Exercises: (§6 and §7)

1. Let A and B be non-empty sets.
   (a) Show that if B is not finite and $B \subset A$, then A is not finite.
   (b) Prove that $A \times B$ is finite if and only if A and B are both finite.
   (c) Show that $A \times B$ and $B \times A$ have the same cardinality.
   (d) Let B^A denote the set of functions $f\colon A \to B$. Show that if A and B are finite, then so is B^A.
   (e) Suppose B^A is finite and B has at least two elements. Show that A and B are finite.
2. Let $\{0,1\}^{\mathbb{N}}$ denote the set of functions $f\colon \mathbb{N} \to \{0,1\}$.
   (a) Show that $\{0,1\}^{\mathbb{N}}$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality.
   (b) Let $\mathcal{C}$ be the collection of *countable* subsets of $\{0,1\}^{\mathbb{N}}$. Show that $\mathcal{C}$ and $\{0,1\}^{\mathbb{N}}$ have the same cardinality. [**Hint:** first construct an injection from $\mathcal{C}$ to $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, and then use Exercise 5 from Homework 1.]
3. Let $n \in \mathbb{N}$. In this problem we will prove that $\mathbb{R}^n$ has the same cardinality as $(0,1) \subset \mathbb{R}$.
   (a) Show that $\mathbb{R}$ has the same cardinality as $(0,1)$.
   (b) Show that $(0,1) \times (0,1)$ has the same cardinality as $(0,1)$. Conclude that $\mathbb{R}^2$ has the same cardinality as $\mathbb{R}$ as a result.
   (c) Use, e.g., induction to prove that $\mathbb{R}^n$ has the same cardinality as $\mathbb{R}$. Conclude that $\mathbb{R}^n$ has the same cardinality as $(0,1)$ as a result.
4. Let $\epsilon > 0$. Prove that $(-\epsilon, \epsilon) \times \mathbb{N} \times \{0,1\}$ has the same cardinality as $\mathbb{R}^{10^{1000}}$.