Quiz 5/Solutions

1. Suppose $xy + \cos y = x^2$. Find y' and y'' when $x = 0$ and $y = \frac{\pi}{2}$.

Differentiating both sides of the $xy + \cos y = x^2$ with respect to x we obtain:

$$\begin{aligned} 1y + xy' - \sin y y \cdot y' &= 2x \\ xy' - \sin y \cdot y' &= 2x - y \\ (x - \sin y)y' &= 2x - y \\ y' &= \frac{2x - y}{x - \sin y} \end{aligned}$$

Next we use the quotient rule to compute y''.

$$\begin{aligned} y'' &= \left(\frac{2x - y}{x - \sin y}\right)' \\ &= \frac{(2x - y)'(x - \sin y) - (2x - y)(x - \sin y)'}{(x - \sin y)^2} \\ &= \frac{(2 - y')(x - \sin y) - (2x - y)(1 - \cos y \cdot y')}{(x - \sin y)^2} \end{aligned}$$

When $x = 0$ and $y = \frac{\pi}{2}$ we get

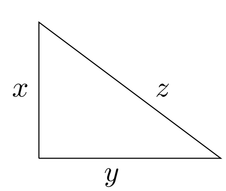
$$y' = \frac{2 \cdot 0 - \frac{\pi}{2}}{0 - \sin\frac{\pi}{2}} = \frac{-\frac{\pi}{2}}{-1} = \boxed{\frac{\pi}{2}}$$

and

$$\begin{aligned} y'' &= \frac{(2 - \frac{\pi}{2})(0 - \sin\frac{\pi}{2}) - (2 \cdot 0 - \frac{\pi}{2})(1 - \cos\frac{\pi}{2} \cdot \frac{\pi}{2})}{(0 - \sin\frac{\pi}{2})^2} \\ &= \frac{(2 - \frac{\pi}{2}) \cdot (-1) - (-\frac{\pi}{2})(1 - 0 \cdot \frac{\pi}{2})}{(-1)^2} \\ &= \frac{\frac{\pi}{2} - 2 + \frac{\pi}{2}}{1} \\ &= \boxed{\pi - 2}. \end{aligned}$$

□

2. A two piece extension ladder leaning against a wall is collapsing at a rate ot 2 feet per second at the same time as its foot is moving away from the wall at a rate of 3 feet per second. How fast is the top of ladder moving down the wall when the top is 8 feet from the ground and the foot is 6 feet from the wall.



The ladder is collapsing at a rate of $2\frac{\text{ft}}{\text{sec}}$. So

$$\frac{dz}{dt} = -2$$

The foot of the ladder is moving away from the wall at rate of $3\frac{\text{ft}}{\text{sec}}$. So

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3$$

We need to compute $\frac{\mathrm{d}x}{\mathrm{d}t}$ when $x = 6$ and $y = 8$. From the Pythagorean Theorem

$$x^2 + y^2 = z^2$$

and differentiating with respect to t gives

$$\begin{aligned} 2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} &= 2z\frac{\mathrm{d}z}{\mathrm{d}t} \\ x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t} &= z\frac{\mathrm{d}z}{\mathrm{d}t} \\ x\frac{\mathrm{d}x}{\mathrm{d}t} &= z\frac{\mathrm{d}z}{\mathrm{d}t} - y\frac{\mathrm{d}y}{\mathrm{d}t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{z\frac{\mathrm{d}z}{\mathrm{d}t} - y\frac{\mathrm{d}y}{\mathrm{d}t}}{x} \end{aligned}$$

For $x = 8$ and $y = 6$ we get

$$z^2 = x^2 + y^2 = 8^2 + 6^2 = 64 + 16 = 100 = 10^2$$

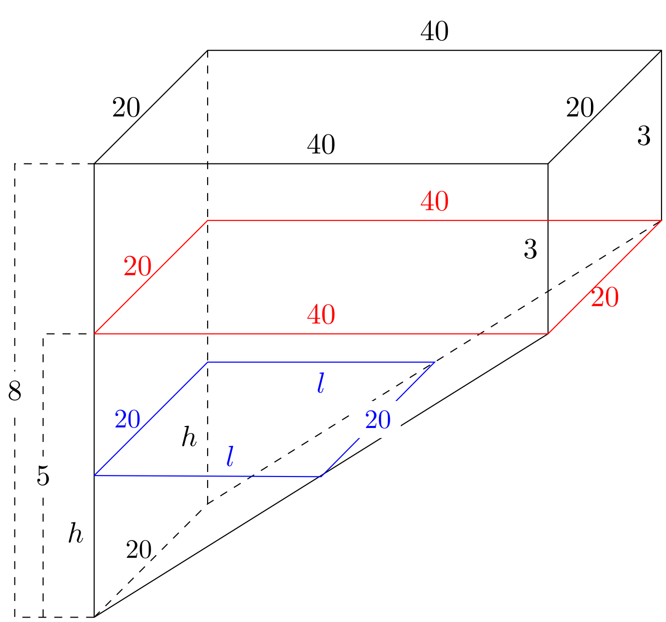
and so $z = 10$. Thus

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{z\frac{\mathrm{d}z}{\mathrm{d}t} - y\frac{\mathrm{d}y}{\mathrm{d}t}}{x} = \frac{10\cdot(-2) - 6\cdot 3}{8} = -\frac{20+18}{8} = -\frac{38}{8} = -\frac{19}{4} = -4.75$$

Hence the top of the ladder is moving down the fall at a rate of

$$\boxed{4.75\frac{\text{ft}}{\text{sec}}}.$$

3. A swimming pool is 40ft long, 20ft wide, 8ft deep at the deep end and 3ft deep at the shallow end. The bottom is rectangular. If the pool is filled at a rate of $40\frac{\text{ft}^3}{\text{min}}$, how fast is the water level rising when it is 3ft deep at the deep end.



Let V be the volume of the water in the pool and h depth of the water at the deep end. We know that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 40\frac{\mathrm{ft}^3}{\mathrm{min}}$$

and we need to find $\frac{\mathrm{d}h}{\mathrm{d}t}$ when $h = 3\mathrm{ft}$. The top surface of the water forms a rectangle of length say l and width 20. Then the water forms a triangular prism whose base triangle has area $\frac{1}{2}hl$ and has witdh 20ft. So the volume of the water is

$$V = \frac{1}{2}hl \cdot 20 = 10hl$$

The right triangle with height h and length l is similar to the right triangle with height 5 and length 40 (where 5 is the difference in depth of the deep and shallow end). Thus

$$\frac{h}{5} = \frac{l}{40}$$
$$l = 40\frac{h}{5} = 8h$$

Substituting into the formula for the volume gives

$$V = 10h \cdot 8h = 80h^2$$

Differentiating with respect to t gives

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 80 \cdot 2h\frac{\mathrm{d}h}{\mathrm{d}t} = 160h\frac{\mathrm{d}h}{\mathrm{d}t}$$

and so

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\frac{\mathrm{d}V}{\mathrm{d}t}}{160h}$$

Recall that $\frac{\mathrm{d}V}{\mathrm{d}t} = 40$. So when $h = 3$ we get

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{40}{160 \cdot 3}\frac{\mathrm{ft}}{\mathrm{min}} = \boxed{\frac{1}{12}\frac{\mathrm{ft}}{\mathrm{min}}} = \boxed{1\frac{\mathrm{in}}{\mathrm{min}}}$$