3.2 Problems

Bread and Butter Problems

Example 1. Consider the function $f(x) = \sqrt[3]{x}$ on the interval $[0, 1]$.

(a) Why does f satisfy the hypotheses for the MVT?

f is continuous + differentiable on $\mathbb{R}$.

(b) Find all c values that satisfies the conclusion of the MVT.

MVT: $\exists c$ so that $f(1) - f(0) = f'(c)(1-0)$.

$f(1) - f(0) = 1 \quad \therefore \Rightarrow f'(c) = 1$.

$f'(x) = \frac{1}{3}x^{-2/3} = 1 \Rightarrow \cancel{x} = \frac{1}{\pm 3^{3/2}}$ but $c \in (0,1)$

$\frac{1}{3} = \cancel{1} x^{2/3}$ $\quad \therefore c = \frac{1}{3^{3/2}}$

Example 2. Consider the function $f(x) = x^3 - 3x + 2$ on the interval $[-2, 2]$.

(a) Why does f satisfy the hypotheses for the MVT?

f is continuous + differentiable on $\mathbb{R}$.

(b) Find all c values that satisfies the conclusion of the MVT.

$0 - 4 = f(2) - f(-2) = f'(c)(2-(-2)) \Rightarrow f'(c) = -1$.

$f'(x) = 3x^2 - 3 = -1$ $\quad \therefore$ 2 points satisfy conclusions

$x^2 = \frac{2}{3}$ $\quad \cancel{c} = \pm\sqrt{2/3}$

$x = \pm\sqrt{2/3}$

Head scratchers

Example 3. Suppose f is differentiable, and suppose $f(2) = -3$ and $f'(x) \leq 6$ on $[0, 20]$. What is the maximum value of $f(10)$? *You have more information here than you need, focus only on the interval* $[2, 10]$

f is differentiable on $[2,10]$

$\therefore$ By MVT $\exists\ c \in [2,10]$ so that

$$f(10) + 3 = f(10) - f(2) = f'(c)\,(10-2)$$

$\exists c \in (2,10)$ $\quad f(10) = f'(c)\,8 - 3$ $\quad$ but $f'(x) \leq 6$ $\forall x \in (2,10)$

$$\therefore\ f(10) \leq 6 \cdot 8 - 3 = 45.$$

Example 4. Let $f(x) = (x-3)^{-2}$. Show that there is no value $c \in (1, 4)$ such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

$f'(x) = -2\,(x-3)^{-3}$

$f'(1) = -2(-2)^{-3} = \frac{2}{8} = \frac{1}{4}$

$f'(x) > \frac{1}{4}$ for $x \in (1,3)$

$f'(x) < 0$ for $x \in (3,4)$

$f'(3)$ does not exist.

$\therefore$ There is no $c \in (1,4)$ so that ~~MVT~~ $f(4) - f(1) = f'(c)(4-1)$

But f is not diff or continuous @ $x=3$

So this is not a contradiction.

Example 5. (a) Let $f(x) = (x+1)x(x-1)$. Notice f has 3 roots, use the MVT to show f' has 2 roots.

1st. Apply MVT to interval $[-1,0]$.

f is continuous + diff $\Rightarrow$ there is $c \in (-1,0)$ so that

$$0 = f(0) - f(-1) = f'(c)\,(0-(-1)) = f'(c)$$

$\therefore$ f' has a root in $(-1,0)$.

2nd Apply MVT to interval $[0,1]$.

then $$0 = f(1) - f(0) = f'(c)\,(1-0) = f'(c)$$

$\therefore$ f' has a root in $(0,1)$ $\therefore$ f has 2 roots

(b) Generalize your reasoning to show that if f is differentiable and has 2 roots, then f' has a root.

Suppose f has a root @ a + b, $a \neq b$.

Apply MVT to interval $[a,b]$

there exists $c \in (a,b)$ so that

$$f'(c) = \frac{f(b)-f(a)}{(b-a)} = 0.$$

$\therefore$ f' has a root.

A Beautiful Final Exam Problem

Example 6. Show that the equation $3x + 2\cos x + 5 = 0$ has exactly one real root by:

(a) Using the IVT to show that $3x + 2\cos x + 5 = 0$ has at least one root.

Let $f(x) = 3x + 2\cos x + 5$

(i) $a = -3$. $f(-3) = -9 + 2\cos(-3) + 5 \leq -9 + 2 + 5 = -2 < 0$

(ii) $b = 0$ $f(0) = 3\cdot 0 + 2\cos(0) + 5 = 7 > 0$

$\therefore$ there is $c \in (-3, 0)$ so that

$f(c) = 0$

(b) Using the MVT to show that $3x + 2\cos x + 5 = 0$ has at most one root.

$f(x) := 3x + 2\cos x + 5$.

$f'(x) = 3 + 2(-\sin x)$

$\therefore$ for all $x \in \mathbb{R}$, $f'(x) \geq 3 - 2 = 1$.

But if f has 2 roots then MVT implies

f' has a root. Since f' does not have

a root f has at most 1 roots.