MTH 132 **Exam 1/Solutions** **Fall 2018**

Standard Response Questions. Error Carried Forward

#1. (9 pts) Calculate the following limits or show that they do no exist:

(a) (4 pts) $\lim_{x\to-1}\frac{x^2-1}{x+1}=$

Solution:

$$\lim_{x\to-1}\frac{x^2-1}{x+1}=\lim_{x\to-1}\frac{(x-1)(x+1)}{x+1}=\lim_{x\to-1}x-1=-1-1=\boxed{-2}$$

(b) (5 pts) $\lim_{x\to0}\left(\frac{1}{x}-\frac{1}{|x|}\right)=$

Solution: We will compute the left- and right-hand limit.

If $x\to0^+$, then $x>0$ and so $|x|=x$. Thus

$$\lim_{x\to0^+}\left(\frac{1}{x}-\frac{1}{|x|}\right)=\lim_{x\to0^+}=\left(\frac{1}{x}-\frac{1}{x}\right)=\lim_{x\to0^+}0=0$$

If $x\to0^-$, then $x<0$ and so $|x|=-x$. Thus

$$\begin{aligned}\lim_{x\to0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)&=\lim_{x\to0^-}\left(\frac{1}{x}-\frac{1}{-x}\right)\\&=\lim_{x\to0^-}\left(\frac{1}{x}+\frac{1}{x}\right)\\&=\lim_{x\to0^-}\frac{2}{x}\quad\left(=\frac{\text{positive}}{\text{small negative}}\right)\\&=-\infty\end{aligned}$$

As the left- and the right-hand limit are different,
$\boxed{\text{the limit does not exist and also is neither }\infty\text{ nor }-\infty}$

#2. (5 pts) Find the value of a that makes the function continuous at $x=0$.

$$f(x)=\begin{cases}\frac{\sin(-8x)}{x} & \text{if } x<0\\ 3x+6a-7 & \text{if } x\geq0\end{cases}$$

Solution: f is continuous at 0 if and only if the left- and right-hand limit at 0 exists and are equal. We compute

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(-8x)}{x} = \lim_{x\to 0^-} -8\frac{\sin(-8x)}{-8x} = -8\lim_{y\to 0^+}\frac{\sin y}{y} = -8\cdot 1 = -8$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 3x + 6a - 7 = 3\cdot 0 + 6a - 7 = 6a - 7$$

So the limit exists if and only if

$$\begin{aligned} 6a - 7 &= -8 \\ 6a &= -1 \\ a &= \boxed{-\frac{1}{6}} \end{aligned}$$

#3. (7 pts) The length of a rectangle is decreasing at a rate of 4cm/s and its width is increasing at a rate of 5cm/s. When the length is 12cm and the width 10cm, how fast is the area of the rectangle changing? Is the area increasing or decreasing at that time? (*Include units*)

Solution: Let A, l, w be the area, length and width of the rectangle, respectively. We know that

$$\frac{\mathrm{d}l}{\mathrm{d}t} == -4 \qquad \text{and} \qquad \frac{\mathrm{d}w}{\mathrm{d}t} = 5$$

and need to find $\frac{\mathrm{d}A}{\mathrm{d}t}$ when $l = 12$ and $w = 10$.

The area of a rectangle is length times width. So

$$A = lw.$$

Differentiating with respect to t and using the product rule gives

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}lw}{\mathrm{d}t} = \frac{\mathrm{d}l}{\mathrm{d}t}w + l\frac{\mathrm{d}w}{\mathrm{d}t}$$

For $l = 12$ and $w = 10$, and using that $\frac{\mathrm{d}l}{\mathrm{d}t} = -4$ and $\frac{\mathrm{d}w}{\mathrm{d}t} = 5$ we get

$$\frac{\mathrm{d}A}{\mathrm{d}t} = (-4)\cdot 10 + 12\cdot 5 = -40 + 60 = \boxed{20\frac{\mathrm{cm}^2}{\mathrm{s}}}$$

#4. (7 pts) Given $f(x) = x^2 + 10\sin x$.

(a) (1 pts) Indicate the interval where the function is continuous.

Solution: Both x^2 and $\sin x$ are continuous functions with domain $(-\infty, \infty)$. So $f(x)$ is defined and is continuous on $\boxed{(-\infty, \infty)}$. □

(b) (4 pts) Prove that there is a number c such that $f(c) = 1$, using the Intermediate Value Theorem.

Solution: We need to find a value of f that is smaller than 0 and a value that is larger than 1. We compute

$$f(0) = 0^2 + 10\sin(0) = 0 + 10 \cdot 0 = 0 < 1$$
$$f(\pi) = \pi^2 + 10 \cdot \sin(\pi) = \pi^2 + 10 \cdot 0 = \pi^2 > 3^2 = 9 > 1$$

Thus 1 is between $f(0)$ and $f(\pi)$. Since f is continuous on $[0, \pi]$ the Intermediate Value Theorem now shows that there exists a number c in $(0, \pi)$ with $f(c) = 1$.

(c) (2 pts) Using (b) state an interval where c can be found.

Solution: As seen in (b) the number c can be found in the open interval

$$\boxed{(0, \pi)}$$

#5. (8 pts) Compute the derivative of the following functions: (**DO NOT SIMPLIFY**)

(a) (4 pts) $f(x) = x \sec x$

Solution: Using the product rule we compute

$$f'(x) = (x \sec x)' = x' \sec x + x \sec' x = \boxed{1 \sec x + x \tan x \sec x}$$

(b) (4 pts) $g(x) = \frac{x^3+1}{6x^2+7}$.

Solution: Using the quotient rule we compute

$$\begin{aligned} g'(x) &= \left(\frac{x^3+1}{6x^2+7}\right)' \\ &= \frac{(x^3+1)'(6x^2+7) - (x^3+1)(6x^2+7)'}{(6x^2+7)^2} \\ &= \boxed{\frac{3x^2(6x^2+7) - (x^3+1)12x}{(6x^2+7)^2}} \end{aligned}$$

#6. (6 pts) Find the equation of the tangent line to the curve $y = \sin\left(\frac{\pi x^2}{4}\right)$ at the point $(1, \frac{\sqrt{2}}{2})$.

Solution: Using the Chain Rule we compute

$$y' = \left(\sin\left(\frac{\pi x^2}{4}\right)\right)' = \sin'\left(\frac{\pi x^2}{4}\right)\left(\frac{\pi x^2}{4}\right)' = \cos\left(\frac{\pi x^2}{4}\right)\frac{\pi 2x}{4} = \frac{\pi x}{2}\cos\left(\frac{\pi x^2}{4}\right)$$

At $x = 1$ we have

$$y = \sin\left(\frac{\pi 1^2}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

and

$$y' = \frac{\pi 1}{2}\cos\left(\frac{\pi 1^2}{4}\right) = \frac{\pi}{2}\cos\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\frac{1}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

Thus the equation of the tangent line is

$$\boxed{y - \frac{1}{\sqrt{2}} = \frac{\pi}{2\sqrt{2}}(x-1)}.$$

#7. (7 pts) Given $y = \sqrt{x}$, use the definition of the derivative to compute y'.

Solution:

$$\begin{aligned} y' &= \lim_{z\to x} \frac{f(z) - f(x)}{z - x} \\ &= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{z - x} \\ &= \lim_{z\to x} \frac{\sqrt{z} - \sqrt{x}}{z - x}\frac{\sqrt{z} + \sqrt{x}}{\sqrt{z} + \sqrt{x}} \\ &= \lim_{z\to x} \frac{\sqrt{z}^2 - \sqrt{x}^2}{(z-x)(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z\to x} \frac{z - x}{(z-x)(\sqrt{z} + \sqrt{x})} \\ &= \lim_{z\to x} \frac{1}{\sqrt{z} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} \\ &= \boxed{\frac{1}{2\sqrt{x}}} \end{aligned}$$

#8. (6 pts) Consider $y^2 + xy + \frac{3}{y} = 4 + x^2$. Use implicit differentiation to find y'.

Solution: Differentiating both sides of

$$y^2 + xy + \frac{3}{y} = 4 + x^2$$

with respect to x gives

$$
\begin{aligned}
2yy' + (1y + xy') - \frac{3}{y^2}y' &= 2x \\
2yy' + xy' - \frac{3}{y^2}y' &= 2x - y \\
\left(2y + x - \frac{3}{y^2}\right) y' &= 2x - y \\
y' &= \boxed{\frac{2x - y}{2y + x - \frac{3}{y^2}}}
\end{aligned}
$$

Multiple Choice Circle the single best answer. No work needed. No partial credit available.

#9. (4 pts) Given that $f(x) = |x - 5|$, which of the following statements is true.

A. $f(x)$ is continuous and differentiable on $(-\infty, \infty)$.

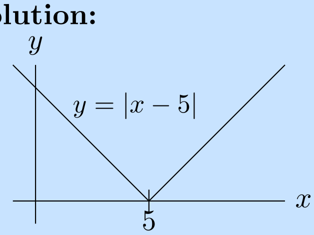
B. $\boxed{f(x) \text{ is continuous on } (-\infty, \infty) \text{ and differentiable on } (-\infty, 5) \cup (5, \infty)}$

C. $f(x)$ is continuous and differentiable on $(-\infty, 5) \cup (5, \infty)$.

D. $f(x)$ is differentiable on $(-\infty, \infty)$, but not continuous at $x = 5$.

E. $f(x)$ is not defined at $x = 5$.

Solution:



From the graph we see that $|x - 5|$ is continuous everywhere and differentiable at all points except 5. Thus B. is the best answer.

#10. (4 pts) Suppose that $f(x)$ is continuous and differentiable and that $f'(x) > 0$ always and $f(0) = 3$. What is true about $f(1)$?

A. It is possible that $f(1) = 3$.

B. It must be that $f(1) < 3$.

C. $\boxed{\text{It must be that } f(1) > 3.}$

D. There is not enough information.

Solution: Since $f'(x)$ (that is, the instantaneous rate of change) is always positive, also any average rate of change is positive. In particular, the average rate of change on the interval $[0, 1]$ is positive. Hence $\frac{f(1)-f(0)}{1-0} > 0$ and so $f(1) > f(0) = 3$. Thus C. is the correct answer.

Remark: This was not a rigorous argument. A real proof will be provided later in the semester

#11. (4 pts) Given that $x^2 + y^2 = 9$, which of the following is true?
(**Hint**: *Use implicit differentiation or sketch the graph.*)

A. $y' > 0$ always.

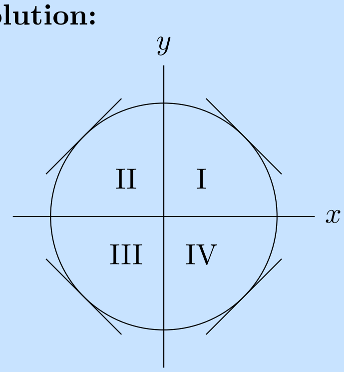
B. $y' < 0$ always.

C. $y' > 0$ in the I and III quadrants.

D. $\boxed{y' > 0 \text{ in the II and IV quadrants.}}$

E. None of the above.

Solution:



Note the y' is the slope of the tangent line to the circle $x^2 + y^2 = 9$. From the picture we see that the slope is negative in the first and third quadrant, and positive in the second and fourth quadrant. Thus D. is the correct answer.

#12. (12 pts) Suppose the height of an object is modeled by $h(t) = 10t - 2t^2$ m, with time measured in seconds.

(a) (4 pts) When does the object reach its maximum height?

A. $\boxed{2.5\text{s}}$
B. 25s
C. 10s
D. 4s.
E. None of the above.

Solution: The maximum height will be reached at a point in time when the velocity is 0. We compute

$$v(t) = h'(t) = (10t - 2t^2)' = 10 - 4t = 4(2.5 - t).$$

So $v(t) = 0$ only for $t = 2.5$. Thus A. is the correct answer.

(b) (4 pts) What is the maximum height of the object?

A. 25m

B. 12.5m

C. 50m

D. 2.5m

E. None of the above.

Solution: By part a) the maximum height is reached when $t = 2.5$. So the maximum height is

$$h(2.5) = 10 \cdot 2.5 - 2 \cdot (2.5)^2 = 25 - 2 \cdot 2.5 \cdot 2.5 = 25 - 5 \cdot 2.5 = 25 - 12.5 = 12.5$$

So B. is the correct answer.

(c) (4 pts) What is the direction of the object at the time $t = 4$s

A. downward

B. upward

C. There is not enough information.

Solution: Recall from part a) that $v(t) = 10 - 4t$. So

$$v(4) = 10 - 4 \cdot 4 = 10 - 16 = -6 < 0$$

Thus the object is moving downwards, that is A. is the correct answer.

#13. (4 pts) Use the squeeze theorem to evaluate $\lim_{x\to 0} \sqrt{\frac{x^3 + x^2}{\pi}} \sin\frac{\pi}{x}$.

A. 1

B. 0

C. DNE

D. $\frac{1}{\pi}$

Solution: Put

$$f(x) = \sqrt{\frac{x^3 + x^2}{\pi}} \sin\frac{\pi}{x}, \quad g(x) = \sqrt{\frac{x^3 + x^2}{\pi}} \quad \text{and} \quad h(x) = \sin\frac{\pi}{x}$$

Then $f(x) = g(x)h(x)$.

Observe that $x^3 + x^2 = x^2(x+1)$ and so $x^3 + x^2 \geq 0$ for $x \geq -1$. So $g(x) = \sqrt{\frac{x^3+x^2}{\pi}}$ is defined on $[-1, \infty]$. Also $h(x) = \sin \frac{\pi}{x}$ is defined for all $x \neq 0$. So the domain of $f(x) = g(x)h(x)$ is $[-1, 0) \cup (1, \infty)$.

Note that $-1 \leq \sin(x) \leq 1$ for all x and so $|\sin(x)| \leq 1$. Thus also

$$|h(x)| = \left|\sin \frac{\pi}{x}\right| \leq 1$$

for all $x \neq 0$.

If follows that

$$|f(x)| = |g(x)h(x)| = |g(x)||h(x)| \leq |g(x| \cdot 1 = |g(x)|$$

for all x in the domain of f.

Since $-|a| \leq a \leq |a|$ for any number a we get

$$-|g(x)| \leq -|f(x)| \leq f(x) \leq |f(x)| \leq |g(x|$$

and so

$$-|g(x)| \leq f(x) \leq |g(x)|$$

Note that $g(x)$ and so also $|g(x)|$ is a continuous function. Hence

$$\lim_{x\to 0} |g(x)| = |g(0)| = \left|\sqrt{\frac{0^3 + 0^2}{\pi}}\right| = 0$$

and so

$$\lim_{x\to 0} -|g(x)| = -\lim_{x\to 0} |g(x)| = 0$$

We proved that

$$-|g(x)| \leq f(x) \leq |g(x)|$$

and

$$\lim_{x\to 0} -|g(x)| = 0 = \lim_{x\to 0} |g(x)|$$

The Squeeze Theorem now implies that also

$$\lim_{x\to 0} f(x) = 0$$

Thus B. is the correct answer.

#14. (4 pts) Calculate the derivative of $f(x) = \cos(\tan x)$.

A. $f'(x) = -\sin(\sec^2 x)$

B. $f'(x) = \sin(\tan x) \sec^2 x$

C. $\boxed{f'(x) = -\sin(\tan x) \sec^2 x}$

D. $f'(x) = \cos(\sec^2 x)$

E. $f'(x) = -\sin x \sec^2 x$

Solution: Using the chain rule we compute:

$$f'(x) = \big(\cos(\tan x)\big)' = \cos'(\tan x)\tan'(x) = -\sin(\tan x)\sec^2 x.$$

So the correct answer is C.

#15. (4 pts) The velocity of a particle moving back and forth along a straight line is given by $v(t) = 2\sin(\pi t) + 3\cos(\pi t)$, where time is measured in seconds.
What does $v'(t) = 2\pi\cos(\pi t) - 3\pi\sin(\pi t)$ represent?

A. The average rate of change of the position of the particle over any 1-second interval.

B. The instantaneous rate of change of the velocity.

C. The speed at which the particle is moving.

D. The average rate of change of the velocity of the particle over any 1-second interval.

E. The instantaneous rate of change of the position.

Solution: If f is any function, then f' is the instantaneous rate of change of f. Thus $v'(t)$ is the instantaneous rate of change of the velocity $v(t)$. Hence B. is the correct answer.

More Challenging Question(s). Error Carried Forward.

#16. (14 pts) Newton's Law of Gravitation says the magnitude of the force, F, exerted by a body of mass m on the body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

(a) (4 pts) Calculate $\frac{\mathrm{d}F}{\mathrm{d}r}$.

Solution: Observe that G, m and M are constant. Thus

$$\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{\mathrm{d}}{\mathrm{d}r}\frac{GmM}{r^2} = GmM\frac{\mathrm{d}}{\mathrm{d}r}r^{-2} = GmM(-2)r^{-3} = \boxed{\frac{-2GmM}{r^3}}$$

(b) (2 pts) Explain the physical meaning of $\frac{\mathrm{d}F}{\mathrm{d}r}$.

Solution: $\frac{dF}{dr}$ is the instantaneous rate of change of the force F with the respect to the distance r. So $\frac{dF}{dr}$ determines how the exerted force F changes when the distance r between the planets changes.

(c) (4 pts) Suppose the earth attracts an object with a force that decreases at a rate of 2N/km when $r = 20,000$km. How is this force changing when $r = 10,000$km? (*Include units*).

Solution: Let $r_1 = 20,000$ and $r_2 = 10,000$. Then $r_1 = 2r_2$. We know that $\frac{dF}{dr} = -2$ when $r = r_1$. So using the formula for $\frac{dF}{dr}$ from part (a):

$$-2 = \left.\frac{dF}{dr}\right|_{r=r_1} = \frac{-2GmM}{r_1^3} = \frac{-2GmM}{(2r_2)^3} = \frac{1}{2^3}\frac{-2GmM}{r_2^3} = \frac{1}{8}\left.\frac{dF}{dr}\right|_{r=r_2}$$

Thus when $r = 10,000 = r_2$ we have

$$\frac{dF}{dr} = 8 \cdot (-2) = \boxed{-16\text{N/km}}$$