) we conclude that there exits \(\beta \in C_x\) with \(\overline{r} = \beta \overline{r}'\). Thus

\[
\left( \begin{array}{c|c} \alpha_s(\overline{t}) & \overline{r} \\ \hline \overline{t} & \alpha_x(\overline{r}) \end{array} \right) = \left( \begin{array}{c} \text{sgn}(ts^{-1}) e_x \\ \hline \text{sgn}(ts^{-1}) e_t \end{array} \right) = \text{sgn}(ts^{-1}) \text{sgn}(\beta).
\]

Since \(\overline{r} = \beta \overline{r}'\), there exists \(\delta \in R_x\) with \(\delta r' = \beta r\). Observe that \(\delta \in C_x, \beta \in R_t\) and \(\delta r = \beta t\). Thus \(\overline{t} = \overline{r} = \overline{t}\). Hence \(\overline{t}\) is involved in \(e_t\) and

\[
\left( \begin{array}{c} \overline{t} \\ \hline \alpha_x(\overline{r}) \end{array} \right) = \text{sgn}(rs^{-1}) \text{sgn}(\delta).
\]

From \(\delta r = \beta t\) we get \(\delta rs^{-1} = \beta ts^{-1}\). Thus

\[
\text{sgn}(rs^{-1}) \text{sgn}(\delta) = \text{sgn}(ts^{-1}) \text{sgn}(\beta)
\]

and so \((+\ast\ast\ast)\) holds.

By \((+\ast\ast\ast)\) we can apply \ref{2.3.15} and conclude that

\[
\ker \alpha_s = (\text{Im } \alpha_x)^\perp \cong S^{\lambda'_x}_{\text{sgn}}
\]

Hence the Isomorphism Theorem gives

\[
M^d/S^{\lambda'_x}_{\text{sgn}} = M^d/\ker \alpha_s \cong \text{Im } \alpha_s \cong S^{\lambda'_x}_{\text{sgn}}
\]

\[\square\]
Lemma 2.8.3. Let $G$ be a group, let $F$ a commutative ring and let $V$ and $W$ be $F[G]$-modules. Then $V \otimes_F W$ is an $F[G]$-module via
\[ g(v \otimes w) = gv \otimes gw \]
for all $g \in G$, $v \in V$ and $w \in W$.

Proof. Readily verified. □

Lemma 2.8.4. Let $F$ be a commutative ring, $G$ a group, $M$ a $F[G]$-module and $\epsilon : G \to \mathbb{Z}$ a multiplicative homomorphism. View $F$ as an $F[G]$-module via $gr = r$ for all $g \in G$, $r \in F$. Then
\[ M_\epsilon \cong F_\epsilon \otimes_F M \]
as an $F[G]$-module.

Proof. Observe first that there exists an $F$-isomorphism
\[ \alpha : F_\epsilon \otimes_F M \to M, \quad r \otimes m \mapsto rm. \]
Moreover, if $g \in G$, $r \in F$ and $m \in M$ then
\[ \alpha(g(r \otimes m)) = \alpha((g \cdot \epsilon r) \otimes gm) = \alpha(\epsilon(g)r \otimes gm) = \epsilon(g)rgm = \alpha(r \otimes m) \]
and so $\alpha$ is an $F[G]$-isomorphism. □

Corollary 2.8.5. (a) $M^{(n)} = S^{(n)} \cong F$ as an $F[\text{Sym}(n)]$-module

(b) $S^{(1^r)} \cong F_{\text{sgn}}$ as an $F[\text{Sym}(n)]$-module

(c) $(S^\lambda)^* \cong S^{(1^r)} \otimes_F S^\lambda$ as an $F[\text{Sym}(n)]$-module

Proof. (a) Observe $12\ldots n$ is the unique $(n)$-tabloid. Also $e_t = \underbrace{12\ldots n}$ for all $(n)$-tableaux $t$.

By 2.8.2
\[ F \cong F^* \cong S^{(n)*} \cong S^{(n)^*}_{\text{sgn}} = S^{(1^r)}_{\text{sgn}}. \]

\[ S^\lambda \cong S^\lambda_{\text{sgn}} \cong F_\epsilon \otimes_F S^\lambda \cong S^{(1^r)} \otimes_F S^\lambda. \]
Bibliography


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