MTH 132 **Exam 1** **Fall 2016**

Standard Response Questions. Show all your work to receive credit. Please $\boxed{\textbf{BOX}}$ your final answer.

#1. (18 pts)

(a) (10 pts) Consider the curve $y^2 + xy + x^3 = 3$. Find the slope of the curve at the point $(1, -2)$.
Computing $\frac{\mathrm{d}}{\mathrm{d}x}$ on both sides of the equation we get

$$\begin{aligned} 2yy' + 1y + xy' + 3x^2 &= 0 \\ 2yy' + xy' &= -(y + 3x^2) \\ (2y + x)y' &= -(y + 3x^2) \\ y' &= -\frac{y + 3x^2}{2y + x} \end{aligned}$$

For $x = 1$ and $y = -2$ we get

$$y' = -\frac{-2 + 3 \cdot 1^2}{2 \cdot (-2) + 1} = -\frac{1}{-3} = \frac{1}{3}$$

So the slope is $\boxed{\frac{1}{3}}$. □

(b) (8 pts) If $f(x) = \sec(\sin(x^2 + x))$, what is $f'(x)$? (*Do not simplify your answer!*).

$$\begin{aligned} f'(x) &= \big(\sec(\sin(x^2 + x))\big)' \\ &= \sec'(\sin(x^2 + x)) \sin'(x^2 + x)(x^2 + x)' \\ &= \boxed{\tan(\sin(x^2 + x)) \sec(\sin(x^2 + x)) \cos(x^2 + x)(2x + 1)} \end{aligned}$$

□

#2. (18 pts) Consider $f(x) = \sqrt{1-2x}$.

(a) (12 pts) Use the definition of the derivative to find $f'(x)$.

$$\begin{aligned}
f'(x) &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x} \\
&= \lim_{z\to x} \frac{\sqrt{1-2z}-\sqrt{1-2x}}{z-x} \\
&= \lim_{z\to x} \frac{\left(\sqrt{1-2z}-\sqrt{1-2x}\right)\left(\sqrt{1-2z}+\sqrt{1-2x}\right)}{(z-x)\left(\sqrt{1-2z}+\sqrt{1-2x}\right)} \\
&= \lim_{z\to x} \frac{(\sqrt{1-2z})^2-(\sqrt{1-2x})^2}{(z-x)\left(\sqrt{1-2z}+\sqrt{1-2x}\right)} \\
&= \lim_{z\to x} \frac{(1-2z)-(1-2x)}{(z-x)\left(\sqrt{1-2z}+\sqrt{1-2x}\right)} \\
&= \lim_{z\to x} \frac{-2(z-x)}{(z-x)\left(\sqrt{1-2z}+\sqrt{1-2x}\right)} \\
&= \lim_{z\to x} \frac{-2}{\left(\sqrt{1-2z}+\sqrt{1-2x}\right)} \\
&= \frac{-2}{\left(\sqrt{1-2x}+\sqrt{1-2x}\right)} \\
&= \frac{-2}{2\sqrt{1-2x}} \\
&= \boxed{-\frac{1}{\sqrt{1-2x}}}
\end{aligned}$$

(b) (6 pts) Use part (a) to find an equation of the tangent line of $f(x)$ at $x=-4$.

$$\begin{aligned}
f(-4) &= \sqrt{1-2\cdot(-4)} = \sqrt{9} = 3 \\
f'(-4) &= -\frac{1}{\sqrt{1-2\cdot(-4)}} = -\frac{1}{\sqrt{9}} = -\frac{1}{3}
\end{aligned}$$

So the equation of the tangent line is

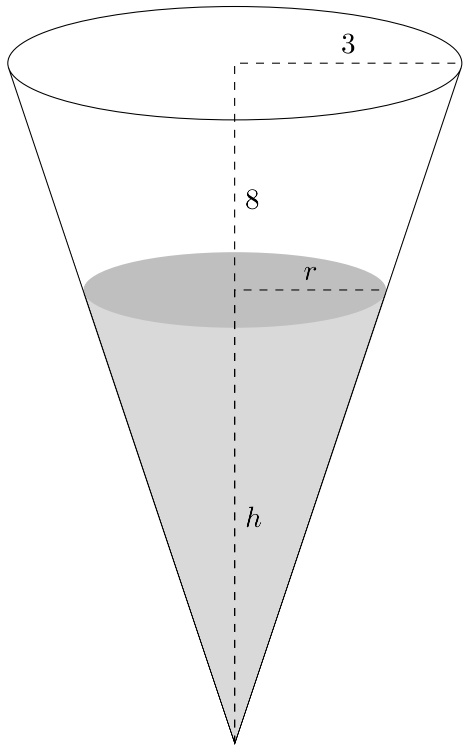
$$y-4 = -\frac{1}{3}(x-(-4)),$$

that is

$$\boxed{y-4 = -\frac{1}{3}(x+4)}$$

□

#3. (18 pts) A filter filled with liquid is in the shape of a vertex-down cone with a height of 8 inches and a diameter of 6 inches at its open (upper) end. If liquid drips out of the bottom of the filter at the constant rate of 7 in^3/min, how fast is the level of the liquid dropping when the liquid is 5 inches deep?



Note first that the radius of the filter cone is $\frac{6}{2} = 3$. Let V be the volume, h the height and r the radius of the cone formed by the water. Then we know that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -7$$

and we want to compute $\frac{\mathrm{d}h}{\mathrm{d}t}$ when $h = 5$.
Considering similar triangles we obtain a equation relating r and h:

$$\frac{r}{h} = \frac{3}{8}$$

and so

$$r = \frac{3}{8}h$$

Observe that the volume of a cone of height h and radius r is

$$V = \frac{\pi}{3}hr^2 = \frac{\pi}{3}h(\frac{3}{8}h)^2 = \frac{3\pi}{64}h^3$$

Differentiating with respect to t gives

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{3\pi}{64}3h^2\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{9\pi h^2}{64}\frac{\mathrm{d}h}{\mathrm{d}t}$$

and so

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{64}{9\pi h^2}\frac{\mathrm{d}V}{\mathrm{d}t}$$

Recall that $\frac{\mathrm{d}V}{\mathrm{d}t} = -7$. So for $h = 5$ we conclude that

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{64}{9\pi 5^2}(-7) = -\frac{448\pi}{225}$$

The question asked how fast the level of the water is **dropping**, so the final answer is

$$\boxed{\frac{448\pi}{225}\frac{\text{in}}{\text{min}}}$$

Remark: The answer $-\frac{448\pi}{225}\frac{\text{in}}{\text{min}}$ *also would have received full credit* □

#4. (18 pts) A particle moves according to the law of motion $s = t^3 - 6t^2 + 5t, t \geq 0$, where t is measured in seconds and s in feet.

(a) (3 pts) Find the average velocity over the interval $[0, 2]$.

The average velocity on the interval $[0, 2]$ is

$$\frac{s(2) - s(0)}{2 - 0} = \frac{(2^3 - 6 \cdot 2^2 + 5 \cdot 2) - 0}{2 - 0} = \frac{8 - 24 + 10}{2} = \frac{-6}{2} = \boxed{-3 \frac{\text{ft}}{\text{sec}}} \quad \square$$

(b) (4 pts) Find the velocity at the time t.

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} = (t^3 - 6t^2 + 5t)' = \boxed{3t^2 - 12t + 5} \quad \square$$

(c) (3 pts) Whats is the acceleration after 6 seconds?

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = (3t^2 - 12t + 5)' = 6t - 12 = 6(t - 2) \quad \square$$

So at $t = 6$:

$$a = 6(6 - 2) = 6 \cdot 4 = \boxed{24 \frac{\text{ft}}{\text{sec}^2}}. \quad \square$$

(d) (3 pts) What is the speed of the particle when the acceleration is zero? We first compute t when $a = 0$:

$$\begin{aligned} a &= 0 \\ 6(t - 2) &= 0 \\ t - 2 &= 0 \\ t &= 2 \end{aligned}$$

When $t = 2$ we have

$$v = 3 \cdot 2^2 - 12 \cdot 2 + 5 = 12 - 24 + 5 = -7$$

and the speed is

$$\boxed{7 \frac{\text{ft}}{\text{sec}}}. \quad \square$$

(e) (5 pts) For $t \geq 0$, when is the particle moving in the positive direction?

We first compute t when $v = 0$ by using the quadratic formula to solve $3t^2 - 12t + 5 = 0$:

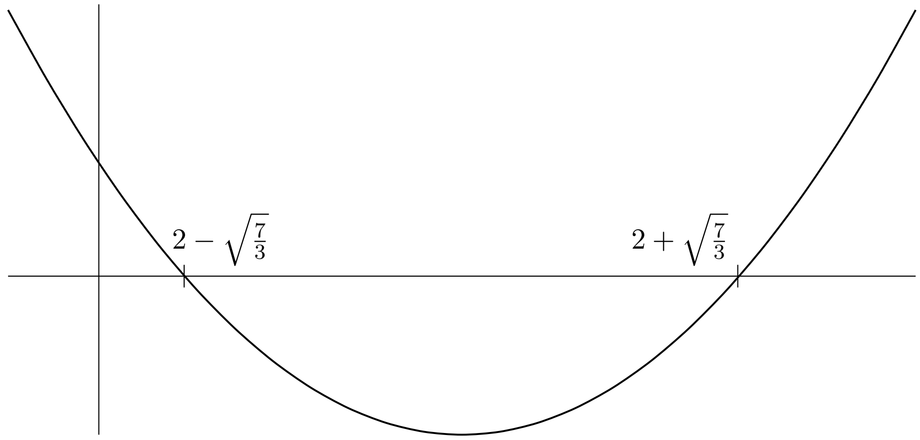
$$t = \frac{-(-12) \pm \sqrt{-(-12)^2 - 4 \cdot 3 \cdot 5}}{2 \cdot 3} = \frac{12 \pm \sqrt{144 - 60}}{6} = \frac{12 \pm \sqrt{84}}{6} = \frac{12 \pm \sqrt{4 \cdot 21}}{6}$$

$$= \frac{12 \pm 2\sqrt{21}}{6} = 2 \pm \frac{\sqrt{21}}{3} = 2 \pm \sqrt{\frac{21}{9}} = 2 \pm \sqrt{\frac{7}{3}}$$

Observe that $\frac{7}{3} < 3 < 4$ and so $\sqrt{\frac{7}{3}} < \sqrt{4} = 2$. Thus

$$0 < 2 - \sqrt{\frac{7}{3}} < 2 + \sqrt{\frac{7}{3}}$$

So the graph of v looks like:



and v will be positive on the intervals $\left(0, 2-\sqrt{\frac{7}{3}}\right)$ and on $\left(2+\sqrt{\frac{7}{3}}\right)$. Hence the particle is moving in the positive direction for all t in

$$\boxed{\left(0, 2-\sqrt{\frac{7}{3}}\right) \cup \left(2+\sqrt{\frac{7}{3}}, \infty\right)} \quad \square$$

#5. (18 pts)

(a) (10 pts) Use the Intermediate Value Theorem to show that there is a solution to the equation $\cos x = \sqrt{x}$. (*Make sure to justify why you can apply the IVT*).

Consider the function $f(x) = \cos x - \sqrt{x}$. Both $\cos x$ and $\sqrt{x}$ are continuous functions, so also $f(x)$ is continuous on its domain, $[0, \infty)$.

We compute

$$\begin{aligned} f(0) &= \cos 0 - \sqrt{0} = 1 - 0 = 1 > 0 \\ f(\pi) &= \cos \pi - \sqrt{\pi} = -1 - \sqrt{\pi} < 0 \end{aligned}$$

Thus 0 is between $f(0)$ and $f(\pi)$ and the IVT shows that there exists a number c in the interval $(0, \pi)$ with $f(c) = 0$. Then $\cos c - \sqrt{c} = 0$ and so $\cos c = \sqrt{c}$. Hence c is a solution of $\cos x = \sqrt{x}$. $\square$

(b) (8 pts) Consider the function $f(x) = \frac{x+2}{\cos(x)}$.

Where is the function continuous on $[0, 2\pi]$? (*Express your answer in interval notation*)

Both $x+2$ and $\cos x$ are continuous function. Hence also the quotient $f(x) = \frac{x+2}{\cos x}$ is a continuous function, that is $f(x)$ is continuous at each point of its domain. The domain of f consists of all x with $\cos(x) \neq 0$. In the interval $[0, 2\pi]$ we have $\cos x = 0$ for $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$. Hence (in the interval $[0, 2\pi]$) $f(x)$ is continuous at each point of

$$\boxed{\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]}$$

Multiple Choice Circle the best answer. No work needed. No partial credit given.

#6. (7 pts) For which real number c does $\lim_{x\to 2} \frac{cx^2+4}{x-2}$ exist and is finite?

A. $c=-2$, $\boxed{\text{B. } c=-1}$, C. $c=0$, D. $c=1$ E. $c=2$.

Note that the denominator of $\frac{cx^2+4}{x-2}$ is 0 for $x=2$. So for the $\lim_{x\to 2} \frac{cx^2+4}{x-2}$ to exists and be finite, also the numerator needs to be 0 at $x=2$. We compute

$$\begin{aligned} c\cdot 2^2+4 &= 0 \\ 4c &= -4 \\ c &= -1 \end{aligned}$$

Thus $c=-1$ is the only value for c where the limit might exists. If $c=-1$, then

$$\lim_{x\to 2} \frac{cx^2+4}{x-2} = \lim_{x\to 2} \frac{-x^2+4}{x-2} = \lim_{x\to 2} -\frac{x^2-4}{x-2} \lim_{x\to 2} -\frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} -(x+2) = -(2+2) = -4$$

So the limit does exist for $c=-1$, and B. is the correct answer. □

#7. (7 pts) Compute $\lim_{x\to -2^-} \frac{|x^2-4|}{x+2}$.

A. $-\infty$, $\boxed{\text{B. } -4}$, C. 0, D. 4 E. ∞.

If $x<-2$, then $|x|>2$ and so $x^2>4$ and $x^2-4>0$. Hence $|x^2-4| = x^2-4$ for $x<-2$. It follows that

$$\lim_{x\to -2^-} \frac{|x^2-4|}{x+2} = \lim_{x\to -2^-} \frac{x^2-4}{x+2} = \lim_{x\to -2^-} \frac{(x+2)(x-2)}{x+2} = \lim_{x\to -2^-} x-2 = (-2)-2 = -4$$

Hence B. is the correct answer. □

#8. (7 pts) Compute the limit: $\lim_{x\to -3^+} \frac{x-2}{x^2(x+3)}$.

$\boxed{\text{A. } -\infty}$, B. -3, C. -1, D. 1 E. ∞.

If x is close to -3 but larger than -3, then $x+3$ is a small positive number. Also $x-2$ is close to -5 (and so negative) and x^2 is close to 9 (and so positive). Hence $\frac{x-2}{x^2(x+3)}$ is a large negative number. Thus

$$\lim_{x\to -3^+} \frac{x-2}{x^2(x+3)} = -\infty$$

So A. is the correct answer. □

#9. (7 pts) For what value of k will $f(x)$ be continuous for all values of x?

$$f(x) = \begin{cases} \frac{x^2-3k}{x-3} & \text{if } x \leq 2 \\ 8x - k & \text{if } x > 2 \end{cases}$$

A. $k = 2$, B. $k = 3$, C. $k = 4$, D. $k = 5$ E. No value of k.

If $x < 2$, then $f(x) = \frac{x^2-3k}{x-3}$ and $x - 3 \neq 0$. Thus $f(x)$ is continuous at values of x less than 2.
If $x > 2$, then $f(x) = 8x - k$ and thus $f(x)$ is continuous at all values of x larger than 2.
So we just need to determine for which values of k the function $f(x)$ is continuous at 2. For this we compute the left and right hand limit if $f(x)$ at $x = 2$:

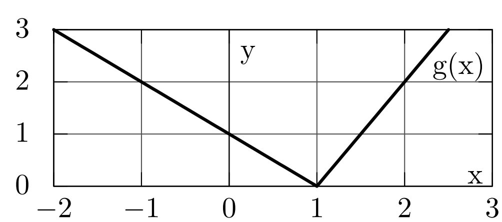
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 3k}{x - 3} = \frac{2^2 - 3k}{2 - 3} = \frac{4 - 3k}{-1} = 3k - 4$$

and

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} 8x - k = 8 \cdot 2 - k = 16 - k.$$

So $f(x)$ is continuous at 2 if and only if $3k - 4 = 16 - k$. and so if and only if $4k = 20$, that is if and only of $k = 5$. Thus D. is the correct answer. □

#10. (7 pts) Given the graph $y = g(x)$ below, find the limit $\lim_{h \to 0} \frac{g(1+h) - g(1)}{h}$.



A. 0, B. 1, C. 2, D. −1 E. Does not exist.

Observe the limit in question is the derivative of g at 1. From the graph we see that the derivative does not exist:
The left hand limit is the slope of the line at the left and so equal to -1.
The right hand limit is the slope of the line at the right and so equal to 2.
Since the left and right hand limits are different, the limit does not exist. Thus E. is the correct answer. □

#11. (7 pts) Let $h(x) = \frac{2G(x)}{1+F(x)}$. Calculate $h'(2)$ if $F(2) = -3$, $G(2) = 5$, $F'(2) = 2$ and $G'(2) = 6$.

A. 6, B. 4, C. 44/9, D. 22 E. −11.

We use the quotient rule to compute $h'(x)$:

$$\begin{aligned} h'(x) &= \left(\frac{2G(x)}{1+F(x)}\right)' \\ &= \frac{(2G(x))'(1+F(x)) - 2G(x)(1+F(x))'}{(1+F(x))^2} \\ &= \frac{2G'(x)(1+F(x)) - 2G(x)F'(x)}{(1+F(x))^2} \end{aligned}$$

Hence

$$\begin{aligned} h'(2) &= \frac{2G'(2)(1+F(2)) - 2G(2)F'(2)}{(1+F(2))^2} \\ &= \frac{2\cdot 6\cdot(1+(-3)) - 2\cdot 5\cdot 2}{(1+(-3))^2} \\ &= \frac{12\cdot(-2)-20}{(-2)^2} \\ &= \frac{-24-20}{4} \\ &= \frac{-44}{4} \\ &= -11 \end{aligned}$$

So E. is the correct answer. □

#12. (7 pts) If $T(x) = 2\sqrt{x} - \frac{1}{2\sqrt{x}}$, then $T'(x) =$

A. $x + \frac{1}{x^{\frac{3}{2}}}$, B. $\frac{1}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}}$, $\boxed{\text{C. } \frac{1}{x^{\frac{1}{2}}} + \frac{1}{4x^{\frac{3}{2}}}}$, D. $\frac{4x-1}{4x^{\frac{3}{2}}}$ E. $\frac{4}{x^{\frac{1}{2}}} + \frac{1}{x^{\frac{3}{2}}}$

$$\begin{aligned} T'(x) &= \left(2\sqrt{x} - \frac{1}{2\sqrt{x}}\right)' \\ &= \left(2x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}\right)' \\ &= 2\frac{1}{2}x^{\frac{1}{2}-1} - \frac{1}{2}(-\frac{1}{2})x^{-\frac{1}{2}-1} \\ &= x^{-\frac{1}{2}} + \frac{1}{4}x^{-\frac{3}{2}} \\ &= \frac{1}{x^{\frac{1}{2}}} + \frac{1}{4x^{\frac{3}{2}}} \end{aligned}$$

Thus C. is the correct answer. □

#13. (7 pts) Find y'' if $y = \sin(x^2)$.

$\boxed{\text{A. } 2\cos(x^2) - 4x^2\sin(x^2)}$, B. $\cos(x^2) - \sin(x^2)$ C. $2x\cos(x^2) - 4x^2\sin(x^2)$
D. $2x\cos(x^2) + 2x\sin(x^2)$, E. $-\sin(x^2)$.

$$y' = (\sin(x^2))' = \sin'(x^2)(x^2)' = \cos(x^2)2x^2 = 2x\cos(x^2).$$

and so

$$\begin{aligned} y'' &= \big(2x\cos(x^2)\big)' \\ &= (2x)'\cos(x^2) + 2x\big(\cos(x^2)\big)' \\ &= 2\cos(x^2) + 2x\cos'(x^2)(x^2)' \\ &= 2\cos(x^2) + 2x \cdot (-\sin(x^2)) \cdot 2x \\ &= 2\cos(x^2) - 4x^2\sin(x^2) \end{aligned}$$

Hence A. is the correct answer. □

#14. (7 pts) Find the limit $\lim_{x\to 0} \dfrac{\sin(x^2+6x)}{x}$.

A. 0, B. 1, C. −1, D. 6 E. Does not exist.

$$\begin{aligned} \lim_{x\to 0} \frac{\sin(x^2+6x)}{x} &= \lim_{x\to 0} \frac{\sin(x^2+6x)}{x} \, \frac{x^2+6x}{x^2+6x} \\ &= \lim_{x\to 0} \frac{x^2+6x}{x} \, \frac{\sin(x^2+6x)}{x^2+6x} \\ &= \lim_{x\to 0} \frac{x^2+6x}{x} \, \lim_{x\to 0} \frac{\sin(x^2+6x)}{x^2+6x} \\ &= \lim_{x\to 0}(x+6) \, \lim_{y\to 0} \frac{\sin(y)}{y} \\ &= 6 \cdot 1 \\ &= 6 \end{aligned}$$

Thus D. is the correct answer. □