1 Overview

In the last lecture, we discussed feasibility and PCA (i.e., subspace approximation with $\tau = 2$). In this lecture, we discuss subspace approximation with $\tau = \infty$.

2 Subspace Approximation

- In the case of $\tau = \infty$.
- Illustrate using SDP as part of “convex relaxation + rounding” strategy.

2.1 Definitions

Definition 1. A $d$-dimensional affine subspace, $A$, is a set of points $\{\bar{a} + \bar{x}| \bar{x} \in S_A\}$, where $S_A$ is a $d$-dimensional subspace and $\bar{a} \in \mathbb{R}^D$. 
Definition 2. Given a $d$-dimensional affine subspace, $A$, we define
\[ \vec{a}_A := \arg\min_{\vec{x} \in A} \|\vec{x}\|_2. \]
Thus, $A - \vec{a}_A = S_A$ and $\vec{a}_A \in S_A^\perp$.

Definition 3. Given an affine subspace $A$, we can define a projection onto $A$ to be
\[ \Pi_A \vec{x} := \Pi_{S_A} \vec{x} + \vec{a}_A, \forall \vec{x} \in \mathbb{R}^D \]
where $\Pi_{S_A}$ is the projection onto $S_A$.

Definition 4. Given a subspace, $S$, $\tilde{d}$-dimensional for $\tilde{d} \geq d \geq 1$, we will define
\[ \Pi_d(S) = \{\text{all } d-\text{dimensional affine subspaces of } S\}. \]

2.2 Main Problem

Given $P = \{\vec{x}_1, \cdots, \vec{x}_N\} \subseteq \mathbb{R}^D$, we want to estimate
\[ R_d(P) = \inf_{A \in \Pi_d(\mathbb{R}^D)} R_\infty(A, P) := \inf_{A \in \Pi_d(\mathbb{R}^D)} (\max_{j=1,\cdots,N} \|\vec{x}_j - \Pi_A \vec{x}_j\|_2) \]
- How quickly can we find a $\tilde{A} \in \Pi_d(\mathbb{R}^D)$ such that $R_\infty(\tilde{A}, P) \approx R_d(P)$?
- NOTE: This is related to bounding box/shape problems in computational geometry.

- Assumptions about $P$:
  1. $\vec{0} \in P$
  2. $\vec{x}_j \in P \iff -\vec{x}_j \in P$
2.3 Solving the Problem

Note that $R_d(P)$ can be found by solving the following optimization problem:

$$R_d^2(P) := \min \alpha$$

satisfying the constraints:

1. $\sum_{i=1}^{D-d} \langle \bar{x}_j, \bar{y}_i \rangle^2 \leq \alpha, \forall \bar{x}_j \in P$

2. $\|\bar{y}_i\| = 1$, $i = 1, \cdots, D - d$.

3. $\langle \bar{y}_i, \bar{y}_k \rangle = 0$, $i \neq k$

This problem finds $R_d(P)$ because:

- An optimal $d$-dimensional subspace $A$ with $R_\infty(A, P) = R_d(P)$ will be given by (span{$\bar{y}_1, \cdots, \bar{y}_{D-d}$})$^\perp$. That is, we are finding an orthonormal basis for $A^\perp$.

- We are trying to minimize $\|\Pi_{A^\perp} \bar{x}_j\|^2 = \sum_{i=1}^{D-d} \langle \bar{x}_j, \bar{y}_i \rangle^2$ over all $j = 1, \ldots, N$

- Here, $\alpha$ and the entries of $\bar{y}_1, \cdots, \bar{y}_{D-d} \in \mathbb{R}^D$ are the variables. There are $D(D - d) + 1$ total.

2.4 A convex relaxation of the problem [SDP(2)]

Consider this related optimization problem:

Calculate $\tilde{\alpha} := \min \alpha \in \mathbb{R}^+$ satisfying the following constraints for some $Y \in S^D$

1. $\bar{x}_j^T Y \bar{x}_j = \text{Trace}(\bar{x}_j \bar{x}_j^T Y) \leq \alpha, \forall \bar{x}_j \in P.$

$$= \sum_{k=1}^{D} Y_{kk}(x_j)_k^2 + 2 \sum_{k=1}^{D} \sum_{l=k+1}^{D} Y_{l,k}(x_j)_l(x_j)_k \leq \alpha, \forall \bar{x}_j \in P.$$  

(Notice that this is linear in the entries of $Y$.)

2. $\text{trace}(Y) = D - d.$

3. $I - Y \succeq 0.$

4. $Y \succeq 0.$

This problem can be solved as a semidefinite program! Note that:
• The variables are $\alpha$, and the independent entries of $Y \in S^D$. There are $\frac{D(D+1)}{2} + 1$ total variables.

• Constraint 1 is linear in the variables $\Rightarrow$ it is OK for an SDP.

• Constraint 2 is a linear equality constraint in the variables $\&$ so it is OK for an SDP. It implies that the eigenvalues of $Y$ sum to $D - d$.

• Constraint 3 : $I - Y \succeq 0 \Rightarrow I \succeq Y \Rightarrow$ All the eigenvalues of $Y$ are $\leq 1$.

• Constraint 4 : All the eigenvalues of $Y$ should be nonnegative.

• Constraints 3 and 4 force all eigenvalues of $Y$ to belong to $[0, 1]$.

• Thus, $\tilde{\alpha}$ can be computed via an SDP.

• What’s left : Show that $\tilde{\alpha}$ has something to do with $R_d(P)$!

2.5 Homework Problems (due Feb 11th(Tue.))

homework 1. Let $\bar{P} := (P - \bar{x}_1) \cup (\bar{x}_1 - P)$ where $P = \{\bar{x}_1, \ldots, \bar{x}_N\}$
This is now both symmetric about the origin, and contains $\bar{0}$. Prove that $R_d(\bar{P}) \leq 2R_d(P)$.

homework 2. Prove that any affine subspace $A$ with $R_\infty(A, \bar{P}) = R_d(\bar{P})$ will be a subspace (i.e., will have $\bar{a}_A = \bar{0}$).

Problem 3. Show that any optimal orthonormal basis for first optimization problem in section 2.3, $\{\bar{y}_1, \ldots, \bar{y}_{D-d}\}$, satisfies all four constraints for the optimization problem in section 2.4 if we set $Y = \sum_{i=1}^{D-d} \bar{y}_i\bar{y}_iT$. Conclude that $\tilde{\alpha} \leq R_d^2(P)$.

Hint: The fact that $\tilde{\alpha} \leq R_d^2(P)$ is related to an $\alpha$ you can achieve with this $Y$ in Constraint 1.

2.6 Showing that a solution to [SDP(2)] has something to do with $R_d(P)$

Lemma 1. Let $\tilde{Y} \in S^D$ be an optimal solution to [SDP(2)] in section 2.4. Then $\tilde{Y}$ will have $r \geq D - d$ eigenvalues $\lambda_1, \ldots, \lambda_r \in (0, 1]$ and $r$ (orthogonal unit) eigenvectors $\bar{v}_1, \ldots, \bar{v}_r$ with the property that $\sum_{i=1}^{r} \lambda_i < \bar{x}_j, \bar{v}_i >^2 \leq R_d^2(P), \forall \bar{x}_j \in P$.

Proof. Constraints 2 through 4 of [SDP(2)] in section 2.4 guarantee that we have $\lambda_1, \ldots, \lambda_r \in (0, 1]$ for $r \geq D - d$. Also, their associated eigenvectors $\bar{v}_1, \ldots, \bar{v}_r$ are perpendicular. We have that

$$\sum_{i=1}^{r} \lambda_i < \bar{x}_j, \bar{v}_i >^2 = \text{Trace} \left( \bar{x}_j\bar{x}_j^T \sum_{i=1}^{r} \lambda_i \bar{v}_i\bar{v}_i^T \right)$$
$$= \text{Trace}(\bar{x}_j\bar{x}_j^T \tilde{Y})$$
$$\leq \tilde{\alpha} \quad \text{(by [SDP(2)] Constraint 1)}$$
$$\leq R_d^2(P) \quad \text{(by Homework Problem 3)}$$

holds for all $j = 1, \ldots, N$.  \hfill \blacksquare