1. (2 points each) Please circle either T (true) or F (false) for each of the below statements.
   a) T F An arbitrage opportunity is a trading strategy that costs nothing to begin with (i.e., no initial cost), has no chance of a loss, and is certain to result in a nonzero gain.
   b) T F The binomial tree model admits arbitrage if and only if there exists the risk neutral probability \( \hat{p} \in (0, 1) \).
   c) T F An investment strategy is admissible if it is either self-financing or has non-negative values for all time. It need not be both.

2. (14 total points) At time \( t = 0 \), the value of a risk-free bond \( B_0 = 20 \) and the stock price is \( S_0 = 40 \). Suppose that the annual risk-free rate is \( r = 5\% \). Suppose that at the next (and final) time \( t = T \), you know that

\[
S_T = \begin{cases} 
44 & : \text{probability 30\%} \\
32 & : \text{probability 70\%} 
\end{cases}
\]

   a) (4 points) What holdings \( x \) in the stock \( S \) and \( y \) in the risk-free bond \( B \) replicate a contract \( C_T \) that pays the holder $1200 if the stock goes up and $900 if the stock goes down (at time \( t = T \))?

\[
\begin{align*}
44x + 21y &= 1200 \\
32x + 21y &= 900
\end{align*}
\]

   b) (2 points) Use your answer to part (a) to find the arbitrage-free price of the contract \( C_T \).

\[
C_0 - B_0 = 25 \cdot 40 + \left( \frac{100}{21} \right) \cdot 20 \approx 1095.24
\]

   c) (4 points) Find the risk-free probability \( \hat{p} \) for parts (a)-(b). Use your answer to verify that 

\[
E[S_T/B_T] = S_0/B_0.
\]

   d) (4 points) If, instead of increasing to \( S_T = 44 \), the stock only increases to \( S_T = 41 \), is there an arbitrage opportunity? If not, why not? If so, describe an arbitrage portfolio.

\[
\text{Yes, Arbitrage Exists. Consider portfolio}
\]

\[
\begin{align*}
40x + 20y &= 0 \quad \Rightarrow \quad y = -2x \\
\frac{S_T}{B_T} &= x \cdot \frac{S_T}{2} + y \cdot 21 = \frac{S_T}{21} - 2x \\
S_T &= 40 \quad \text{so} \quad x < 0 \quad \text{i.e., short stock}
\end{align*}
\]

\[
\Rightarrow \quad \frac{S_T}{B_T} > 0 \quad \forall \omega \in \Omega.
\]