1. (4 points) Please circle either T (true) or F (false) for each of the below statements.

A) \( \text{F} \) In an \( N \times M \) one-period market model, with \( N \) assets and \( M \) economic states, a payoff \( X \in \mathbb{R}^M \) is \textit{attainable} if there exists a portfolio \( \phi \in \mathbb{R}^N \) such that \( D^T \phi = X \).

B) \( \text{F} \) In a one-period \( 3 \times 4 \) model with 3 base assets and 4 economic states, the 3\textsuperscript{rd} Arrow-Debreu security has payoff vector \( e^3 = (0, 0, 1, 0) \).

C) \( \text{T} \) In an \( N \times M \) one-period model, the Law of one Price guarantees no arbitrage opportunities.

D) \( \text{T} \) An \( N \times M \) is \textit{complete} if there exists a payoff \( X \in \mathbb{R}^M \) that has no replicating portfolio \( \phi \in \mathbb{R}^N \).

2. (16 total points) A bond \( B_t \) has initial price \( B_0 = 9 \) and terminal value \( B_T = 10 \). A stock \( S_t \) has initial price \( S_0 = 30 \) and possible future values of 30, 40, or 20. A put on the stock with strike \( K = 25 \) has initial price \( P = 1 \).

A) (4 points) Write the payoff matrix \( D \) and initial price vector \( S_0 \) for this market model.

\[
S_0 = (9, 30, 1).
\]

\[
D = \begin{bmatrix}
10 & 10 & 10 \\
20 & 40 & 20 \\
0 & 0 & 5
\end{bmatrix}
\quad \land \quad (S) = (25 - \frac{S}{T})^+
\]

B) (4 points) Show that the market model is complete and has no redundant assets. Be sure to explain your reasoning.

\[
\det D = 10(200) - 10(150) + 10(0) = 2000 - 1500 = 500 \neq 0.
\]

\[\therefore \text{All rows are L.I. so } \not\exists \text{ redundant assets.}\]

\[\therefore \text{Columns of } D^T \text{ are L.I. so } \text{Range}(D^T) = \mathbb{R}^3.\]
C) (4 points) Compute the state vector \( \Psi \) for this problem by solving \( D\Psi = S_0 \). Explain why the model does not allow for arbitrage.

\[
D\Psi = S_0 \Rightarrow \begin{bmatrix}
10 & 10 & 10 & 9 \\
20 & 40 & 20 & 30 \\
0 & 0 & 5 & 1
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 1 & 1 & 6 \\
0 & 10 & -10 & 3 \\
0 & 0 & 1 & 3/5
\end{bmatrix} \\
\rightarrow \begin{bmatrix}
1 & 1 & 0 & 6/10 \\
0 & 1 & -1 & 3/10 \\
0 & 0 & 1 & 3/5
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 2 & 6/10 \\
0 & 1 & -1 & 3/10 \\
0 & 0 & 1 & 3/5
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix} \\
\Rightarrow \Phi = \left( \frac{1}{5}, \frac{1}{2}, \frac{1}{5} \right).
\]

Since \( \Phi > 0 \) \( \forall \lambda \), FTAP \( \Rightarrow \) \( \text{no arbitrage} \).

D) (4 points) Find a portfolio \( \phi \) that replicates the payoff vector \( X = (10, 0, 30) \). Show that the arbitrage free price of this portfolio satisfies

\[
\phi \cdot S_0 = X \cdot \Psi.
\]

Need \( \phi \) s.t. \( D\phi = X \rightarrow \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 1 & 0 & -1 \\
0 & -10 & 5 & 20
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -1 \\
0 & -10 & 5 & 20
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & 0 & 0 & 4 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & 2
\end{bmatrix} \\
\Rightarrow \phi = (4, -1, 2) \sim 4 \text{ Bonds, Short 1 stock, 2 Puts}.
\]

\[
\phi \cdot S_0 = 4 \left( \frac{3}{5} \right) -1 (30) + 2(1) = \frac{36}{5} - \frac{60}{5} + 2 = 8
\]

\[
\Phi \cdot X = \frac{1}{5} \cdot 10 + \frac{1}{2} \cdot 0 + \frac{1}{5} \cdot 30 = 20 + 6 = 8.
\]