Quiz 11/Solutions

1. Compute the average value of $\cos x$ on the interval $[-\frac{\pi}{4}, \frac{\pi}{4}]$.

$$\operatorname{av}(\cos x) = \frac{1}{\frac{\pi}{4} - (-\frac{\pi}{4})} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos x \, \mathrm{d}x = \frac{1}{\frac{\pi}{2}} \Big[\sin x\Big]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \frac{2}{\pi}\left(\sin\left(\frac{\pi}{4}\right) - \sin\left(-\frac{\pi}{4}\right)\right)$$

$$= \frac{2}{\pi}\left(\frac{1}{\sqrt{2}} - (-\frac{1}{\sqrt{2}})\right) = \frac{2}{\pi}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{2}{\pi}\frac{2}{\sqrt{2}} = \frac{2}{\pi}\sqrt{2} = \boxed{\frac{2\sqrt{2}}{\pi}}$$

2. A population is growing at a rate of $n'(t) = 4t^3$ (where time is measured in years). Compute the net change of the population from $t = 0$ to $t = 2$.

$$n(2) - n(0) = \int_0^2 n'(t) \, \mathrm{d}t = \int_0^2 4t^3 \, \mathrm{d}t = \left[t^4\right]_0^2 = 2^4 - 0^4 = \boxed{16}$$

3. Compute each of the following (definite or indefinite) integrals:

(a) $\int x \sec^2(x^2) \, \mathrm{d}x$.

$u = x^2$

$\mathrm{d}u = 2x \, \mathrm{d}x$

$\frac{1}{2} \, \mathrm{d}u = x \, \mathrm{d}x$.

$$\int x \sec^2(x^2) dx = \int \sec^2(x^4) x \, \mathrm{d}x = \int \sec^2(u) \frac{1}{2} \, \mathrm{d}u = \frac{1}{2} \tan u + C = \boxed{\frac{1}{2} \sec^2(x^2) + C}.$$

(b) $\int_0^1 (3x^2+1)\sqrt{x^3+x+1} \, \mathrm{d}x$.

$u = x^3 + x + 1$

$\mathrm{d}u = (x^3 + x + 1)' \, \mathrm{d}x = (3x^2 + 1) \, \mathrm{d}x$

$x = 1: \; u = 1^3 + 1 + 1 = 3$

$x = 0: \; u = 0^3 + 0 + 1 = 1$

$$\int_0^1 (3x^2+1)\sqrt{x^3+x+1} \, \mathrm{d}x = \int_0^2 \sqrt{x^3+3x+1}(3x^2+1) \, \mathrm{d}x = \int_1^3 \sqrt{u} \, \mathrm{d}u$$

$$= \left[\frac{2}{3}u^{3/2}\right]_1^3 = \boxed{\frac{2}{3}\left(3^{3/2} - 1^{3/2}\right)} = \boxed{\frac{2}{3}(\sqrt{27} - 1)}$$

(c) $\int_{-2}^2 x \cos(x) \, \mathrm{d}x$.

Let $f(x) = x \cos x$. Then

$$f(-x) = (-x)\cos(-x) = -x \cos x = -f(x).$$

Thus $x \cos x$ is an odd function and so

$$\int_{-2}^2 x \cos x \, \mathrm{d}x = \boxed{0}.$$