Multiple Choice. Circle the best answer. No work needed. No partial credit available.

1. Evaluate $\lim\limits_{x \to -1} \dfrac{x^2 + 2x + 1}{x + 1}$

   (a) 1
   (b) -1
   (c) 2
   (d) $\boxed{0}$
   (e) None of the above.

2. Evaluate $\lim\limits_{x \to -1} \dfrac{x + 1}{|x + 1|}$

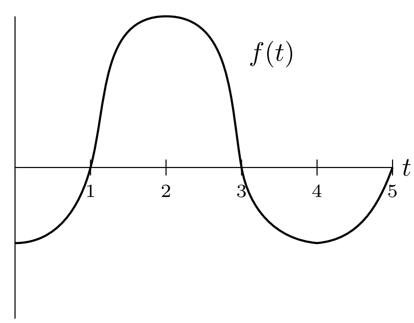
   (a) 1
   (b) -1
   (c) 2
   (d) 0
   (e) $\boxed{\text{None of the above.}}$

3. Find c so that $f(x) = \begin{cases} \frac{\sin(3x+x^2)}{x} & \text{if } x \neq 0 \\ c & \text{if } x = 0 \end{cases}$ is continuous.

   (a) $\boxed{c = 3}$
   (b) $c = 2$
   (c) $c = 1$
   (d) $c = -1$
   (e) None of the above.

4. The graph of $f(t)$ is shown below. If $F(x) = \displaystyle\int_0^x f(t)\, dt$, for what value of x is $F(x)$ an absolute maximum on $[0, 5]$?

   (a) $x = 0$
   (b) $x = 1$
   (c) $x = 2$
   (d) $\boxed{x = 3}$
   (e) None of the above.



5. Evaluate $\displaystyle\int x(2x^2 + 1)^3\, dx$

   (a) $\boxed{\dfrac{1}{16}(2x^2 + 1)^4 + C}$
   (b) $\frac{1}{12}(2x^2 + 1)^4 + C$
   (c) $\frac{1}{8}(2x^2 + 1)^4 + C$
   (d) $\frac{1}{4}(2x^2 + 1)^4 + C$
   (e) None of the above.

Fill in the Blanks. No work needed. Only possible scores given are 0, 3, and 5.

6. If $f(x) = x^2 \sin x$ then $f'(x) = \underline{(2x)(\sin x) + (x^2)(\cos x)}$

7. The vertical asymptote(s) of $f(x) = \dfrac{1-x}{x+2}$ are $\underline{x = -2}$. The horizontal asymptote(s) of $f(x)$ are $\underline{y = -1}$.

8. $\dfrac{d}{dt}\left(\dfrac{(t+\tan t)^2}{\sqrt{t}}\right) = \underline{\dfrac{2(t+\tan t)(1+\sec^2 t)(\sqrt{t}) - (t+\tan t)^2(\frac{1}{2\sqrt{t}})}{t}}$

9. A function which satisfies $y'(x) = 4x$ and $y(1) = 5$ is given by $y(x) = \underline{2x^2 + 3}$

10. Let f and g be differentiable functions such that

$$\begin{array}{lll} f(0) = 2 & f'(0) = 3 & f'(2) = 4 \\ g(0) = 2 & g'(0) = -1 & g'(2) = 5 \end{array}$$

If $h(x) = g(f(x))$ then $h'(0) = \underline{15}$

Extra Work Space.

Standard Response Questions. Show all work to receive credit. Please put your final answer in the $\boxed{\textbf{BOX}}$.

11. (8+4=12 points) Throughout this problem y is defined implicitly as a function of x.

(a) Find the slope of the tangent line to the curve $y^2 + 7x = x^2y + 9$ at the point $(1, 2)$.

Solution.

$$\begin{aligned} 2yy' + 7 &= 2xy + x^2y' \\ 2(2)y' + 7 &= 2(1)(2) + (1)y' \\ 4y' + 7 &= 4 + y' \\ 3y' &= -3 \\ y' &= \boxed{-1} \end{aligned}$$

(b) Write the equation of the tangent line to the curve at the point $(1, 2)$.

Solution.

$$\boxed{y - 2 = -1(x - 1)}$$

or

$$\boxed{y = -x + 3}$$

12. (10 points) Use the *definition* of the derivative *as a limit* to calculate $f'(x)$ for $f(x) = \dfrac{2}{x-3}$. (**There will be no credit for other methods.**)

Solution.

$$\begin{aligned} f(x+h) - f(x) &= \frac{2}{x+h-3} - \frac{2}{x-3} \\ f(x+h) - f(x) &= \frac{2(x-3)}{(x+h-3)(x-3)} - \frac{2(x+h-3)}{(x+h-3)(x-3)} \\ f(x+h) - f(x) &= \frac{-2h}{(x+h-3)(x-3)} \\ \implies \frac{f(x+h) - f(x)}{h} &= \frac{-2}{(x+h-3)(x-3)} \\ \implies \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \frac{-2}{(x-3)(x-3)} \\ \implies f'(x) &= \boxed{\frac{-2}{(x-3)^2}} \end{aligned}$$

13. (6+6+6+4=22 points) Let $f(x) = x + 2\cos(x)$ on the interval $[0, 2\pi]$.

(a) Find the intervals within $[0, 2\pi]$ on which f is increasing and the intervals on which f is decreasing.

Solution. $f'(x) = 1 - 2\sin x$ giving us crit points at $\pi/6$ and $5\pi/6$. Choosing test points and sketching a number line we get our answer:

$$\boxed{f \text{ is increasing on } [0, \pi/6) \cup (5\pi/6, 2\pi]}$$

$$\boxed{f \text{ is decreasing on } (\pi/6, 5\pi/6)}$$

(b) Find the intervals within $[0, 2\pi]$ on which f is concave up and the intervals on which f is concave down.

Solution. $f''(x) = -2\cos x$. giving us possible inflection points at $\pi/2$ and $3\pi/2$. Choosing test points and sketching a number line we get our answer:

$$\boxed{f \text{ is con up on } (\pi/2, 3\pi/2)}$$

$$\boxed{f \text{ is con down on } [0, \pi/2) \cup (3\pi/2, 2\pi]}$$

(c) Indicate the y-intercept, any local maxes/mins, and inflection points.

Solution. Based on the info above we can determine:

$$\begin{aligned} y - \text{int} &: \boxed{(0, 2)} \\ \text{local max} &: \boxed{\pi/6} \\ \text{local min} &: \boxed{5\pi/6} \\ \text{inflection points} &: \boxed{\pi/2, 3\pi/2} \end{aligned}$$

(d) Locate the absolute maximum of f on $[0, 2\pi]$.

Solution. Need to test the points $\pi/6$ and 2π as our only maximum candidates.

$$\begin{aligned} f(\pi/6) &= \pi/6 + 2(\sqrt{3}/2) \approx .5 + 1.7 = 2.2 \\ f(2\pi) &= 2\pi + 2 \approx 8 \end{aligned}$$

Giving us a clear winner of $\boxed{x = 2\pi}$ as the absolute max.

14. (8+8+8=24 points) Evaluate the following integrals:

(a) $\displaystyle\int \frac{x^3}{\sqrt{1+2x^4}}\,dx$

Solution.

$$\begin{aligned}\int \frac{x^3}{\sqrt{1+2x^4}}\,dx &= \int \frac{x^3}{\sqrt{u}}\,dx && (u = 1+2x^4)\\ &= \int \frac{x^3}{\sqrt{u}}\left(\frac{du}{8x^3}\right) && (du = 8x^3dx)\\ &= \left[2\sqrt{u}\right]\left(\frac{1}{8}\right)+C = \boxed{\frac{\sqrt{1+2x^4}}{4}+C}\end{aligned}$$

(b) $\displaystyle\int \frac{\sec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)}{\sqrt{\sec\left(\frac{x}{2}\right)}}\,dx$

Solution.

$$\begin{aligned}\int \frac{\sec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)}{\sqrt{\sec\left(\frac{x}{2}\right)}}\,dx &= \int \frac{\sec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)}{\sqrt{u}}\,dx && (u=\sec\left(\tfrac{x}{2}\right))\\ &= \int \frac{\sec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)}{\sqrt{u}}\left(\frac{2du}{\sec\left(\frac{x}{2}\right)\tan\left(\frac{x}{2}\right)}\right) && (du = \sec\left(\tfrac{x}{2}\right)\tan\left(\tfrac{x}{2}\right)\left(\tfrac{1}{2}\right)\ dx)\\ &= \left[2\sqrt{u}\right](2)+C = \boxed{4\sqrt{\sec\left(\frac{x}{2}\right)}+C}\end{aligned}$$

(c) $\displaystyle\int_8^{11} x\sqrt{x-7}\,dx$

Solution. Consider

$$\begin{aligned}\int x\sqrt{x-7}\,dx &= \int (u+7)\sqrt{u}\,dx && (u=x-7 \implies x=u+7)\\ &= \int u^{3/2}+7u^{1/2}\,du && (du=dx)\\ &= \left[\frac{2u^{5/2}}{5}+\frac{14u^{3/2}}{3}\right]\\ &= \left[\frac{2(x-7)^{5/2}}{5}+\frac{14(x-7)^{3/2}}{3}\right]+C\end{aligned}$$

Giving us the solution

$$\begin{aligned}\int_8^{11} x\sqrt{x-7}\,dx &= \left[\frac{2(4)^{5/2}}{5}+\frac{14(4)^{3/2}}{3}\right]-\left[\frac{2(1)^{5/2}}{5}+\frac{14(1)^{3/2}}{3}\right]\\ &= \left[\frac{2(32)}{5}+\frac{14(8)}{3}\right]-\left[\frac{2}{5}+\frac{14}{3}\right]\\ &= \boxed{\frac{62}{5}+\frac{98}{3}}\end{aligned}$$

15. (8 points) Use the Fundamental Theorem of Calculus to find the derivative of

$$F(x) = \int_{\pi}^{\sqrt{x}} \frac{2\sin(t^2) - 1}{\sqrt{t^4 + 1}}\, dt$$

Solution.

$$\begin{aligned} F'(x) &= \frac{2\sin(x) - 1}{\sqrt{x^2 + 1}} \frac{d}{dx}\left(\sqrt{x}\right) \\ &= \boxed{\left(\frac{2\sin(x) - 1}{\sqrt{x^2 + 1}}\right)\left(\frac{1}{2\sqrt{x}}\right)} \end{aligned}$$

16. (8 points) Estimate $\int_1^4 \frac{2x - 1}{\sqrt{x}}\, dx$ using areas of 3 rectangles of equal width, with heights of the rectangles determined by the height of the curve at left endpoints (Do not simplify).

Solution.

x	1	2	3
$f(x)$	1	$3/\sqrt{2}$	$5/\sqrt{3}$

Giving us the final solution:

$$\begin{aligned} \int_1^4 \frac{2x - 1}{\sqrt{x}}\, dx &\approx 1\left(1 + \frac{3}{\sqrt{2}} + \frac{5}{\sqrt{3}}\right) \\ &\approx \boxed{1 + \frac{3}{\sqrt{2}} + \frac{5}{\sqrt{3}}} \end{aligned}$$

17. (8 points) Use a linear approximation to estimate $\sqrt[3]{26}$.
Hint: Is 26 close to a number whose cube root is well-known?

Solution. Take $f(x) = \sqrt[3]{x}$ and $a = 27$ because $f(a) = f(27) = 3$. The final component we need is $f'(a) = \dfrac{1}{3(\sqrt[3]{27})^2} = \dfrac{1}{27}$ Therefore

$$\begin{aligned} L(x) &= f(a) + f'(a)(x-a) \\ &= 3 + \frac{1}{27}(x-27) \\ L(26) &= 3 + \frac{1}{27}(26-27) \\ &= 3 + \frac{1}{27}(-1) \\ &= \frac{81}{27} - \frac{1}{27} = \boxed{\frac{80}{27}} \end{aligned}$$

18. (12 points) Find the area of the region enclosed by the graphs of the equations $y = 2x^2 + x - 2$ and $y = x^2 - x + 1$.

Solution. First lets find where these intersect by solving:

$$\begin{aligned} 2x^2 + x - 2 &= x^2 - x + 1 \\ x^2 + 2x - 3 &= 0 \\ (x+3)(x-1) &= 0 \\ \implies x &= -3, 1 \end{aligned}$$

Now we check to see which is greater in this interval by taking a test point (0 is a good choice). $0^2 + 0 - 2 < 0^2 - 0 + 1$ so we get the integral:

$$\begin{aligned} Area &= \int_{-3}^{1} \left[(x^2 - x + 1) - (2x^2 + x - 2)\right] \, dx \\ &= \int_{-3}^{1} \left[-x^2 - 2x + 3\right] \, dx \\ &= \left[-\frac{x^3}{3} - x^2 + 3x\right]_{-3}^{1} \\ &= \left[-\frac{1}{3} - 1 + 3\right] - [9 - 9 - 9] \\ &= \boxed{11 - \frac{1}{3}} \end{aligned}$$

19. (12 points) The top of a 13 foot ladder, leaning against a vertical wall, is slipping down the wall at the rate of 2 feet per second. How fast is the bottom of the ladder sliding along the ground away from the wall when the bottom of the ladder is 5 feet away from the base of the wall?

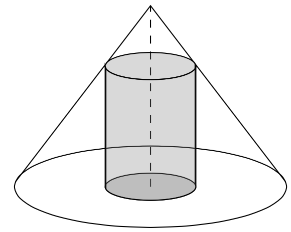
Solution. The information above can be transformed into the following mathematical statements.

$$\begin{aligned} x(t)^2 + y(t)^2 &= h(t)^2 \\ h(t) &= 13 \\ y'(t) &= -2 \end{aligned}$$

We can define t_0 to be the time at which the bottom of the ladder is 5 feet away from the base of the wall (aka $x(t_0) = 5$). It is then our goal to find $|x'(t_0)|$ according to the statement of the problem.

$$\begin{aligned} x(t)^2 + y(t)^2 &= h(t)^2 \\ x(t)^2 + y(t)^2 &= 13^2 \\ 2x(t)x'(t) + 2y(t)y'(t) &= 0 \\ x(t)x'(t) + y(t)(-2) &= 0 \\ x(t_0)x'(t_0) - 2y(t_0) &= 0 \\ 5x'(t_0) - 2y(t_0) &= 0 \\ x'(t_0) &= \frac{2y(t_0)}{5} \\ x'(t_0) &= \frac{2(12)}{5} = \boxed{\frac{24}{5}} && \text{(since } 5^2 + 12^2 = 13^2\text{)} \end{aligned}$$

20. (12 points) A cylinder is inscribed in a right circular cone of height 4 inches and radius (at the base) equal to 3 inches. What are the dimensions of such a cylinder that has maximum volume?
You MUST verify that you have found the maximum.
Hint: Recall the formula for volume of a cylinder: $V = \pi r^2 h$

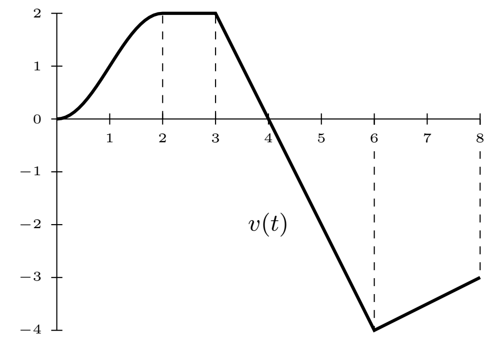


Solution. The main restriction is that we would like the cylinder to push right up against the cone to maximize the volume. Therefore we need the restriction $h = 4 - \frac{4}{3}r$, giving us the volume equation:

$$\begin{aligned} V &= \pi r^2 h && (r \text{ is in } [0,3]) \\ &= \pi r^2 (4 - \frac{4}{3} r) \\ &= 4\pi \left[r^2 - \frac{r^3}{3} \right] \\ \implies V' &= 4\pi \left[2r - r^2 \right] \end{aligned}$$

Giving us critical points are $r = 0$ and $r = 2$. Choosing test points for V' we can see that V is increasing on $(0,2)$ and decreasing on $(2,3]$. Our only candidate for the absolute maximum is $\boxed{r = 2}$. Giving the corresponding $\boxed{h = 4/3}$.

21. (2+2+2+2+2+4+4+4=22 points)
The graph to the right shows the velocity $v(t)$ in meters per second of a particle moving on a horizontal coordinate line, for t in seconds within the closed interval $[0, 8]$.



(a) When is the particle moving forward?

Solution.

$$\boxed{t \in [0, 4]}$$

(b) When is the particle's speed decreasing?

Solution.

$$\boxed{t \in [3, 4] \cup [6, 8]}$$

(c) When is the particle's acceleration positive?

Solution.

$$\boxed{t \in [0, 2] \cup [6, 8]}$$

(d) When is the particle's acceleration the greatest?

Solution.

$$\boxed{t = 1}$$

(e) When does the particle move at its greatest speed?

Solution.

$$\boxed{t = 6}$$

(f) What is the change in the particle's position from $t = 2$ to $t = 6$?

Solution. $2 + 1 - 4 = -1$

$$\boxed{-1}$$

(g) What is the total distance the particle travels from $t = 2$ to $t = 6$?

Solution. $2 + 1 + 4 = 7$

$$\boxed{7}$$

(h) If the particle is at the origin at $t = 2$ use linear approximation to estimate its position at $t = 3/2$

Solution. $L(x) = 0 + 2(x - 2)$

$$\boxed{s(3/2) \approx -1}$$