Standard Response Questions. Show all your work to receive credit. Please **BOX** your final answer.

#1. (18 pts)

(a) (6 pts) Find the most general antiderivative of $f(x) = 4\cos x + 8$.

Solution:

$$\boxed{4\sin x + 8x + C}$$

(b) (6 pts) Evaluate $\int_1^2 \frac{5-7x^6}{x^4}\,\mathrm{d}x$.

Solution:

$$\begin{aligned}
\int_1^2 \frac{5-7x^6}{x^4}\,\mathrm{d}x &= \int_1^2 5x^{-4} - 7x^2\,\mathrm{d}x \\
&= \left[-\frac{5}{3}x^{-3} - \frac{7}{3}x^3\right]_1^2 \\
&= \left(-\frac{5}{3}2^{-3} - \frac{7}{3}2^3\right) - \left(-\frac{5}{3}1^{-3} - \frac{7}{3}1^3\right) \\
&= \boxed{\left(-\frac{5}{24} - \frac{56}{3}\right) - \left(-\frac{5}{3} - \frac{7}{3}\right)} \\
&= \frac{-5}{24} + \frac{-56+5+7}{3} \\
&= \frac{-5}{24} + \frac{-44}{3} \\
&= -\frac{5+8\cdot 44}{24} \\
&= -\frac{5+352}{24} \\
&= \boxed{-\frac{357}{24}}
\end{aligned}$$

This answer will get you full credit, since you were not asked to simplify.

(c) (6 pts) Let $F(x) = \int_{x^3}^1 \frac{1}{t^2+1}\,\mathrm{d}x$. Find $F'(x)$.

Solution:

$$\begin{aligned}
F'(x) &= \left(\int_{x^3}^1 \frac{1}{t^2+1}\,\mathrm{d}x\right)' \\
&= \left(-\int_1^{x^3} \frac{1}{t^2+1}\,\mathrm{d}x\right)' \\
&= -\frac{1}{(x^3)^2+1}\cdot (x^3)' \\
&= \boxed{-\frac{3x^2}{x^6+1}}
\end{aligned}$$

#2. (18 pts)

(a) (8 pts) A particle is moving along a line with acceleration (in $\mathrm{m/s^2}$) given by $a(t) = 4t^3 + 2\sin t$. Given that the initial velocity is $v(0) = 5\,\mathrm{m/s}$, find the velocity at the time $t = \pi$ seconds.

Solution: Since $v' = a$, we know that v is an antiderivative of $4t^3 + 2\sin t$. Hence

$$v = t^4 - 2\cos t + C$$

and so

$$5 = v(0) = 0^4 - 2\cos 0 + C = -2 + C$$

and

$$C = 5 + 2 = 7$$

Thus

$$v = t^4 - 2\cos t + 7$$

and

$$v(\pi) = \pi^4 - 2\cos\pi + 7 = \pi^4 - 2\cdot(-1) + 7 = \boxed{(\pi^4 + 9)\,\mathrm{m/s}}$$

(b) (10 pts) Use a linearization to find a good approximation of $\sqrt{9.01}$.

Solution: Since we can compute $\sqrt{9}$ we use the linearization $L(x)$ of $f(x) = \sqrt{x}$ at $a = 9$.

$$\begin{aligned} a &= 9 \\ f(x) &= \sqrt{x} \\ f'(x) &= \frac{1}{2\sqrt{x}} \\ L(x) &= f(a) + f'(a)(x-a) \\ &= \sqrt{9} + \frac{1}{2\sqrt{9}}(x-9) \\ &= 3 + \frac{1}{6}(x-9) \end{aligned}$$

and so

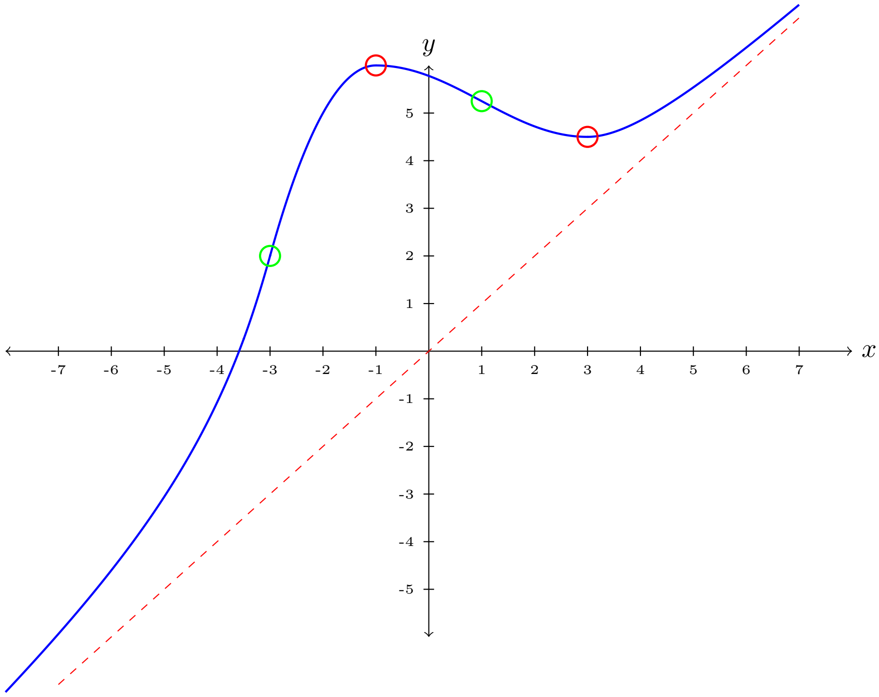
$$\sqrt{9.01} \approx L(9.01) = 3 + \frac{1}{6}(9.01 - 9) = 3 + \frac{0.01}{6} = 3 + \frac{1}{600} = \boxed{3\frac{1}{600}}$$

#3. (18 pts) sx

(a) (10 pts) There are many curves $y = f(x)$ which satisfy the following conditions:

- f is continuous and the curve $y = f(x)$ has a slant (or oblique) asymptote given by $y = x$.
- $f'(x) > 0$ for $x \in (-\infty, -1) \cup (3, \infty)$ and $f'(x) < 0$ for $x \in (-1, 3)$.
- $f''(x) > 0$ for $x \in (-\infty, -3) \cup (1, \infty)$ and $f''(x) < 0$ for $x \in (-3, 1)$.

Sketch the graph of one such curve, making sure that all the above conditions are demonstrated by the curve you draw. Identify with a large dot • the locations of any local max/min or points of inflections on your graph, and give the x-values of these points in the boxes below.



Fill in this information:

f has local minimums at these x-values:

$\boxed{x = 3}$

f has local maximums at these x-values:

$\boxed{x = -1}$

f has inflection points at these x-values:

$\boxed{x = -3 \quad \text{and} \quad x = 1}$

(b) (8 pts) Find the critical numbers (i.e., critical points) of the function $f(x) = x^{3/2} + \frac{6}{\sqrt{x}}$.

Solution: Observe first that the domain of f is $(0, \infty)$.

$$\begin{aligned} f'(x) &= \left(x^{3/2} + \frac{6}{\sqrt{x}}\right)' \\ &= \left(x^{3/2} + 6x^{-1/2}\right)' \\ &= \frac{3}{2}x^{1/2} - \frac{1}{2}6x^{-3/2} \\ &= \frac{3x^{1/2}}{2} - \frac{3}{x^{3/2}} \\ &= \frac{3x^{1/2}x^{3/2} - 3 \cdot 2}{2x^{3/2}} \\ &= \frac{3}{2}\frac{x^2 - 2}{x^{3/2}} \end{aligned}$$

$f'(x) = 0$ when $x^2 - 2 = 0$ and so for $x = \pm\sqrt{2}$. Of these two only $\sqrt{2}$ is the domain of f.

$f'(x)$ is not defined when $x^{3/2} = 0$ and so at $x = 0$. But 0 is not in the domain of f, so the only critical number is

$$\boxed{x = \sqrt{2}}$$

#4. (18 pts) Suppose $f(x) = \frac{x}{x^2+1}$, $\quad f'(x) = \frac{1-x^2}{(x^2+1)^2}$, $\quad f''(x) = \frac{2(x^3-3x)}{(x^2+1)^3}$.

Answer the following questions or enter **none** *in the case of no answer.*

(a) (4 pts) Does f have symmetry about the y-axis (even function), symmetry about the origin (odd function), both, or neither? Justify your answer.

Solution:

$$f(-x) = \frac{-x}{(-x)^2 + 1} = -\frac{x}{x^2 + 1} = -f(x) \neq f(x)$$

Hence $\boxed{f \text{ is an odd function, but not an even function}}$.

(b) (7 pts) Find the largest interval(s) where f is increasing and the largest interval(s) where f is decreasing. Express you answer using interval notation.

Solution: We know that $f'(x) = \frac{1-x^2}{(x^2+1)^2}$.

Note that $1 - x^2 = 0$ for $x = \pm 1$. Also $x^2 + 1 \geq 1$ and so $x^2 + 1$ is never 0. Thus $f'(x)$ is defined for all x, and $f'(x) = 0$ for $x = \pm 1$. We compute

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$(x^2 - 1)$	$+$	$-$	$+$
$(x^2 + 1)^2$	$+$	$+$	$+$
f'	$+$	$-$	$+$
f	$\nearrow$	$\searrow$	$\nearrow$

Since f is continuous we conclude that

$$\boxed{f(x) \text{ is increasing on } (-\infty, -1] \text{ and } [1, \infty)}$$

and

$$\boxed{f(x) \text{ is decreasing on } [-1, 1]}$$

(c) (7 pts) Find the interval(s) where f is concave up and the largest interval(s) where f is concave down. Express you answer using interval notation.

Solution: We know that $f''(x) = \frac{2(x^3-3x)}{(x^2+1)^3}$. Note that

$$x^3 - x = (x^2 - 3)x$$

so $f''(x) = 0$ at $x = \pm\sqrt{3}$ and at $x = 0$. $x^2 + 1$ is never zero, so $f''(x)$ is defined for all x.

We compute

	$(-\infty, -\sqrt{3})$	$(-\sqrt{3}, 0)$	$(0\ \sqrt{3})$	$(\sqrt{3}, \infty)$
$x^2 - 3$	$+$	$-$	$-$	$+$
x	$-$	$-$	$+$	$+$
$(x^2 + 1)^3$	$+$	$+$	$+$	$+$
f''	$-$	$+$	$-$	$+$
f	$\cap$	$\cup$	$\cap$	$\cup$

Thus

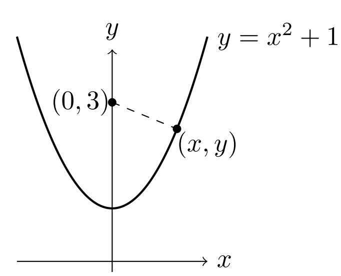
$$\boxed{f(x) \text{ is concave up on } \ (-\sqrt{3}, 0) \text{ and } (\sqrt{3}, \infty)}$$

and

$$\boxed{f(x) \text{ is concave down on } \ (-\infty, -\sqrt{3}) \text{ and } (0, \sqrt{3})}$$

#5. (18 pts)

(a) (8 pts) Consider the problem of finding the point (x, y) that lies on the curve $y = x^2 + 1$ in the first quadrant and which is closest to the point $(0, 3)$.
Define a function f of x which, if minimized, will give the x-coordinate of the point on the curve closest to $(0, 3)$. Also give the domain of this function which is *appropriate for the minimization problem.*



Solution: The distance from $(0,3)$ to (x,y) is

$$d = \sqrt{(x-0)^2 + (y-3)^2} = \sqrt{x^2 + (y-3)^2}$$

Since (x,y) is on the curve $y = x^2 + 1$, we get

$$d = \sqrt{x^2 + (x^2+1-3)^2} = \sqrt{x^2 + (x^2-2)^2}$$

So we can minimize the function

$$d(x) = \boxed{\sqrt{x^2 + (x^2-2)^2}}$$

But for easier computation it would be better to minimize the function

$$d^2(x) = \boxed{x^2 + (x^2-2)^2}$$

Both functions are defined for all real numbers x. But we are supposed to find a point in the first quadrant. Some people define the first quadrant by $x \geq 0$ and $y \geq 0$, but others as $x > 0$ and $y > 0$. So both

$$\boxed{[0,\infty)} \quad \text{and} \quad \boxed{(0,\infty)}$$

are acceptable answers for the domain of the function.

(b) (10 pts) Find the absolute maximum and the absolute minimum values of

$$f(x) = x^2(2x-8)$$

on the interval $[-1,2]$.

Solution: Since f is continuous, we can use the closed interval method.

$$f(x) = x^2(2x-8) = 2x^2(x-4) = 2(x^3-4x^2)$$

and so

$$f'(x) = 2\left(x^3-4x^2\right)' = 2(3x^2-8x) = 2x(3x-8)$$

Hence $f'(x)$ is defined for all x. Also $f'(x) = 0$ for $x = 0$ and $x = \frac{8}{3} = 2\frac{2}{3}$. Of these two only $x = 0$ is in the interval $(-1,2)$. So we need to compute $f(x)$ for $x = -1, 0$ and 2. Using $f(x) = 2x^2(x-4)$ we compute

$$\begin{aligned} f(-1) &= 2\cdot(-1)^2\cdot(-1-4) = 2\cdot 1\cdot(-5) = -10 \\ f(0) &= 2\cdot 0^2\cdot(0-4) = 0 \\ f(2) &= 2\cdot 2^2\cdot(2-4) = 2\cdot 4\cdot(-2) = -16 \end{aligned}$$

Of these three, 0 is the largest and -16 the smallest. Thus

$\boxed{0}$ is the absolute maximum value and $\boxed{-16}$ is the absolute minimum value of f on $[-1,2]$.

Multiple Choice Circle the best answer. No work needed.
No partial credit available. No credit will be given for choices not clearly marked.

#6. (7 pts) If the Mean Value Theorem is applied to the function $f(x) = x^2 - 2x$ on the interval $[1,4]$ which of the following values of c satisfy the conclusion of the Mean Value Theorem in this case?

A. $c = 1$ B. $c = \frac{3}{2}$ $\boxed{\text{C. } c = 2}$ D. $c = \frac{5}{2}$ E. $c = 3$

Solution: We need to find c in $(1,4)$ with

$$f'(c) = \frac{f(4) - f(1)}{4 - 1} = \frac{(4^2 - 2 \cdot 4) - (1^2 - 2 \cdot 1)}{3} = \frac{(16 - 8) - (-1)}{3} = \frac{9}{3} = 3$$

Note that

$$f'(x) = \left(x^2 - 2x\right)' = 2x - 1$$

So

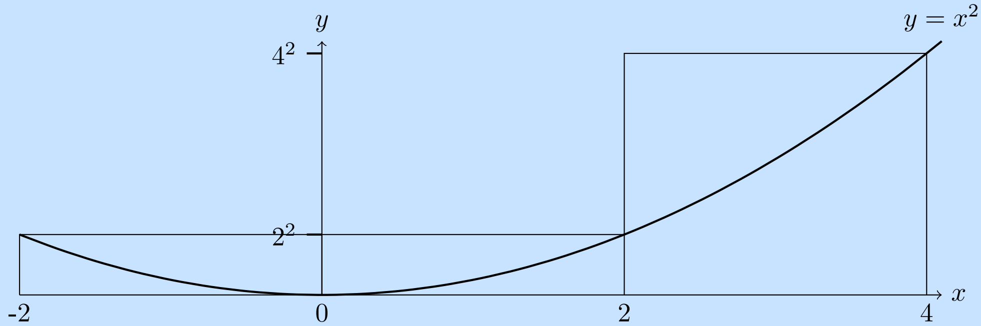
$$2c - 1 = 3$$
$$2c = 3 + 1 = 4$$
$$c = 2$$

Thus C. is the correct answer.

#7. (7 pts) Using three equally-spaced rectangles of equal width, find the upper sum approximation of the area between the curve $y = x^2$ and the x-axis from $x = -2$ to $x = 4$.

A. 8 B. 16 C. 24 D. 40 E. 48

Solution:



Each interval has length 2. So the upper sum approximation is

$$2^2 \cdot 2 + 2^2 \cdot 2 + 4^2 \cdot 2 = 8 + 8 + 32 = 48$$

So the correct answer is E.

#8. (7 pts) Evaluate $\int_0^3 \sqrt{9-x^2}$. (*Hint: A definite integral represents an area.*)

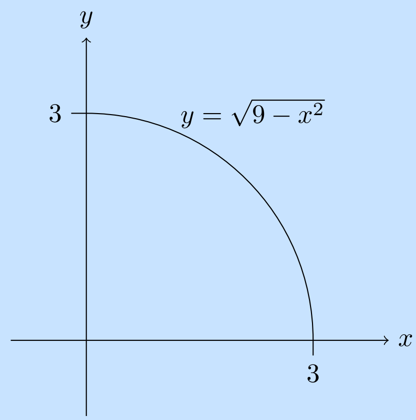
A. $\frac{3\pi}{4}$ B. $\frac{3\pi}{2}$ **C. $\frac{9\pi}{4}$** D. $\frac{9\pi}{2}$ E. 9π

Solution:

The definite integral is the area of the region between $y = 0$ and $y = \sqrt{9-x^2}$ from $x = 0$ to $x = 3$. This region is exactly one quarter of a circle of radius $r = 3$. The area of a circle of radius r is πr^2. Thus

$$\int_0^3 \sqrt{9-x^2}\,\mathrm{d}x = \frac{1}{4}\pi 3^2 = \frac{9}{4}$$

Hence C. is the correct answer.



#9. (7 pts) Which of the following is the equation for the horizontal asymptote for the curve $y = \frac{9x-2}{5-2x}$?

A. $y = \frac{9}{2}$ **B. $y = -\frac{9}{2}$** C. $y = 0$ D. $y = -\frac{5}{2}$ E. $y = \frac{5}{2}$

Solution:

$$\lim_{x\to\pm\infty} \frac{9x-2}{5-2x} = \lim_{x\to\pm\infty} \frac{x(9-\frac{2}{x})}{x(\frac{5}{x}-2)} = \lim_{x\to\pm\infty} \frac{9-\frac{2}{x}}{\frac{5}{x}-2} = \frac{9-0}{0-2} = -\frac{9}{2}$$

So B. is the correct answer.

#10. (7 pts) $\int_0^4 |3-x|\,\mathrm{d}x =$?

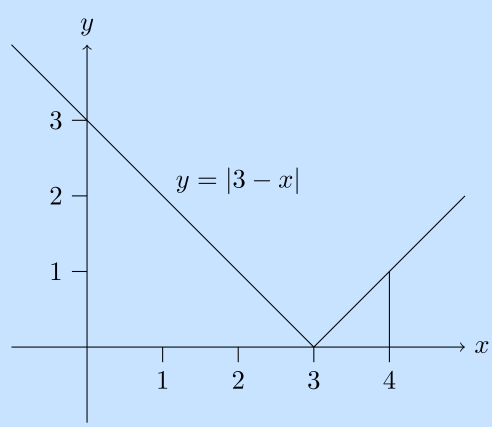
A. 6 **B. 5** C. 4 D. 3 E. 2

Solution 1:

The definite integral is the area of the region between $y = 0$ and $y = |3-x|$ from $x = 0$ to $x = 4$. This region consists of two triangles: One of height 3 and base 3 and one of height 1 and base 1. The area of a triangle with height h and base b is $\frac{1}{2}hb$. So

$$\int_0^4 |3-x|\,\mathrm{d}x = \frac{1}{2}\cdot 3\cdot 3 + \frac{1}{2}\cdot 1\cdot 1 = \frac{9+1}{2} = \frac{10}{2} = 5$$

Hence B. is the correct answer.



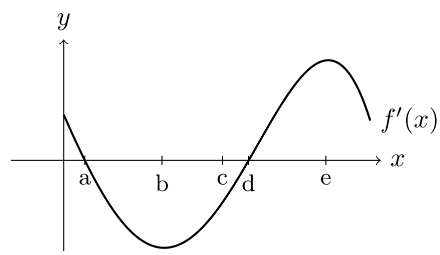
Solution 2: If $x < 3$, then $3 - x > 0$ and so $|3-x| = 3 - x$. If $x > 3$, then $3 - x < 0$ and so $|3-x| = -(3-x) = x - 3$. Thus

$$
\begin{aligned}
\int_0^4 |3-x|\,\mathrm{d}x &= \int_0^3 |3-x|\,\mathrm{d}x \;+\; \int_3^4 |3-x|\,\mathrm{d}x \\
&= \int_0^3 3-x\,\mathrm{d}x \;+\; \int_3^4 x-3\,\mathrm{d}x \\
&= \left[3x-\frac{1}{2}x^2\right]_0^3 + \left[\frac{1}{2}x^2-3x\right]_3^4 \\
&= (3\cdot 3-\frac{1}{2}3^2)-(3\cdot 0-\frac{1}{2}0^2)+(\frac{1}{2}4^2-3\cdot 4)-(\frac{1}{2}3^2-3\cdot 3) \\
&= (9-\frac{9}{2})-0+(8-12)-(\frac{9}{2}-9) \\
&= \frac{9}{2}-4+\frac{9}{2} \\
&= 9-4 \\
&= 5
\end{aligned}
$$

So B. is the correct answer.

#11. (7 pts) The graph of the first derivative $f'(x)$ of a function $f(x)$ is shown. At what value of x does f have a local maximum?

[A. $x=a$] B. $x=b$ C. $x=c$ D. $x=d$ E. $x=e$



Solution: We use the first derivative test: f has a local maximum at $x=a$ if f changes from increasing to decreasing, and so if f' changes from positive to negative. According to the graph of $f'(x)$ this happens only at $x=a$. So A. is the correct answer.

#12. (7 pts) Which of the following definite integrals is equivalent to the following limit of Riemann sums?

$$\lim_{n\to\infty}\sum_{i=1}^{n}\sqrt{8+\frac{5i}{n}}\cdot\frac{5}{n}$$

A. $\int_8^{13}\sqrt{8+5x}\,\mathrm{d}x$ B. $\int_0^5\sqrt{8+x}\,\mathrm{d}x$ C. $\int_0^1\sqrt{8+5x}\,\mathrm{d}x$ D. $\int_0^5 5\sqrt{8+x}\,\mathrm{d}x$ E. none of the above

Solution: According to the Right End Point Rule

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. So we need to find f, a and b such that

$$f(a+i\Delta x)\cdot\Delta x = \sqrt{8+\frac{5i}{n}}\cdot\frac{5}{n}$$

To identify the various terms better it is best to isolate the coefficient of i:

$$
\begin{aligned}
&f(a+i\Delta x)\cdot\Delta x \\
=&\sqrt{8+i\frac{5}{n}}\cdot\frac{5}{n}
\end{aligned}
$$

So we can choose $f(x)=\sqrt{x}$, $a=8$ and $\Delta x=\frac{5}{n}$. It remains to compute b. From $\Delta x=\frac{b-a}{n}$ we see that

$$b = a + n\Delta x = 8 + n\frac{5}{n} = 8 + 5 = 13$$

Thus the limit of Rieman sums is equal to

$$\int_8^{13} \sqrt{x}\,\mathrm{d}x$$

Unfortunately this is not one of the answers. Lets get back to the equation to solve

$$f(a + i\Delta) \cdot \Delta x$$
$$= \sqrt{8 + i\frac{5}{n}} \cdot \frac{5}{n}$$

and see whether we can find a different solution. If we look at the possible answers, a is either 8 or 0. We tried $a = 8$, but it did not work. To try $a = 0$ we write the equation as

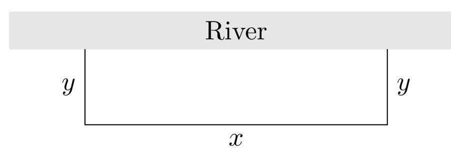
$$f(a + i\Delta) \cdot \Delta x$$
$$= \sqrt{8 + (0 + i\frac{5}{n})} \cdot \frac{5}{n}$$

So we can choose $f(x) = \sqrt{8 + x}$, $a = 0$ and $\Delta x = \frac{5}{n}$. Thus $b = a + n\Delta x = 0 + n\frac{5}{n} = 5$ and so the limit of Riemann sums is equal to

$$\int_0^5 \sqrt{8 + x}\,\mathrm{d}x$$

Thus B. is the correct answer.

#13. (7 pts) A farmer wants to build a rectangular pen which will be bounded on one side by a river and on the other three sides by a wire fence. If the farmer has 60 m of wire to use, what is the largest area the farmer can enclose.



A. $200\,\mathrm{m}^2$ B. $400\,\mathrm{m}^2$ C. $450\,\mathrm{m}^2$ D. $600\,\mathrm{m}^2$ E. $900\,\mathrm{m}^2$

Solution: We are trying to maximize the area

$$A = xy$$

The farmer has 60 m of wire so

$$x + 2y = 60$$

It is easier to solve for x than for y. So I choose y as the independent variable.

$$x = 60 - 2y \quad \text{and} \quad A = xy = (60 - 2y)y = 60y - 2y^2$$

Both x and y can not be negative. So $60 - 2y = x \geq 0$ and $y \geq 0$. Hence $0 \leq y \leq 30$.
We compute

$$\frac{\mathrm{d}A}{\mathrm{d}y} = \frac{\mathrm{d}}{\mathrm{d}y}(60y - 2y^2) = 60 - 4y = 4 \cdot (15 - y)$$

Hence $\frac{\mathrm{d}A}{\mathrm{d}y}$ is defined for all y, and $\frac{\mathrm{d}A}{\mathrm{d}y} = 0$ for $y = 15$.

To justify that maximum value occurs at $y = 15$. one can use the closed interval method. If $y = 0$, then $A = xy = 0$. If $y = 15$, then $x = 60 - 2 \cdot 15 = 30$ and $A = xy = 15 \cdot 30 = 450$. If $y = 30$ then $x = 0$ and $A = xy = 0$. 450 is the largest of these three numbers, so the maximal area is $450\,\mathrm{m}^2$. Thus C. is the correct answer.

(One also could have use the First Derivative Test for Extreme Values: Since $\frac{\mathrm{d}A}{\mathrm{d}y} = 4 \cdot (15 - y)$ we see that $\frac{\mathrm{d}A}{\mathrm{d}y} > 0$ for $y < 15$, and $\frac{\mathrm{d}A}{\mathrm{d}y} < 0$ for $y > 15$. Hence A has a maximum value at $y = 15$.)

#14. (7 pts) Suppose $\int_2^5 f(x)\,\mathrm{d}x = 3$ and $\int_2^3 f(x)\,\mathrm{d}x = -4$. Find $\int_3^5 2f(x)\,\mathrm{d}x$.

A. -1 B. 7 C. -2 D. 14 E. 28

Solution: We have

$$3 = \int_2^5 f(x)\,\mathrm{d}x = \int_2^3 f(x)\,\mathrm{d}x + \int_3^5 f(x)\,\mathrm{d}x = -4 + \int_3^5 f(x)\,\mathrm{d}x$$

and so

$$\int_3^5 f(x)\,\mathrm{d}x = 3 - (-4) = 7$$

and

$$\int_3^5 2f(x)\,\mathrm{d}x = 2\int_3^5 f(x)\,\mathrm{d}x = 2 \cdot 7 = 14$$

Thus D. is the correct answer.