Quiz 7/Solutions

1. $f(x) = \frac{x^3+1}{x^3-8}$.

A)

$$x^3 - 8 = 0$$
$$x^3 = 8$$
$$x^3 = 2^3$$
$$x = 2$$

So $f(x)$ is defined for $x \neq 2$ and the domain of f is

$$\boxed{(-\infty, 2) \cup (2, \infty)}$$

B)

$$y = 0$$
$$x^3 + 1 = 0$$
$$x^3 = -1$$
$$x^3 = (-1)^3$$
$$x = -1$$

So the x-intercept is

$$\boxed{(-1, 0)}$$

$$f(0) = \frac{0^3 + 1}{0^3 - 8} = \frac{1}{-8} = -\frac{1}{8}$$

and so the y-intercept is

$$\boxed{\left(0, -\frac{1}{8}\right)}$$

C)

$$f(-x) = \frac{(-x)^3 + 1}{(-x)^3 - 8} = \frac{-x^3 + 1}{-x^3 - 8} \neq \pm \frac{x^3 + 1}{x^3 - 8} = f(x)$$

So f has $\boxed{\text{no symmetries}}$. Since f has exactly one x-intercept, f is $\boxed{\text{not periodic}}$

D) The denominator of f is 0 for $x = 2$. So we compute

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} \frac{x^3 + 1}{x^3 - 8} \left(= \frac{\text{positive}}{\text{small positive}} \right) = +\infty$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^3 + 1}{x^3 - 8} \left(= \frac{\text{positive}}{\text{small negative}} \right) = -\infty$$

So

$$\boxed{x = 2 \text{ is a vertical asymptote}}$$

$$\boxed{\lim_{x \to 2^+} f(x) = +\infty}$$

$$\boxed{\lim_{x \to 2^-} f(x) = -\infty}$$

Horizontal asymptotes:

$$\lim_{x \to \pm\infty} \frac{x^3 + 1}{x^3 - 8} = \lim_{x \to \pm\infty} \frac{x^3(1 + \frac{1}{x^3})}{x^3(1 - \frac{8}{x^3})} = \lim_{x \to \pm\infty} \frac{1 + \frac{1}{x^3}}{1 - \frac{8}{x^3}} = \frac{1 + 0}{1 + 0} = 1$$

Thus

$$\boxed{y = 1 \text{ is a horizontal asymptote}}$$

E) We compute

$$\begin{aligned}
f'(x) &= \left(\frac{x^3 + 1}{x^3 - 8} \right)' \\
&= \frac{(x^3 + 1)'(x^3 - 8) - (x^3 + 1)(x^3 - 8)'}{(x^3 - 8)^2} \\
&= \frac{3x^2 \cdot (x^3 - 8) - (x^3 + 1) \cdot 3x^2}{(x^3 - 8)^2} \\
&= \frac{3x^2 \cdot \big((x^3 - 8) - (x^3 + 1)\big)}{(x^3 - 8)^2} \\
&= \frac{3x^2 \cdot (-9)}{(x^3 - 8)^2} \\
&= -27 \frac{x^2}{(x^3 - 8)^2}
\end{aligned}$$

$f'(x) = 0$ when $x^2 = 0$ and so for $x = 0$.
$f'(x)$ is not defined when $x^3 - 8 = 0$ and so for $x = 2$. Hence the critical numbers are

$$\boxed{x = 0} \quad \text{and} \quad \boxed{x = 2}$$

Note that $f'(x) \le 0$ for all x. f is continuous at $x = 0$, but not defined at $x = 2$. Thus

$$\boxed{f(x) \text{ is decreasing on } (-\infty, 2) \quad \text{and on} \quad (2, \infty)}.$$

and

$$\boxed{f(x) \text{ is nowhere increasing}}$$

F) Since f is decreasing everywhere,

$$\boxed{f \text{ has no local extrema}}$$

G) We compute

$$
\begin{aligned}
f''(x) &= \left(-27\frac{x^2}{(x^3-8)^2}\right)' \\
&= -27\frac{(x^2)'\cdot(x^3-8)^2 - x^2\cdot\left((x^3-8)^2\right)'}{\left((x^3-8)^2\right)^2} \\
&= -27\frac{2x\cdot(x^3-8)^2 - x^2\cdot 2\cdot(x^3-8)\cdot 3x^2}{(x^3-8)^4} \\
&= -27\frac{2x\cdot(x^3-8)\left((x^3-8)-3x^3\right)}{(x^3-8)^4} \\
&= -54\frac{x\cdot(-2x^3-8)}{(x^3-8)^3} \\
&= 108\frac{x\cdot(x^3+4)}{(x^3-8)^3}
\end{aligned}
$$

So f'' is not defined at $x = 2$. Also $f'' = 0$ when $x(x^3+4) = 0$ and so for $x = 0$ and $x = -\sqrt[3]{4}$.

	$(-\infty, -\sqrt[3]{4})$	$(-\sqrt[3]{4}, 0)$	$(0, 2)$	$(2, \infty)$
x^3+4	$-$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$
$(x^3-8)^3$	$-$	$-$	$-$	$+$
f''	$-$	$+$	$-$	$+$
f	$\cap$	$\cup$	$\cap$	$\cup$

Hence

$$\boxed{f \quad \text{is concave up on} \quad (-\sqrt[3]{4}, 0) \quad \text{and} \quad (2, \infty)}$$

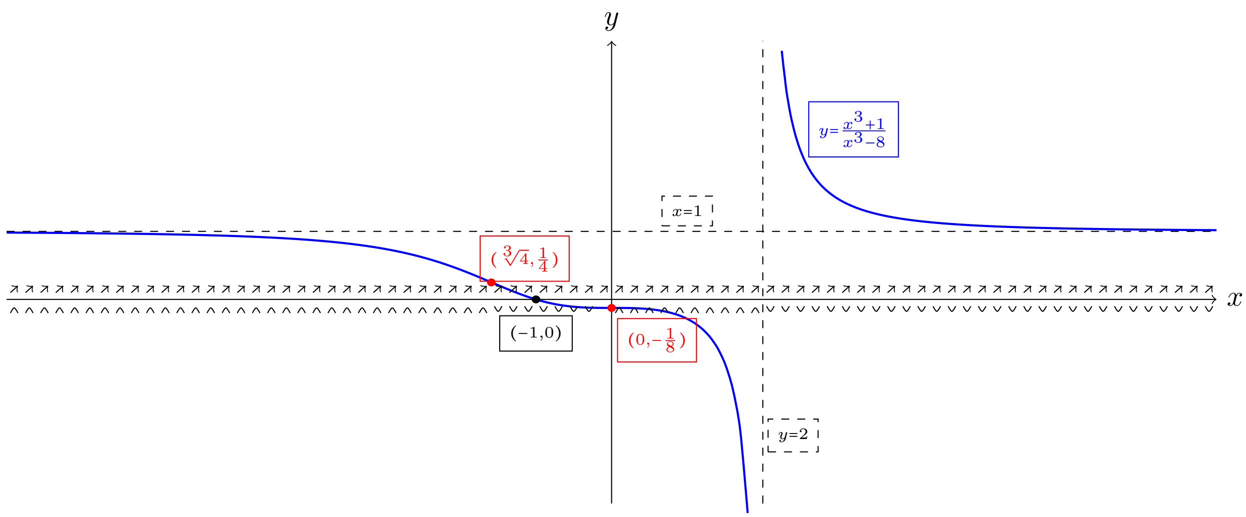
and

$$\boxed{f \quad \text{is concave down on} \quad (-\infty, -\sqrt[3]{4}) \quad \text{and} \quad (0, 2)}$$

In particular, f changes concavity at $x = -\sqrt[3]{4}$, at $x = 0$ and at $x = 2$.
If $x = -\sqrt[3]{4}$, then $y = \frac{(-\sqrt[3]{4})^3+1}{(-\sqrt[3]{4})^3-8} = \frac{-4+1}{-4-8} = \frac{-3}{-12} = \frac{1}{4}$.
If $x = 0$, then $y = \frac{0^3+1}{0^3-8} = \frac{1}{-8} = -\frac{1}{8}$.
If $x = 2$, then y is not defined. Thus

$$\boxed{\text{the inflections points of } f \text{ are} \quad \left(-\sqrt[3]{4}, \frac{1}{4}\right) \quad \text{and} \quad \left(0, -\frac{1}{8}\right)}$$

H)



2. $f(x) = x^{1/3}(x-4)$

A) f is defined for all real numbers. So

$$\boxed{\text{The domain of } f \text{ is} \quad (-\infty, \infty)}$$

B) $f(x) = 0$ if $x^{1/3} = 0$ or $x - 4 = 0$, that is if $x = 0$ or $x = 4$. So

$$\boxed{\text{The } x\text{-intercepts of } f \text{ are} \quad (0,0) \quad \text{and} \quad (4,0)}$$

$f(0) = 0$. So

$$\boxed{\text{The } y\text{-intercept of } f \text{ is} \quad (0,0)}$$

C) $f(-x) = (-x)^{1/3}(-x-4) = x^{1/3}(x+4) \neq \pm x^{1/3}(x-4) = \pm f(x)$. Thus

$$\boxed{f \text{ has no symmetries}}$$

Since f has only two x-intercepts:

$$\boxed{f \text{ is not periodic}}$$

D) f is continuous at all real numbers, so

$$\boxed{f \text{ has no vertical asymptotes}}$$

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\infty} x^{1/3}(x-4) = \lim_{x\to\pm\infty} x^{4/3}\left(1 - \frac{4}{x^{1/3}}\right) = +\infty$$

So

$$\boxed{\lim_{x\to\pm\infty} f(x) = +\infty}$$

and

$$\boxed{f \text{ has no horizontal asymptotes}}$$

E)

$$\begin{aligned} f'(x) &= \left(x^{1/3}(x-4)\right)' \\ &= \left(x^{4/3} - 4x^{1/3}\right)' \\ &= \frac{4}{3}x^{1/3} - 4\cdot\frac{1}{3}x^{-2/3} \\ &= \frac{4}{3}\left(x^{1/3} - \frac{1}{x^{2/3}}\right) \\ &= \frac{4}{3}\frac{x-1}{x^{2/3}} \end{aligned}$$

Thus $f' = 0$ for $x = 1$ and f' is not defined at $x = 0$. We have

$$f(0) = 0 \quad \text{and} \quad f(1) = 1^{1/3}(1-4) = -3$$

$$\boxed{\text{The critical points of } f \text{ are } \quad (0,0) \quad \text{and} \quad (1,-3)}$$

To compute the intervals where f is increasing and decreasing, respectively:

	$(-\infty,0)$	$(0,1)$	$(1,\infty)$
$x-1$	$-$	$-$	$+$
$x^{2/3}$	$+$	$+$	$+$
f'	$-$	$-$	$+$
f	$\searrow$	$\searrow$	$\nearrow$

Since f is continuous at 0 and 1 we conclude that

$$\boxed{f \text{is decreasing on } \quad (-\infty,1] \quad \text{and increasing on } \quad [1,\infty)}$$

F) From E) we see that

$$\boxed{-3 \text{ is a local minimum value of } f \text{ attained at } x = 1}$$

and

$$\boxed{f \text{ has no local maximum value}}$$

G)

$$\begin{aligned} f''(x) &= \left(\frac{4}{3}\left(x^{1/3} - \frac{1}{x^{2/3}}\right)\right)' \\ &= \frac{4}{3}\left(\frac{1}{3}x^{-2/3} + \frac{2}{3}\frac{1}{x^{5/3}}\right) \\ &= \frac{4}{3}\cdot\frac{1}{3}\left(\frac{1}{x^{2/3}} + \frac{2}{x^{5/3}}\right) \\ &= \frac{4}{9}\left(\frac{x+2}{x^{5/3}}\right) \end{aligned}$$

So $f'' = 0$ at $x = -2$, and f'' is not defined at $x = 0$.

	$(-\infty,-2)$	$(-2,0)$	$(0,\infty)$
$x+2$	$-$	$+$	$+$
$x^{5/3}$	$-$	$-$	$+$
f''	$+$	$-$	$+$
f	$\cup$	$\cap$	$\cup$

Thus

$$\boxed{f \text{ is concave up on } \quad (-\infty,-2) \text{ and } (0,\infty) \quad \text{and concave down on} \quad (-2,0)}$$

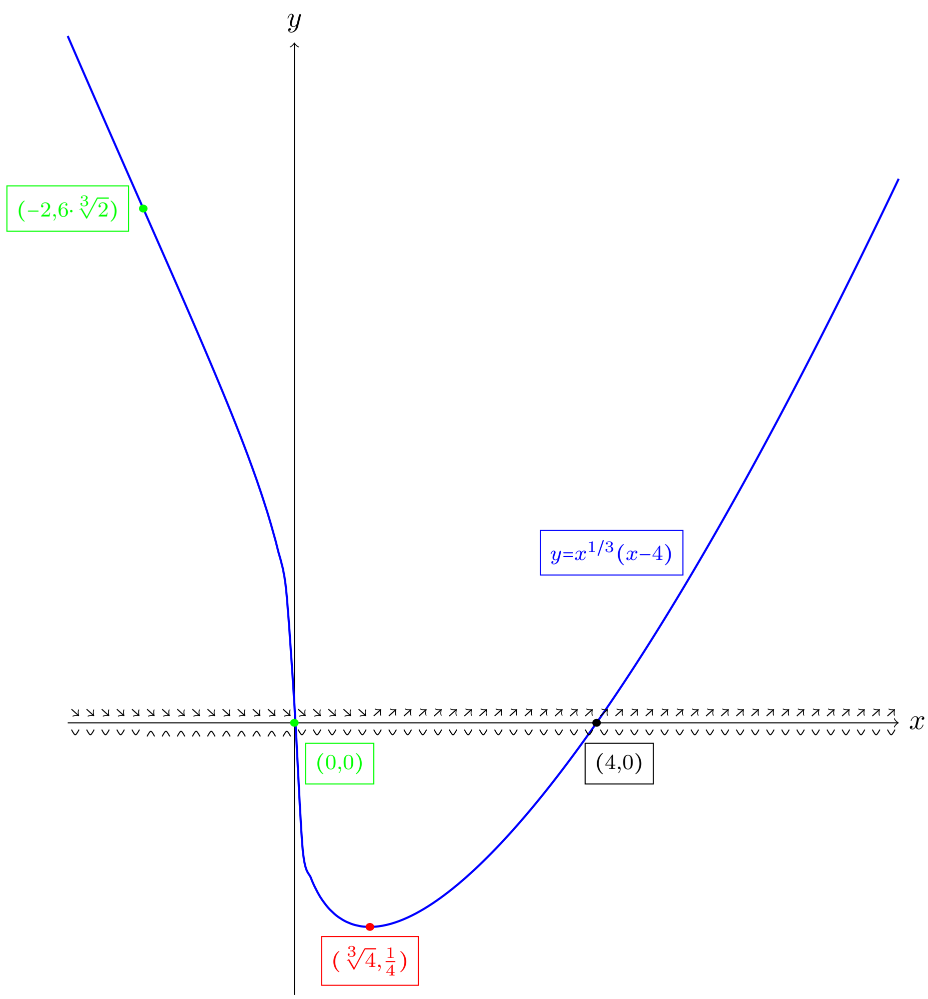
Also f changes concavity at $x=-2$ and at $x=0$.

$$f(-2)=(-2)^{1/3}(-2-4)=6\cdot\sqrt[3]{2}\approx 7.6 \quad \text{and} \quad f(0)=0$$

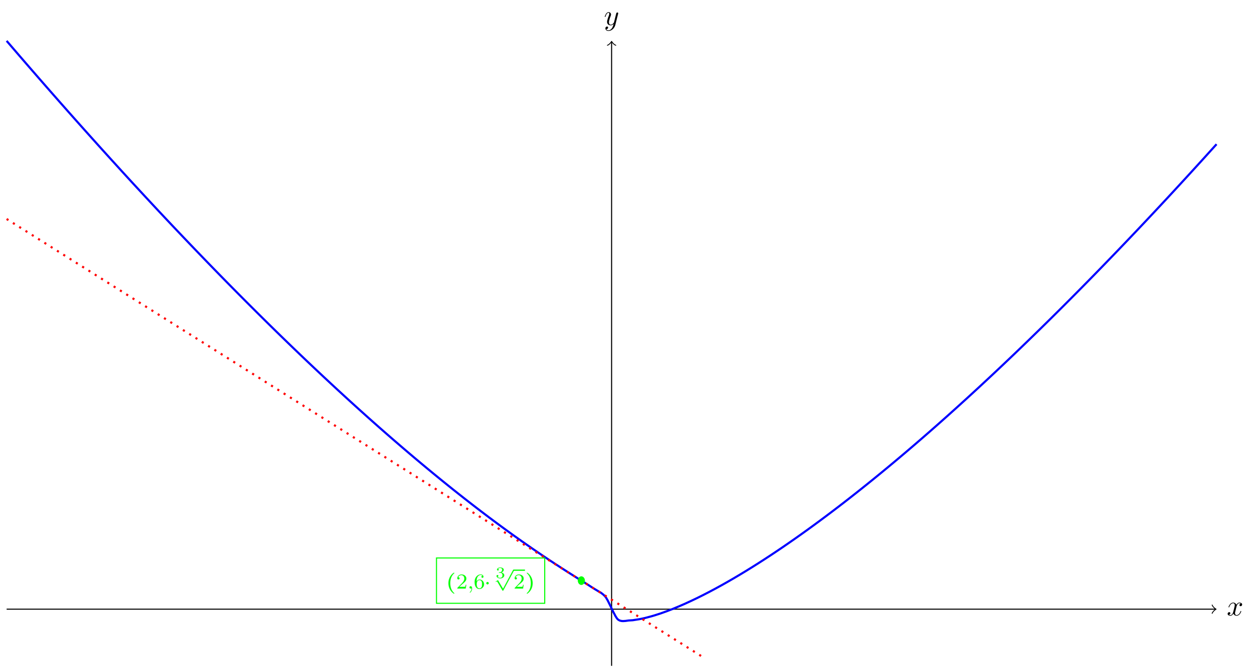
so

$$\boxed{\text{The inflections points of } f \text{ are } \quad (-2,6\cdot\sqrt[3]{2}) \quad \text{and} \quad (0,0)}$$

H)



On the above graph one cannot really see that $(-2,6\cdot\sqrt[3]{2})$ is a inflection point. The next graph uses a different view and scale and I also added the tangent line to the curve at the inflection point (the dotted red line) to make the inflection point more visible:



3. $f(x) = \frac{\sin x}{2+\cos x}$

A) Since $\cos x \geq -1$ we have $2 + \cos x \geq 2 + (-1) = 1 > 0$ for all x. Thus f is defined for all real numbers. Hence

$$\boxed{\text{The domain of } f \text{ is } \quad (-\infty, \infty)}$$

B) $f(x) = 0$ if $\sin x = 0$ and so when $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$$\boxed{\text{The } x\text{-intercepts of } f \text{ are } \quad (0,0), (\pm\pi, 0), (\pm 2\pi, 0), (\pm 3\pi, 0), \ldots}$$

$f(0) = 0$. So

$$\boxed{\text{The } y\text{-intercept of } f \text{ is } \quad (0,0)}$$

C) $f(-x) = \frac{\sin(-x)}{2+\cos(-x)} = \frac{-\sin x}{2+\cos x} = -\frac{\sin x}{2+\cos x} = -f(x)$ Thus

$$\boxed{f \text{ is an odd function and is symmetric about } (0,0)}$$

Both $\sin x$ and $\cos x$ are periodic with period 2π. Thus also

$$\boxed{f \text{ is periodic with period } 2\pi}$$

Remark: *Since f has period 2π we can restrict our computation to any interval of length 2π. To make use of the fact that f is symmetric about $(0,0)$ it is convenient to choose the interval to be symmetric about 0. So for steps E-G I will restrict the computation to the interval $[-\pi, \pi]$.*

D) f is continuous at all real numbers, so

$$\boxed{f \text{ has no vertical asymptotes}}$$

As $x \to \pm\infty$, the function $\sin x$ goes infinitely often forth and back between -1 and 1, and $2+\cos x$ goes forth and back between 1 and 3. Since $f(x) = \frac{\sin x}{2+\cos x}$ we conclude that $f(x)$, as $x \to \pm\infty$, will infinitely often by larger than $\frac{1}{3}$ and infinitely often be smaller than $\frac{1}{3}$. Thus

$$\boxed{\lim_{x\to\pm\infty} f(x) = \text{DNE}}$$

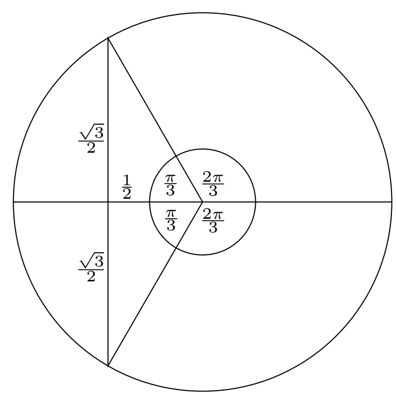
and

$$\boxed{f \text{ has no horizontal asymptotes}}$$

(More generally, any non-constant periodic function does not have a horizontal asymmptote)
E)

$$\begin{aligned}
f'(x) &= \left(\frac{\sin x}{2+\cos x}\right)' \\
&= \frac{\sin'(x)\cdot(2+\cos x) - \sin x\cdot(2+\cos x)'}{(2+\cos x)^2} \\
&= \frac{\cos x\cdot(2+\cos x) - \sin x\cdot(-\sin x)}{(2+\cos x)^2} \\
&= \frac{2\cos x + \cos^2 x + \sin^2 x}{(2+\cos x)^2} \\
&= \frac{2\cos x + 1}{(2+\cos x)^2} \\
&= 2\frac{\cos x - (-\frac{1}{2})}{(2+\cos x)^2}
\end{aligned}$$

Thus $f'(x)$ is defined for all x and $f'(x) = 0$ when $\cos x = -\frac{1}{2}$.
In the interval $[-\pi,\pi]$: $\cos x = -\frac{1}{2}$ at $x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ and at $x = -\frac{2\pi}{3}$.



$$f\left(\frac{2\pi}{3}\right) = \frac{\sin\left(\frac{2\pi}{3}\right)}{2+\cos\left(\frac{2\pi}{3}\right)} = \frac{\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{2}} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

and

$$f\left(-\frac{2\pi}{3}\right) = -f\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}$$

$$\boxed{\text{The critical points of } f \text{ in } [-\pi,\pi] \text{ are } \left(-\frac{2\pi}{3}, -\frac{1}{\sqrt{3}}\right) \text{ and } \left(\frac{2\pi}{3}, \frac{1}{\sqrt{3}}\right)}$$

Observe that $\cos x > -\frac{1}{2}$ on $\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ and $\cos x < -\frac{1}{2}$ on $\left[-\pi, -\frac{2\pi}{3}\right)$ and $\left(\frac{2\pi}{3}, \pi\right]$. Thus

	$\left[-\pi, -\frac{2\pi}{3}\right)$	$\left(-\frac{2\pi}{3}, \frac{2\pi}{3}\right)$	$\left(\frac{2\pi}{3}, \pi\right]$
$\cos x - (-\frac{1}{2})$	$-$	$+$	$-$
$(2+\cos x)^2$	$+$	$+$	$+$
f'	$-$	$+$	$-$
f	↘	↗	↘

Since f is continuous we conclude that

$$\boxed{f \text{ is increasing on } \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right] \text{ and decreasing on } \left[\pi, -\frac{2\pi}{3}\right] \text{ and } \left[\frac{2\pi}{3}, \pi\right]}$$

F) From E) we see that

$$\boxed{-\frac{1}{\sqrt{3}} \text{ is a local minimum value of } f \text{ attained at } x = -\frac{2\pi}{3}}$$

and

$$\boxed{\frac{1}{\sqrt{3}} \text{ is a local maximum value of } f \text{ attained at } x = \frac{2\pi}{3}}$$

G)

$$\begin{aligned}
f''(x) &= \left(\frac{2\cos x + 1}{(2+\cos x)^2}\right)' \\
&= \frac{(2\cos x+1)' \cdot (2+\cos x)^2 - (2\cos x + 1)\cdot \big((2+\cos x)^2\big)'}{((2+\cos x)^2)^2} \\
&= \frac{-2\sin x \cdot (2+\cos x)^2 - (2\cos x+1)\cdot 2 \cdot (2+\cos x)\cdot(-\sin x)}{(2+\cos x)^4} \\
&= \frac{2\cdot(2+\cos x)\cdot \sin x \cdot -(2+\cos x) + 2\cdot(2+\cos x)\cdot \sin x \cdot (2\cos x + 1)}{(2+\cos x)^4} \\
&= 2\frac{(2+\cos x)\cdot \sin x \cdot \big(-(2+\cos x) + (2\cos x+1)\big)}{(2+\cos x)^3} \\
&= 2\frac{\sin x\cdot(\cos x - 1)}{(2+\cos x)^3}
\end{aligned}$$

So $f'' = 0$ when $\sin x = 0$ or $\cos x = 1$. In $[-\pi, \pi]$: $\sin x = 0$ at $x = -\pi, 0, \pi$, and $\cos x = 1$ at $x = 0$.

	$(-\pi, 0)$	$(0, \pi)$
$\sin x$	$-$	$+$
$\cos x - 1$	$-$	$-$
$(2+\cos x)^3$	$+$	$+$
f''	$+$	$-$
f	$\cup$	$\cap$

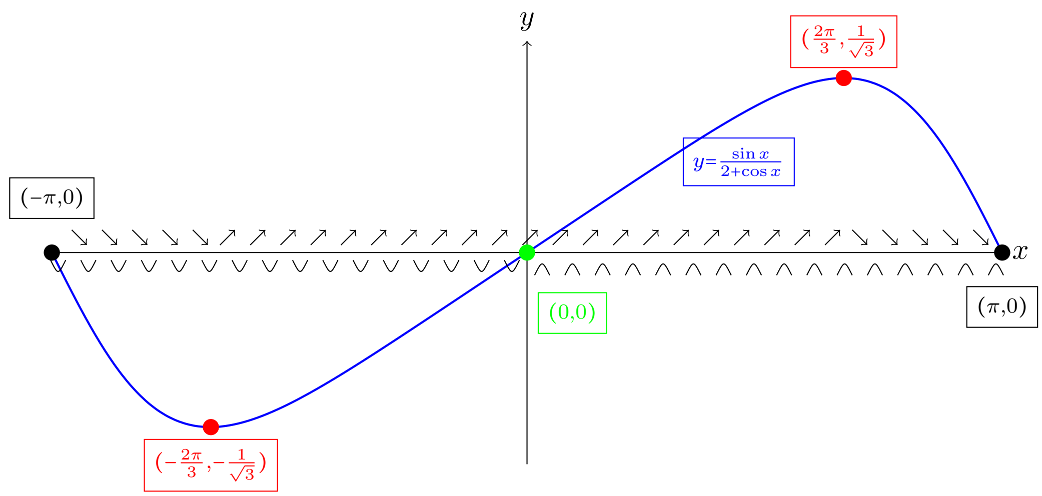
Thus restricted to $[\pi, \pi]$:

$$\boxed{f \text{ is concave up on } \quad (0,\pi) \quad \text{and concave down on } \quad (-\pi, 0)}$$

Also f changes concavity at $x = 0$. Since $f(0) = 0$:

$$\boxed{\text{The inflection point of } f \text{ in } [\pi,\pi] \text{ is } \quad (0,0)}$$

H) The graph of f on the interval $[\pi, \pi]$:



The whole graph of f: