1 Overview

In this lecture, we present the algorithm for fast support identification. We want to design measurements that allow us to quickly find an $S \subset [N]$ such that $S_0(k) \subset S$ for $\bar{x} \in \mathbb{C}^N$.

2 Notational review

Let $A \in \{0, 1\}^{m \times N}$ and $B_N$ be the $N$th bit testing matrix. Let $\{\vec{b}_0, \vec{b}_1, \ldots, \vec{b}_{\lceil \log_2 N \rceil}\} \in \{0, 1\}^N$ be the rows of $B_N$. Given $(A \otimes B_N)\bar{x}$ we also get $(A \otimes \vec{b}_i)\bar{x} \in \mathbb{C}^m$, $\forall i = 0 \ldots \lceil \log_2 N \rceil$. This means that we get $A\vec{x}$ as well as $(A(K, n) \otimes \vec{b}_i)\bar{x}, \forall n \in [N]$ and $\forall i = 0 \ldots \lceil \log_2 N \rceil$.

Example 1.

$$
\begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2
\end{pmatrix}
\otimes
\begin{pmatrix}
1 & 0 & 1 & 0 \\
2 & 0 & 2 & 0
\end{pmatrix}
= 
\begin{pmatrix}
1 & 0 & 1 & 0 \\
2 & 0 & 2 & 0
\end{pmatrix}
$$

Let $n \in [N]$ and assume that matrix $A$ is $(K, \alpha)$-coherent with $K > \frac{4\tilde{k}\alpha}{\varepsilon}$, where $\varepsilon \in (0, 1)$ for sparsity $\tilde{k}$. Theorem 1 (Lecture 27) tells us that more than $1/2$ of the $j \in [K]$ satisfy $(A(K, n)\bar{x})_j \in B(x_n, \delta)$, where

$$
\delta := \varepsilon \frac{\|\bar{x} - \bar{x}_0(\tilde{k})\|_1}{\tilde{k}} \in [N], \forall \varepsilon \in (0, 1)
$$

Definition 1. Given $\bar{x} \in \mathbb{C}^N$, let $|\bar{x}| \in \mathbb{R}^N$ be such that $|\bar{x}|_j := |\bar{x}_j|, \forall j \in [N]$.

Now let’s let $\vec{a}_j \in \{0, 1\}^N$ be the $j$th row of $A(K, n)$ and suppose that

(i) $\langle \vec{a}_j, |\bar{x}| \rangle \in B(|x_n|, \delta)$, and
(ii) $|x_n| > \delta$. 

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From Theorem 1 (Lecture 27) we know that more than $1/2$ of the rows, $\vec{a}_j$, of $A(K, n)$ will satisfy (i). Supposing that $|x_n| > \delta$ and that the $i^{th}$ bit of $n$ in binary is 1:

$$\left| \langle \vec{a}_j \otimes \vec{b}_i, \vec{x} \rangle \right| \geq |x_n| - \sum_{l \in \text{supp}(\vec{a}_j) \text{ s.t. } l \neq n; \text{i}^{th} \text{ bit of } l = 1} |x_l|$$

$$\geq \delta - \sum_{l \in \text{supp}(\vec{a}_j) \text{ s.t. } l \neq n; \text{i}^{th} \text{ bit of } l = 1} |x_l|$$

(2)

$$\geq \sum_{l \in \text{supp}(\vec{a}_j) \text{ s.t. } l \neq n; \text{i}^{th} \text{ bit of } l = 0} |x_l|$$

Essentially the same argument shows that $\left| \langle \vec{a}_j - \vec{a}_j \otimes \vec{b}_i, \vec{x} \rangle \right| > \left| \langle \vec{a}_j \otimes \vec{b}_i, \vec{x} \rangle \right|$, whenever the $i^{th}$ bit of $n$ is zero. We have now shown that the algorithm below will identify all $n \in [N]$ with $|x_n| > \delta$ more than $K/2$ times apiece.

**Algorithm 1.**

1. $S = \emptyset$
2. **For** $j \in [m]$
3. **For** $i = 0 \ldots \lceil \log_2 N \rceil - 1$
4. **If** $\left| \langle \vec{a}_j \otimes \vec{b}_i, \vec{x} \rangle \right| > \left| \langle \vec{a}_j - \vec{a}_j \otimes \vec{b}_i, \vec{x} \rangle \right|$  
   Set $n_i = 1$
5. **Else**  
   Set $n_i = 0$
6. **End For**
7. Set $n = \sum_{i=0}^{\lceil \log_2 N \rceil - 1} n_i \cdot 2^i$ (translate from binary to decimal);
8. $S = S \cup \{n\}$
9. **End For**

It takes $O(m \log N)$ operations to go through steps 1 to 9. Also, we know that, e.g., $m = K^2$ is possible (from Lecture 26). Therefore, the total runtime of Algorithm 1 is generally sublinear in $N$. For example,

$$m = O \left( \frac{k^2 \log^3 N}{\epsilon^2} \right) \ll N$$

works.
Measurements $m$ can be randomized/reduced to get the total runtime of $O\left( \tilde{k} \log \left( \frac{N}{1-p} \right) \log \tilde{k} \right)$, which has the same accuracy as the deterministic variant with probability at least $p$.

It is true that $|S| \leq m$, but we also know that every $n \in [N]$ such that $|x_n| > \delta$ is recovered at least $K/2$ times. Therefore, $|S| = O(K)$, and we expect $S \supset S_0 \left( \frac{2\tilde{k}}{\varepsilon} \right)$, which follows from the Lemma below.

**Lemma 1.** Suppose that $|x_n| > \delta$. Then, $n \in S_0 \left( \frac{2\tilde{k}}{\varepsilon} \right)$. As a result, Algorithm 1 finds all $n \in S_0 \left( \frac{2\tilde{k}}{\varepsilon} \right)$ with $|x_n| > \delta$.

Note that $n \in S_0 \left( \frac{2\tilde{k}}{\varepsilon} \right)$ with $|x_n| \leq \delta$ are “OK to miss”.

Next time we will use results from Lectures 28 and 29 to help construct Sparse Fast Fourier Transforms (SFFTs).