Appendix E Problems

Sigma Notation

Example 1. Prove the formula for the sum of the first n positive integers.

$$
\begin{array}{rl}
 & 1+2+\cdots+n = A \\
+ & n+(n-1)+\cdots+1 = A \\
\hline
 & (n+1)+\cdots+(n+1) = 2A
\end{array}
$$

$$n(n+1) = 2A. \Leftrightarrow 1+2+\cdots+n = \frac{n(n+1)}{2}.$$

$$2\sum_{i=1}^{n} i = 2(1+\cdots+n)$$

$$= \sum_{i=1}^{n} i + \sum_{i=1}^{n} (n-i+1) = \sum_{i=1}^{n} (n+1) = n(n+1)$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Example 2. Find m and n such that $9 + 27 + 81 + 243 = \sum_{i=m}^{n} 3^i$

$m=2$

$n=5$

$3^2 = 9$

$3^3 = 27$

$3^4 = 81$

$3^5 = 243.$

Example 3. Write the sum in sigma notation:

(a) $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\frac{1}{36}$

$$\sum_{i=1}^{6} \frac{1}{i^2}$$

(b) $3-8+15-24+35-48$

$3=4-1=2^2-1$

$8=9-1=3^2-1$

$15=16-1=4^2-1$

$24=25-1=5^2-1$

$35=36-1=6^2-1$

$48=49-1=7^2-1$

$$\sum_{i=2}^{7} (i^2-1)(-1)^i$$

Example 4. Find the number n such that $\sum_{i=1}^{n} i = 78$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = 78 \leftrightarrow n(n+1) = 156$$

$$n = \frac{-1 \pm \sqrt{1+4\cdot 156}}{2} \rightarrow n = \frac{-1+25}{2} = 12.$$

Example 5. Find the value of the sum.

(a) $\sum_{k=0}^{92} \cos k\pi$

$$\cos(2n\pi) + \cos((2n+1)\pi) = 1 + (-1) = 0$$

$$\therefore \sum_{k=0}^{92} \cos k\pi = \left(\sum_{k=0}^{91} \cos(k\pi)\right) + \cos(92\pi)$$

$$= 0 + 1 = 1.$$

(b) $\sum_{j=1}^{n}(j+1)(j+2)$

$$= \sum_{j=1}^{n} j^2 + \sum_{j=1}^{n} 3j + \sum_{j=1}^{n} 2$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{3n(n+1)}{2} + 2n$$

$$= \frac{2n^3+3n^2+n}{6} + \frac{9n^2+9n}{6} + \frac{12n}{6} = \frac{2n^3+12n^2+22n}{6}$$

(c) $\sum_{i=1}^{n} i(4i-3)$

$$= \sum_{i=1}^{n} 4i^2 - 3i$$

$$= 4\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i$$

$$= 4\,\frac{n(n+1)(2n+1)}{6} - 3\,\frac{n(n+1)}{2}$$

(d) $\sum_{i=5}^{25} -3i^2$ $= -3\sum_{i=5}^{25} i^2 = -3\left[\sum_{i=1}^{25} i^2 - \sum_{i=1}^{4} i^2\right]$

$$= -3\left[\frac{25(25+1)(2\cdot 25+1)}{6} - \frac{4(4+1)(2\cdot 4+1)}{6}\right]$$