Quiz 10/Solutions

1. Compute $\sum_{i=1}^{n}(4i^3+6i^2+1)$.

$$\begin{aligned}
\sum_{i=1}^{n}(4i^3+6i^2+1) &= 4\sum_{i=1}^{n} i^3 \;+\; 6\sum_{i=1}^{n} i^2 \;+\; \sum_{i=1}^{n} 1\\
&= 4\left(\frac{n(n+1)}{2}\right)^2 + 6\frac{n(n+1)(2n+1)}{6} + n\\
&= n^2(n+1)^2 + n(n+1)(2n+1) + n\\
&= n^2(n^2+2n+1) + (n^2+n)(2n+1) + n\\
&= (n^4+2n^3+n^2) + (2n^3+2n^2+n^2+n) + n\\
&= \boxed{n^4+4n^3+4n^2+2n}.
\end{aligned}$$

2. Use the Right End Point Rule (and not the Fundamental Theorem of Calculus) to compute $\int_{-1}^{0} x^2\,\mathrm{d}x$. We have

$$f(x) = x^2, \qquad \Delta x = \frac{0-(-1)}{n} = \frac{1}{n}, \qquad x_i = -1 + i\Delta x = -1 + \frac{i}{n} = \frac{-n+i}{n} = \frac{i-n}{n}.$$

So

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n}\left(\frac{i-n}{n}\right)^2\frac{1}{n}\\
&= \sum_{i=1}^{n}\frac{(i-n)^2}{n^3}\\
&= \frac{1}{n^3}\sum_{i=1}^{n}(i^2-2in+n^2)\\
&= \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 \;-2n\sum_{i=1}^{n} i \;+n^2\sum_{i=1}^{n} 1\right)\\
&= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2n\frac{n(n+1)}{2} \;+n^2n\right)\\
&= \frac{1}{n^3}n\left(\frac{(n+1)(2n+1)}{6} - n(n+1) + n^2\right)\\
&= \frac{1}{n^2}\left(\frac{2n^2+2n+n+1}{6} - n^2 - n + n^2\right)\\
&= \frac{1}{n^2}\left(\frac{2n^2+3n+1}{6} - n\right)\\
&= \frac{1}{n^2}\left(\frac{(2n^2+3n+1)-6n}{6}\right)\\
&= \frac{1}{n^2}\left(\frac{2n^2-3n+1}{6}\right)\\
&= \frac{1}{6}\left(\frac{2n^2-3n+1}{n^2}\right)\\
&= \frac{1}{6}\left(2-\frac{3}{n}+\frac{1}{n^2}\right)
\end{aligned}$$

So according to the Right-End-Point Rule

$$\int_{-1}^{0} x^2\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$
$$= \lim_{n\to\infty}\frac{1}{6}\left(2-\frac{3}{n}+\frac{1}{n^2}\right) = \frac{1}{6}(2-0-0) = \boxed{\frac{1}{3}}.$$

To confirm the answer I will evaluate the integral with the FToC:

$$\int_{-1}^{0} x^2\,\mathrm{d}x = \left[\frac{1}{3}x^3\right]_{-1}^{0} = \frac{1}{3}0^3 - \frac{1}{3}(-1)^3 = \frac{1}{3}.$$

3. Evaluate the following definite integrals either by using the Fundamental Theorem of Calculus or by determining the area of the region corresponding to the integral.

(a) $\int_{-\pi}^{\pi}\sin x\,\mathrm{d}x$.

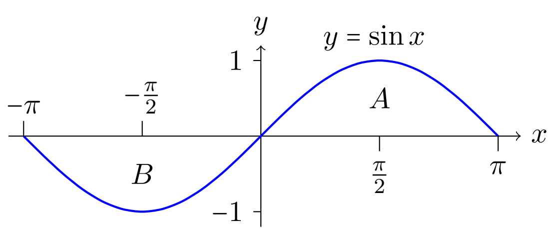
(b) $\int_{1}^{3}\frac{x^5+x+1}{x^3}\,\mathrm{d}x$.

(c) $\int_{-5}^{5}\sqrt{25-x^2}\,\mathrm{d}x$.

(d) $\int_{3}^{5}|x-4|\,\mathrm{d}x$

(a) **Solution 1:** $\int_{-\pi}^{\pi}\sin x\,\mathrm{d}x = [-\cos x]_{-\pi}^{\pi} = -\cos\pi - (-\cos(-\pi)) = -0-(-0) = \boxed{0}$.

Solutions 2:

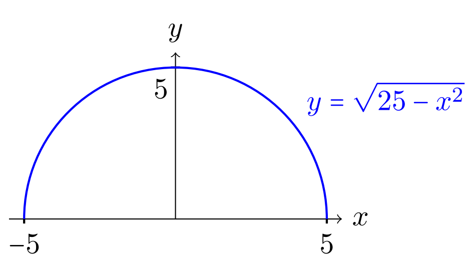


Since $\sin x$ is an odd function, the area of the region A under $y = \sin x$ and above the x-axis is equal to area of the region B above $y = \sin x$ and below the x-axis. As the integral counts the area under the x-axis negative, the integral is $\boxed{0}$.

(b)

$$\int_{1}^{3}\frac{x^5+x+1}{x^3}\,\mathrm{d}x = \int_{1}^{3}\left(x^2+x^{-2}+x^{-3}\right)\,\mathrm{d}x = \left[\frac{1}{3}x^3 - x^{-1} - \frac{1}{2}x^{-2}\right]_1^3 = \left[\frac{x^3}{3}-\frac{1}{x}-\frac{1}{2x^2}\right]_1^3$$
$$= \left(\frac{3^3}{3}-\frac{1}{3}-\frac{1}{2\cdot 3^2}\right) - \left(\frac{1^3}{3}-\frac{1}{1}-\frac{1}{2\cdot 1^2}\right) = \frac{27\cdot 6-6-1}{18} - \frac{2-6-3}{6}$$
$$= \frac{162-7}{18} - \frac{-7}{6} = \frac{155+21}{18} = \frac{176}{18} = \boxed{\frac{88}{9}}$$

(c) $\int_{-4}^{4}\sqrt{16-x^2}\,\mathrm{d}x$.



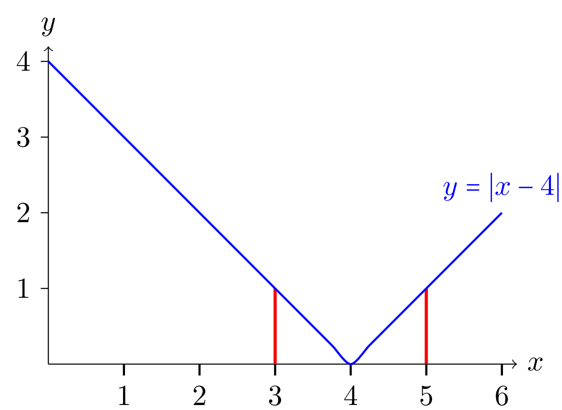
The region corresponding to the integral is a semicircle of radius 5 and so has area $\frac{1}{2}\pi 5^2 = \frac{25\pi}{2}$. Thus

$$\int_{-5}^{5} \sqrt{25 - x^2}\, \mathrm{d}x = \boxed{\frac{25\pi}{2}}.$$

(d) **Solution 1:** If $x \leq 4$, then $x - 4 \leq 0$ and so $|x-4| = -(x-4) = 4 - x$. If $x \geq 4$, then $x - 4 \geq 0$ and so $|x - 4| = x - 4$. Hence

$$\begin{aligned}
\int_3^5 |x-4|\, \mathrm{d}x &= \int_3^4 |x-4|\, \mathrm{d}x + \int_4^5 |x-4|\, \mathrm{d}x \\
&= \int_3^4 (4-x)\, \mathrm{d}x + \int_4^5 (x-4)\, \mathrm{d}x \\
&= \left[4x - \frac{x^2}{2}\right]_3^4 + \left[\frac{x^2}{2} - 4x\right]_4^5 \\
&= \left(4\cdot 4 - \frac{4^2}{2}\right) - \left(4\cdot 3 - \frac{3^2}{2}\right) + \left(\frac{5^2}{2} - 4\cdot 5\right) - \left(\frac{4^2}{2} - 4\cdot 4\right) \\
&= 16 - 8 - 12 + \frac{9}{2} + \frac{25}{2} - 20 - 8 + 16 = -4 + \frac{9+25}{2} - 12 = \frac{34}{2} - 16 = 17 - 16 = \boxed{1}
\end{aligned}$$

Solution 2:



The region corresponding to the integral consist of two triangles of base 1 and height 1. A triangle of base b and h has area $\frac{1}{2}bh$. Thus each the two triangles has area $\frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}$. Hence

$$\int_3^5 |x-4|\, \mathrm{d}x = \frac{1}{2} + \frac{1}{2} = \boxed{1}.$$

4. Compute

$$\lim_{n\to\infty} \sum_{i=1}^{n} \cos\left(\frac{\pi}{2} + \frac{i\pi}{2n}\right)\frac{\pi}{2n}$$

by interpreting the limit as a limit of Riemann sums and then using the Fundamental Theorem of Calculus to evaluate the corresponding definite integral.

Let f be continuous function on the interval $[a, b]$. According to the Right End Point Rule

$$\int_a^b f(x)\, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. So we need to find f, a and b such that

$$f(a + i\Delta x)\cdot \Delta x = \cos\left(\frac{\pi}{2} + \frac{i\pi}{2n}\right)\frac{\pi}{2n}$$

To identify the various terms better it is best to isolate the coefficient of i:

$$f(a+i\Delta x)\cdot\Delta x$$
$$=\cos\left(\frac{\pi}{2}+i\frac{\pi}{2n}\right)\frac{\pi}{2n}$$

So we can choose $f(x)=\cos x$, $a=\frac{\pi}{2}$ and $\Delta x=\frac{\pi}{2n}$. To compute b recall that $\Delta x=\frac{b-a}{n}$ and so

$$b=a+n\Delta x=\frac{\pi}{2}+n\frac{\pi}{2n}=\frac{\pi}{2}+\frac{\pi}{2}=\pi$$

Thus the limit we need to compute is equal to

$$\int_{\pi/2}^{\pi}\cos x\,\mathrm{d}x$$

Since $\sin x$ is an antiderivative of $\cos x$, the FToC gives

$$\int_{\pi/2}^{\pi}\cos x\,\mathrm{d}x=[\sin x]_{\pi/2}^{\pi}=\sin\pi-\sin\frac{\pi}{2}=0-1=\boxed{-1}$$

5. Compute the derivative of the function $f(x)=\int_{x^3}^{\sin x}\frac{\cos t}{t^4}\,\mathrm{d}t$.

Solution 1: (Using Part 1 of the FToC and the chain rule)

$$\begin{aligned}
f'(x)&=\left(\int_{x^3}^{\sin x}\frac{\cos t}{t^4}\,\mathrm{d}x\right)'\\
&=\left(\int_{x^3}^{1}\frac{\cos t}{t^4}\,\mathrm{d}x \;+\; \int_{1}^{\sin x}\frac{\cos t}{t^4}\,\mathrm{d}x\right)'\\
&=\left(-\int_{1}^{x^3}\frac{\cos t}{t^4}\,\mathrm{d}x \;+\; \int_{1}^{\sin x}\frac{\cos t}{t^4}\,\mathrm{d}x\right)'\\
&=-\frac{\cos x^3}{(x^3)^4}(x^3)' \;+\; \frac{\cos(\sin x)}{(\sin x)^4}(\sin x)'\\
&=-\frac{3x^2\cos x^3}{x^{12}} \;+\; \frac{\cos x\cdot\cos(\sin x)}{\sin^4 x}\\
&=\boxed{\frac{\cos x\cdot\cos(\sin x)}{\sin^4 x}-\frac{3\cos x^3}{x^{10}}}
\end{aligned}$$

Solution 2: (Using Part 2 of the FToC and the chain rule)

Let $G(x)$ be an anti derivative of $\frac{\cos t}{t^4}$, so $G'(x)=\frac{\cos t}{t^4}$ We compute

$$\begin{aligned}
f'(x)&=\left(\int_{x^3}^{\sin x}\frac{\cos t}{t^4}\,\mathrm{d}x\right)'\\
&=\left([G(x)]_{x^3}^{\sin x}\right)'\\
&=\left(G(\sin x)-G(x^3)\right)'\\
&=G'(\sin x)(\sin x)'-G'(x^3)(x^3)'\\
&=\frac{\cos(\sin x)}{(\sin x)^4}(\sin x)'-\frac{\cos x^3}{(x^3)^4}(x^3)'\\
&=\frac{\cos x\cdot\cos(\sin x)}{\sin^4 x}-\frac{3x^2\cos x^3}{x^{12}}\\
&=\boxed{\frac{\cos x\cdot\cos(\sin x)}{\sin^4 x}-\frac{3\cos x^3}{x^{10}}}
\end{aligned}$$