Exercises: (§17)

1. Let $A \subset X$ where X is a topological space. Prove that
   (a) $A = A^\circ$ if and only if A is open in X, and that
   (b) $A = \overline{A}$ if and only if A is closed in X.
2. Let X be a topological space with subset $S \subset X$. Recall that $\overline{S}$ denotes the closure of S and S° denotes the interior of S. We will also denote by $S^c := X \setminus S$ the complement of S.
   (a) Show that $\overline{S} = ((S^c)^\circ)^c$ for all $S \subset X$.
   (b) Show that $S^\circ = (\overline{S^c})^c$ for all $S \subset X$.
3. Let X be a topological space and let $A, B \subset X$ be subsets.
   (a) Show that $A \subset B$ implies $\overline{A} \subset \overline{B}$ and $A^\circ \subset B^\circ$.
   (b) For $A, B \subset X$, show that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (c) For $A, B \subset X$, show that $(A \cap B)^\circ = A^\circ \cap B^\circ$.
   (d) Let $\mathbb{R}$ have the standard topology. Find examples of subsets $A, B \subset \mathbb{R}$ such that $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$ and $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.
4. Let X be a topological space. We say a subset $S \subset X$ is **dense** in X if for every $x \in X$ and every neighborhood U of x one has $U \cap S \neq \emptyset$. Show the following are equivalent:
   (i) S is dense in X.
   (ii) $(S^c)^\circ = \emptyset$.
   (iii) $\overline{S} = X$.
5. If $A \subset X$, we define the **boundary** of A by $\mathrm{Bd}A := \overline{A} \cap \overline{X \setminus A}$.
   (i) Show that A° and $\mathrm{Bd}A$ are disjoint.
   (ii) Show that $\overline{A} = A^\circ \cup \mathrm{Bd}A$.
   (iii) Show that $\mathrm{Bd}A = \overline{A} \setminus A^\circ$ also always holds.
   (iv) Show that $\mathrm{Bd}A = \emptyset$ if and only if A is both open and closed.
   (v) Show that A is open if and only if $\mathrm{Bd}A = \overline{A} \setminus A$.