4.5b Problems

Question 1. Evaluate

(a) $\int_0^{\pi/2} \cos x\ \sin(\sin x)dx \quad = I$

$u = \sin x$

$du = \cos x dx$

$x = 0 \leftrightarrow u = 0$

$x = \frac{\pi}{2} \leftrightarrow u = 1.$

$I = \int_0^1 \sin(u)\, du = -\cos u \Big|_0^1 = \cos(0) - \cos(1).$

Question 2. Evaluate

(a) $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$ $\quad u = 1+\sqrt{\cancel{x}}$ $\quad du = \frac{1}{2\sqrt{x}}dx = \frac{1}{2(u-1)}dx$ $\quad$ $x=0 \to u=1$, $x=1 \to u=2$.

$$dx = 2(u-1)\,du$$

$$\int_1^2 \frac{2(u-1)}{u^4}du = \int_1^2 \left(2u^{-3} - 2u^{-4}\right)du = \left(-u^{-2} + \frac{2}{3}u^{-3}\right)\Big|_1^2$$

$$= \left(-2^{-2} + \frac{2}{3}2^{-3}\right) - \left(-1 + \frac{2}{3}\right)$$

(b) $\displaystyle\int_0^1 x\sqrt{1-x^4}\,dx = I$; $\quad x^2 = \sin t \leftrightarrow 2x\,dx = \cos t\,dt \leftrightarrow \begin{cases} x=0 & t=0 \\ x=1 & t=\pi/2 \end{cases}$

$$I = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sqrt{1-\sin^2 t}\,\cos t\,dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2 t\,dt$$

$$= \frac{1}{4}\int_0^{\frac{\pi}{2}} (\cos 2t + 1)\,dt = \frac{1}{4}\left\{\left(\frac{1}{2}\sin 2t + t\right)\Big|_0^{\pi/2}\right\}$$

$$= \pi/2.$$

☆ $\cos^2 t = \dfrac{\cos 2t + 1}{2}$

Question 3. Find the average value of the function on the given interval

(a) $g(t) = \dfrac{t}{\sqrt{3+t^2}}$, $[1,3]$

$u = 3 + t^2$; $du = 2t\,dt$; $x=1 \to u=4$

$x = 3 \to u = 12$

$$\int_1^3 \frac{t}{\sqrt{3+t^2}}\,dt = \frac{1}{2}\int_1^3 \frac{2t}{\sqrt{3+t^2}}\,dt$$

$$= \frac{1}{2}\int_4^{12} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\left(2\sqrt{u}\right)\Big|_4^{12}$$

$$= \sqrt{12} - \sqrt{4}$$

Avg value

$$\frac{1}{2}\int_1^3 \frac{t}{\sqrt{3+t^2}}\,dt = \frac{\sqrt{12}-\sqrt{4}}{2}$$

(b) $h(x) = \cos^4 x \sin x$, $[0,\pi]$

$u = \cos x$

$$\int_0^\pi \cos^4 x \sin x\,dx = \int_1^{-1} -u^4\,du = \int_{-1}^1 u^4\,du$$

$$= \frac{1}{5}u^5\Big|_{-1}^1 = \frac{2}{5}.$$

Question 4. If f is continuous on $[0, 1]$, prove that $\int_0^1 f(x)\,dx = \int_0^1 f(1-x)\,dx$

$$\int_0^1 f(1-x)\,dx \qquad u = 1-x \qquad du = -dx$$

$$x = 0 \to u = 1$$

$$x = 1 \to u = 0 .$$

$$= \int_1^0 -f(u)\,du$$

$$= \int_0^1 f(u)\,du$$

$$= \int_0^1 f(x)\,dx \qquad \boxed{\text{QED}}$$