1 Overview

In the last lecture we proved the Johnson-Lindenstrauss lemma (J-L lemma) implies the RIP property. In this lecture we will discuss the lower bound on the number of rows of a RIP matrix and compare this to the matrix the J-L lemma gives us.

2 Lecture

Lemma 1. Given \( K, N \in \mathbb{N}, K < N \).
Then \( \exists n \geq \left( \frac{N}{4K} \right)^{K/2} \) subsets \( S_1, S_2, \ldots, S_n \subseteq [N] \) such that

\[
\begin{align*}
|S_j| &= K, \forall j \in [n] \\
|S_i \cap S_j| &< \frac{K}{2}, \forall i \neq j 
\end{align*}
\]

Proof: Assume \( K < \frac{N}{4} \)
Let \( C = \{ S \subseteq [N] | K = |S| \} \)
Pick \( S_1 \in C \)
Let \( C_1 \subseteq C : C_1 = \{ S \in C | \frac{K}{2} \leq |S \cap S_1| \} \) i.e. things that looks like \( S_1 \)
\[
|C_1| = \sum_{S=\left\lfloor K/2 \right\rfloor}^{K} \binom{N-K}{S-K} \binom{K}{S} \\
\leq 2^K \max_{\frac{K}{2} \leq S \leq K} \binom{N-K}{K-S}, \text{ by sum of binomial} \leq 2^K \\
= 2^K \binom{N-K}{\left\lfloor K/2 \right\rfloor} \text{ because N-K is small, so it maximize when K-S maximize}
\]

We will pick \( S_2, S_3, \ldots, S_n \) using the following algorithm:

Pick : \( S_1, C_1, C \)
\( n \leftarrow 1 \)
While : \( |C \setminus \bigcup_{i=1}^{n} C_i| > 0 \)
- Choose : \( S_{n+1} \in C \setminus \bigcup_{i=1}^{n} C_i \)
- Set : \( C_{n+1} = \{ S \in C \setminus \bigcup_{i=1}^{n} C_i | \frac{K}{2} \leq |S \cap S_{n+1}| \} \)
- \( n \leftarrow n + 1 \)

Note that by construction: \( |S_i \cap S_j| < \frac{K}{2}, \forall i \neq j \)
Algorithm stop when
\[ n \geq \max_{1 \leq i \leq n} |C_i| \geq \frac{\binom{N}{K}}{2^K \binom{N}{K/2}} \geq \left( \frac{N}{4K} \right)^K \frac{K}{2^K} \] by expanding binomial coefficients.

**Theorem 1.** Given \( A \in \mathbb{R}^{m \times N} \)

*Condition for Basis Pursuit (BP), i.e.:*

\[ \forall x \in \mathbb{R}^N \text{ with } ||x||_0 \leq 2K, \forall z \in \mathbb{R}^N \text{ with } Az = Ax \text{ then } ||x||_1 \leq ||z||_1 \]

*Then*

\[ m \geq \frac{K}{m} \ln \left( \frac{N}{4K} \right) \]

**Proof:** Consider \([x] = x + \text{Ker}(A)\) associated with a norm: \(||x|| := \inf_{v \in \text{Ker}(A)} ||x - v||_1\)

Indentify (i.e., note the existence of the bijection) \( y \in \text{Ker}(A)^\perp \) with \([y]\)

This bijection induces a norm in \(\text{Ker}(A)\): \(||y||_S = ||[y]||\)

Suppose \(||x||_0 \leq 2K\)

Project \(x\) on \(\text{Ker}(A)^\perp\):

\[
|| \prod_{\text{Ker}(A)^\perp} x ||_S = || \prod_{\text{Ker}(A)^\perp} x ||_1 \\
= \inf_{v \in \text{Ker}(A)} || \prod_{\text{Ker}(A)^\perp} x - v ||_1 \\
= \inf_{v \in \text{Ker}(A)} ||x - v||_1 \\
= ||x||_1
\]

The last equality is due to Basis Pursuit condition (note that \(A(x - v) = A(x)\))

Let \(S_j\) be the subsets from lemma 1:

\[
\begin{cases}
|S_j| = K, \forall j \in [n] \\
|S_i \cap S_j| < K^2, \forall i \neq j
\end{cases}
\]

Let \(y_j\) be vector in \(\mathbb{R}^N\) such that:

\[
(y_j)_l = \begin{cases} 
\frac{1}{K} \text{ if } l \in S_j \\
0 \text{ otherwise}
\end{cases}
\]

Note that \(||y_j||_0 = k\) and \(||y_j||_1 = 1\)

Define \(x_j = \prod_{\text{Ker}(A)^\perp} y_j\)

Note:

\[
||x_j||_S = || \prod_{\text{Ker}(A)^\perp} y_j ||_S = ||y_j||_1 = 1
\]

\[
||x_j - x_l||_S = ||y_j - y_l||_1 > 1 \text{ since } K \leq ||y_j - y_l||_0 \leq 2K
\]

Hence:
\[
\left( \frac{N}{4K} \right)^K \leq n \quad \text{by lemma 1}
\]
\[
\leq P_1(\|\cdot\|_S - \text{ball in Ker}(A)^\perp) \quad \text{since } x_j \text{ can be a packing}
\]
\[
\leq (1 + \frac{2}{1})^{\text{rank}(A)}
\]
\[
= 3^m
\]

Take the log of both side and we are done.

This theorem give us a lower bound on the number of rows, m, of matrices with RIP and NSP (null space property), which is in order of k. This means our random matrices from J-L lemma are pretty close to optimal as they are also in order of k.