Quiz 12/Solutions
Take-Home
due 12/7/18 at 10:20AM

1. Compute the following (definite or indefinite) integrals:

(a) $\int_{-1}^{2} x^7\sqrt{x^4+1}\,\mathrm{d}x$.

$u = x^4+1$,

$\mathrm{d}u = (x^4+1)'\,\mathrm{d}x = 4x^3\,\mathrm{d}x$.

$x^3\,\mathrm{d}x = \frac{1}{4}\,\mathrm{d}u$.

$x=-1 : u = (-1)^4+1 = 1+1 = 2$.

$x = 2 : u = 2^4+1 = 17$

$$\begin{aligned}\int_{-1}^{2} x^7\sqrt{x^4+1}\,\mathrm{d}x &= \int_{-1}^{2} x^4\sqrt{x^4+1}x^3\,\mathrm{d}x\\ &= \int_{2}^{17}(u-1)\sqrt{u}\frac{1}{4}\,\mathrm{d}u\\ &= \frac{1}{4}\int_{2}^{17}(u\sqrt{u}-\sqrt{u})\,\mathrm{d}u\\ &= \frac{1}{4}\int_{2}^{17}(u^{\frac{3}{2}}-u^{\frac{1}{2}})\,\mathrm{d}u\\ &= \frac{1}{4}\left[\frac{2}{5}u^{\frac{5}{2}}-\frac{2}{3}u^{\frac{3}{2}}\right]_2^{17}\\ &= \frac{1}{2}\left[\frac{1}{5}u^{\frac{5}{2}}-\frac{1}{3}u^{\frac{3}{2}}\right]_2^{17}\\ &= \boxed{\frac{1}{2}\left(\left(\frac{1}{5}17^{\frac{5}{2}}-\frac{1}{3}17^{\frac{3}{2}}\right)-\left(\frac{1}{5}2^{\frac{5}{2}}-\frac{1}{3}2^{\frac{3}{2}}\right)\right)}\end{aligned}$$

(b) $\int_{-3}^{3}\sin^{99}(x)\,\mathrm{d}x$.

$\sin^{99}(-x) = (\sin(-x))^{99} = (-\sin(x))^{99} = -\sin^{99}(x)$. So $\sin^{99}(x)$ is an odd function and

$$\int_{-3}^{3}\sin^{99}(x)\,\mathrm{d}x = \boxed{0}.$$

(c) $\int \sec^2(x)\tan^5(x)\,\mathrm{d}x$.

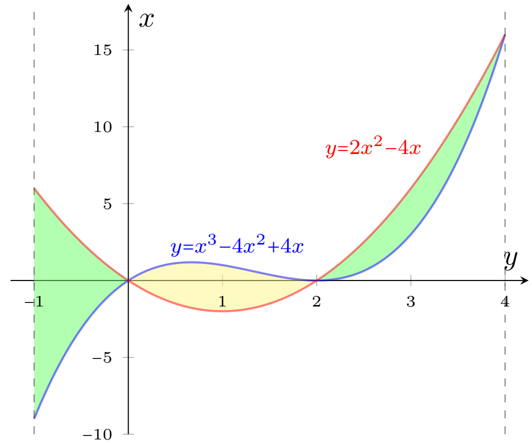
$u = \tan x$

$\mathrm{d}u = (\tan x)'\,\mathrm{d}x = \sec^2(x)\,\mathrm{d}x$.

$$\int \sec^2(x)\tan^5(x)\,\mathrm{d}x = \int(\tan x)^5\sec^2(x)\,\mathrm{d}x = \int u^5\,\mathrm{d}u = \frac{1}{6}u^6+C = \boxed{\frac{1}{6}\tan^6(x)+C}.$$

2. Compute the area of the region between the curves $y = x^3-4x^2+4x$ and $y = 2x^2-4x$ from $x=-1$ to $x=4$.

We first sketch the graph of both curves:



Next we compute the intersection points:

$$x^3 - 4x^2 + 4x = 2x^2 - 4x$$
$$x^3 - 6x^2 + 8x = 0$$
$$x(x^2 - 6x + 8) = 0$$
$$x(x-2)(x-4) = 0$$

So the graphs intersect at $x = 0$, $x = 2$ and $x = 4$. From the sketch of the graph, $y = 2x^2 - 4x$ is the larger function on $(-1,0)$ and on $(2,4)$, while $y = x^3 - 4x^2 + 4x$ is the larger function on $(0,2)$. Thus

$$\left|(x^3 - 4x^2 + 4x) - (2x^2 - 4)\right| = \left|x^3 - 6x^2 + 8x\right| = \begin{cases} -x^3 + 6x^2 - 8x & \text{if } x \text{ is in } [-1,0] \\ x^3 - 6x^2 + 8x & \text{if } x \text{ is in } [0,2] \\ -x^3 + 6x^2 - 8x & \text{if } x \text{ is in } [2,4] \end{cases}$$

Hence

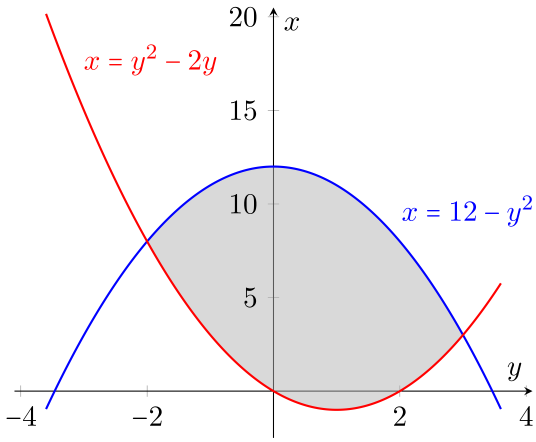
$$\begin{aligned}
\text{Area} &= \int_{-1}^{4} \left|(x^3 - 4x^2 + 4x) - (2x^2 - 4)\right| \mathrm{d}x \\
&= \int_{-1}^{0} (-x^3 + 6x^2 - 8x)\,\mathrm{d}x \; + \int_{0}^{2} (x^3 - 6x^2 + 8x)\,\mathrm{d}x \; + \int_{2}^{4} (-x^3 + 6x^2 - 8x)\,\mathrm{d}x \\
&= \left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_{-1}^{0} \; + \; \left[\frac{1}{4}x^4 - 2x^3 + 4x^2\right]_{0}^{2} \; + \; \left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_{2}^{4} \\
&= \frac{1}{4}\Big(\left[-x^4 + 8x^3 - 16x^2\right]_{-1}^{0} \; + \; \left[x^4 - 8x^3 + 16x^2\right]_{0}^{2} \; + \; \left[-x^4 + 8x^3 - 16x^2\right]_{2}^{4}\Big) \\
&= \frac{1}{4}\Big(0 - (-(-1)^4 + 8(-1)^3 - 16(-1)^2) \; + \; (2^4 - 8\cdot 2^3 + 16\cdot 2^2) - 0 \\
&\quad + (-4^4 + 8\cdot 4^3 - 16\cdot 4^2) - (-2^4 + 8\cdot 2^3 - 16\cdot 2^2)\Big) \\
&= \frac{1}{4}\left(-(-1 - 8 - 16) \; + (16 - 64 + 64) + (-256 + 512 - 256) - (-16 + 64 - 64)\right) \\
&= \frac{1}{4}(25 + 16 + 0 + 16) \\
&= \boxed{\frac{57}{4}}
\end{aligned}$$

3. Find the area of the region enclosed by the curves $y^2 + x = 12$ and $y^2 = 2y + x$.

Since its is easier to solve for x, than for y we will view x as a function of y. So the two curves are

$$x = 12 - y^2 \quad \text{and} \quad x = y^2 - 2y$$

We will first sketch the graph of the two curves:



Next we compute the intersection points:

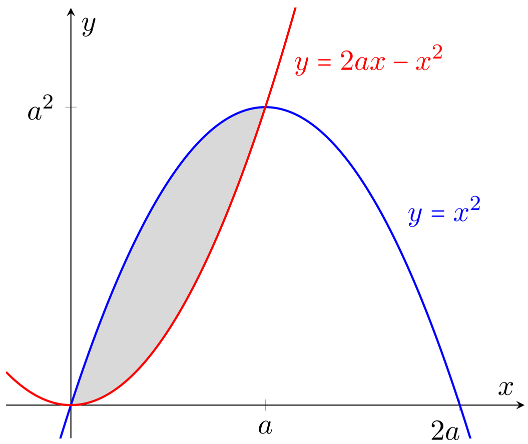
$$\begin{aligned} 12 - y^2 &= y^2 - 2y \\ 2y^2 - 2y - 12 &= 0 \\ y^2 - y - 6 &= 0 \\ (y+2)(y-3) &= 0 \end{aligned}$$

Hence the intersection points are at $y = -2$ and $y = 3$.
The larger function on the interval $[-2, 3]$ is $12 - y^2$. So

$$\begin{aligned} \text{Area} &= \int_{-2}^{3} (12 - y^2) - (y^2 - 2y)\,\mathrm{d}y \\ &= \int_{-2}^{3} -2y^2 + 2y + 12\,\mathrm{d}y \\ &= \left[-\frac{2}{3}y^3 + y^2 + 12y\right]_{-2}^{3} \\ &= \frac{1}{3}\left[-2y^3 + 3y^2 + 36y\right]_{-2}^{3} \\ &= \frac{1}{3}\big((-2\cdot 3^3 + 3\cdot 3^2 + 36\cdot 3) - (-2\cdot(-2)^3 + 3\cdot(-2)^2 + 36\cdot(-2))\big) \\ &= \frac{1}{3}(-54 + 27 + 108) - (16 + 12 - 72) \\ &= \frac{81 - (-44)}{3} \\ &= \boxed{\frac{125}{3}} \end{aligned}$$

4. Find the positive number a such that the area of the region enclosed by the parabolas $y = 2ax - x^2$ and $y = x^2$ is equal to 9.

To help sketching the two two parabolas, note that $2ax - x^2 = x(2a - x)$. So $2ax - x^2 - 0$ at $x = 0$ and $x = a$.



We know compute the intersection point of the two parabolas:

$$
\begin{aligned}
x^2 &= 2ax - x^2 \\
2x^2 - 2ax &= 0 \\
2x(x-a) &= 0
\end{aligned}
$$

So the two parabolas intersect at $x = 0$ and $x = a$. The larger function on the interval $[0, a]$ is $2ax - x^2$. Hence the area of the enclosed region is

$$\int_0^a (2ax - x^2) - x^2 = \int_0^a 2ax - 2x^2 = \left[ax^2 - \frac{2}{3}x^3\right]_0^a = aa^2 - \frac{2}{3}a^3 = a^3 - \frac{2}{3}a^3 = \frac{1}{3}a^3$$

Since the area of the enclosed region is 9 we get

$$
\begin{aligned}
\frac{1}{3}a^3 &= 9 \\
a^3 &= 3 \cdot 9 = 3 \cdot 3^2 = 3^3 \\
&\boxed{a = 3}
\end{aligned}
$$