3.1 Problems

Extreme Values

Example 1. What are the two ways in which a function $f(x)$ can have a critical value?

(i) f has a critical value @ x if $f'(x)$ does not exist

(ii) " " " " " " $f'(x) = 0$

Example 2. Find the critical numbers of the functions

(a) $g(t) = t^4 + t^3 + t^2 + 1$; g' exists everywhere $\therefore$ only critical point is $t=0$.

$g'(t) = 4t^3 + 3t^2 + 2t = (4t^2 + 3t + 2)t$

$g'(t) = 0$ only @ $t=0$ $\leftarrow$ $4t^2 + 3t + 2 = 0$

$t = \frac{-3 \pm \sqrt{9 - 4 \cdot 4 \cdot 2}}{2 \cdot 4}$ not real.

(b) $f(x) = x^{3/4} - 2x^{1/4}$

$f' = \frac{3}{4} x^{-\frac{1}{4}} - \frac{1}{2} x^{-\frac{3}{4}}$

let $y = x^{-\frac{1}{4}}$

$x = \frac{1}{y^4}$

$f' = \frac{3}{4} y - \frac{1}{2} y^3 = 0$

$f' = \frac{3}{4} y \left(1 - \frac{2}{3} y^2\right) \Rightarrow y = 0, \pm\sqrt{3/2}$

$\Rightarrow x = 0, 4/9$.

(c) $f(x) = x + \dfrac{1}{x}$ on $[-1, 1]$

$f' = 1 + -\frac{1}{x^2}$ on $[-1, 0) \cup (0, 1]$. $f' \neq 0$ on $(-1, 0) \cup (0, 1)$.

$\therefore$ f has a critical point @ 0

Example 3. Find the absolute maximum and absolute minimum values of f on the given interval

(a) $f(x) = 2x^3 - 3x^2 - 12x + 1$ on $[-2, 3]$

$f' = 6x^2 - 6x - 12 = 0$

$0 = x^2 - x - 2 = (x-2)(x+1)$

critical points @ $-1, 2$.

$f' < 0$ on $(-1, 2)$.

$f' > 0$ on $(-2, -1) \cup (2, 3)$

$\therefore$ local max @ $x = -1$

" min @ $x = 2$.

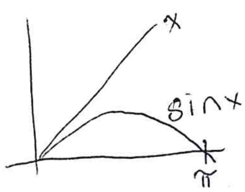
min : $f(2) = 16 - 12 - 24 + 1 = -19$; $f(-2) = -16 - 12 + 24 + 1 = -3$.

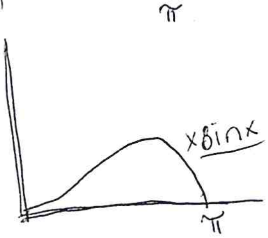
abs min : $f(2)$

max: $f(-1) = -2 - 3 + 12 + 1 = 8$; $f(3) = 54 - 27 - 36 + 1 = -8$

abs max : $(-1, f(-1))$

(b) $f(x) = x \sin x$ on $[0, \pi]$ *start by graphing* x *and* $\sin x$ *then sketch a graph of* $f(x)$
(You won't find the actual maximum for this problem, but you can show it exists!)





$f' = \sin x + x \cos x$

find $f'(x) = 0 \rightarrow$ Show a solution exists using IVT

$f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{\pi}{4}\frac{1}{\sqrt{2}} > 0$

$f'(\pi) = \sin(\pi) + \pi \cos(\pi) = -\pi < 0$

$\exists c \in [\frac{\pi}{4}, \pi]$ so that $f'(c) = 0$.

(c) $f(x) = x + \dfrac{1}{x}$ on $[0.2, 4]$

$f'(x) = 1 + \left(-\frac{1}{x^2}\right)$

$f'(x) = 0 \Rightarrow x = \pm 1.$

$f'(x) \begin{cases} < 0 & x \in (.2, 1) \\ > 0 & x \in (1, 4) \end{cases}$

$\therefore$ absolute min @ $(1, f(1))$

absolute max @ $x = .2$ or 4.

$f(.2) = .2 + \frac{1}{.2} = 5.2$

$f(4) = 4 + \frac{1}{4} = 4.25$

$\therefore$ absolute max at

$(.2, f(.2))$.

$=$

$(.2, 5.2)$

(d) $f(x) = \sin x$ on $\left[\dfrac{-2\pi}{3}, \dfrac{\pi}{6}\right]$

$f' = \cos x$

$f'(x) = 0$ @ $x = -\frac{\pi}{2}$

$\sin\left(\frac{-2\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$

$f(-\frac{\pi}{2}) = -1$

absolute min $\left(-\frac{\pi}{2}, f(-\frac{\pi}{2})\right) = \left(-\frac{\pi}{2}, -1\right)$

absolute max

$\left(\frac{\pi}{6}, \frac{1}{2}\right)$; $\left(-\frac{2\pi}{3}, \frac{1}{2}\right)$.

(e) $f(x) = x\sqrt{4 - x^2}$ on $[-1, 2]$

$f'(x) = \sqrt{4-x^2} + \frac{1}{2} \cdot \frac{x}{\sqrt{4-x^2}}(-2x)$ $\therefore$ $0 = \dfrac{4 - x^2 - x^2}{\sqrt{4-x^2}} = \dfrac{4-2x^2}{\sqrt{4-x^2}}$

$\therefore$ $f'(x) = 0 \Rightarrow x = \sqrt{2}$.

$f(-1) = -\sqrt{5} < f(2) = 0 < f(\sqrt{2}) = 2$.

absolute min: $(-1, f(-1)) = (-1, -\sqrt{5})$

" max: $(\sqrt{2}, f(\sqrt{2})) = (\sqrt{2}, 2)$.

Example 4. Find the critical points and local minima/maxima of f on $[-10, 10]$

$$f(x) = \begin{cases} 2 - 3x^2 + x^4, & x \le 0 \\ -3x + 2, & x > 0 \end{cases}$$

$$f'(x) = \begin{cases} -6x + 4x^3 & x < 0 \\ -3 & x > 0 \end{cases}$$

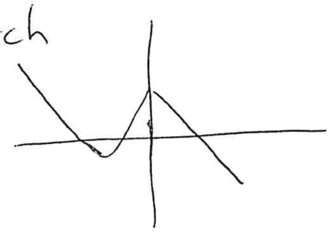
$f'(0)$ does not exist.

$-10 < x < 0$: $f'(x) = -2x(3 - 2x^2) \Rightarrow f'(x) = 0 \Rightarrow x = -\sqrt{\frac{3}{2}}$.

$f'(x) > 0$ on $(-\sqrt{\frac{3}{2}}, 0)$

$f'(x) < 0$ on $(-10, -\sqrt{\frac{3}{2}})$.

sketch



$f(-10) = 2 - 300 + 10000$

$f(-\sqrt{\frac{3}{2}}) = 2 - \frac{9}{2} + \frac{9}{4}$

$f(0) = 2$

$f(10) = -30 + 2$

Absolute max @ $x = -10$

Absolute min @ $x = 10$

local max @ $x = 0$

local min @ ~~…~~ $x = -\sqrt{3/2}$