2.8 Problems

Level 1 Problems

Example 1. A particle is moving along the curve $y = 2\sin(\pi x/2)$. As the particle passes through the point $\left(\frac{1}{3}, 1\right)$ its $x-$coordinate is increasing at a rate of $\sqrt{10}$ cm/s. How fast is the distance from the origin to the particle changing at this instant?

$$D = \text{distance to the origin} = \sqrt{x^2+y^2}$$

$$D^2 = x^2+y^2 = x^2 + \left(2\sin\tfrac{\pi x}{2}\right)^2$$

$$2\,\frac{\sqrt{10}}{3}D' = 2\left(\tfrac{1}{9}+1\right)^{1/2} D' = 2DD' = \frac{d}{dt}D^2 = \frac{d}{dt}\left(x^2 + \left(2\sin\tfrac{\pi x}{2}\right)^2\right)$$

$$= 2\underbrace{x}_{x=\frac{1}{3}}x' + 2\underbrace{\left(2\sin\tfrac{\pi x}{2}\right)}_{y=1}\left(2\sin\tfrac{\pi x}{2}\right)'$$

$$= 2\left(\tfrac{1}{3}\right)x' + 2(1)\left(2\underbrace{\left(\cos\tfrac{\pi x}{2}\right)}_{\sqrt{3}/2}\tfrac{\pi}{2}x'\right)$$

$$= \left(\tfrac{2}{3} + \sqrt{3}\pi\right)x' = \left(\tfrac{2}{3}+\sqrt{3}\pi\right)\sqrt{10}\ \text{cm/s}.$$

cont

$$D' = \frac{3}{2}\left(\frac{2}{3} + \sqrt{3}\pi\right)\ \text{cm/s}.$$

$$= \left(1 + \frac{3\sqrt{3}\pi}{2}\right)\text{cm/s}.$$

Example 2. A snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

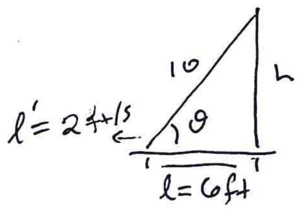
$$A = 4\pi r^2 = \pi d^2$$

$$-1 = A' = 2\pi d d' = 2\pi(10)\,d'$$

$$d' = -\frac{1}{20\pi}\ \text{cm/min}.$$

Level 2 Problems

Example 3. A 10 ft ladder rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 2 ft/s, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 6 ft from the wall?



~~sin~~ $\cos\theta = \frac{l}{10}$

$$(-\sin\theta)\,\theta' = \frac{d}{dt}\cos\theta = \frac{d}{dt}\frac{l}{10} = \frac{l'}{10}$$

$$* \quad \sin\theta = \frac{h}{10} = \frac{\sqrt{10^2-6^2}}{10} = \frac{\sqrt{64}}{10} = \frac{8}{10}$$

$$\therefore \left(-\frac{8}{10}\right)\theta' = \frac{l'}{10} \Rightarrow \theta' = -\frac{l'}{8} = -\frac{2}{8}\ \text{ft/sec.}$$

Example 4. Gravel is being dumped from a conveyor belt at a rate of 3 ft^3/min. It forms a pile in the shape of a cone whose base diameter and height are always the same. How fast is the height of the pile increasing when the pile is 10 ft high?

$$V = \text{volume of cone} = \pi r^2 \frac{h}{3} = \frac{\pi d^2 h}{12}.$$

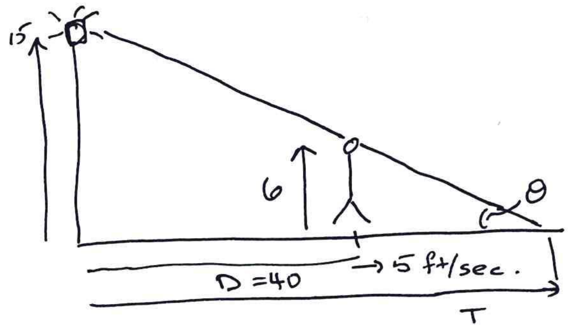
$$d = h \quad \therefore \quad V = \frac{\pi}{12} h^3.$$

$$3\,\frac{ft^3}{min} = V' = \frac{d}{dt}V = \frac{d}{dt}\frac{\pi}{12}h^3 = \frac{\pi}{4}h^2\,h' = \frac{\pi}{4}100\,h'$$

$$\therefore \quad h' = \frac{3}{\pi 25}\ \text{ft/min}$$

Level 3 Problems

Example 5. A street light is mounted at the top of a 15 foot tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?



$$\cot\theta = \frac{1}{6}(T-D) = \frac{1}{15}T$$

$$\therefore \quad \frac{15}{6}(T-D) = T \quad \Rightarrow \quad T = \frac{15}{9}D.$$

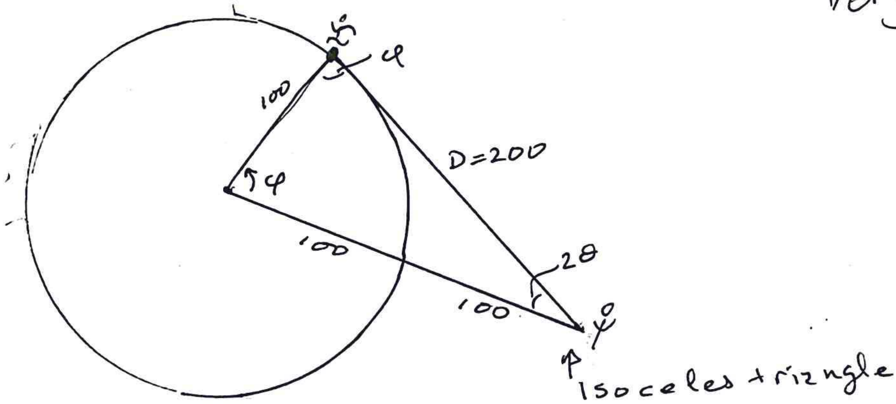
$$\therefore \quad T' = \frac{15}{9}D'$$

But $D' = 5$ ft/s.

$$\therefore \quad T' = \frac{15}{9} \cdot 5 \text{ ft/sec.}$$

Example 6. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/s. The runner's friend is standing at a distance 200 m from the center of the track. How fast is the distance between the friends changing when the distance between them is 200 m?

Very Challenging!

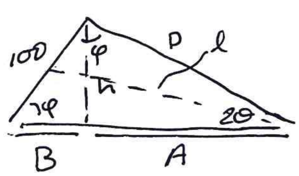


The runner completes a circuit in $\frac{100}{7}$ sec.

$\therefore \quad \varphi' = \frac{2\pi}{100/7} = \frac{7\pi}{50}\ s^{-1}$

$$\underline{2\varphi + 2\theta = \pi}$$

analyze triangle:



$A + B = 200$

$\cos\varphi = \frac{B}{100}$

$\sin\varphi = \frac{h}{100}$

$\boxed{\text{At } t_0 \quad \cos\varphi = \frac{50}{200} = \frac{l}{D}.}$

$\sin\varphi = \sqrt{1^2 - (\frac{1}{4})^2}$

$\sin\varphi = \frac{\sqrt{15}}{4}$

$h = 25\sqrt{15}$

$B = 25$

$D^2 = h^2 + A^2$

$A + B = 200$

$\hookrightarrow \quad D^2 = h^2 + (200 - B)^2$

$2DD' = 2hh' + 2(200 - B)(-B')$

$h' = 100(\sin\varphi)' = 100\cos\varphi\,\varphi'$

$= 25\left(\frac{7\pi}{50}\right)$

$B' = 100(\cos\varphi)' = 100(-\sin\varphi)\varphi'$

$= -25\sqrt{15}\ \frac{7\pi}{50}$

$\therefore$ plug in above

$2\cdot 200\, D' = 2(25\sqrt{15})\, h' + 2(200-25)\left(-(25\sqrt{15})\frac{7\pi}{50}\right)$

$25\left(\frac{7\pi}{50}\right)$