Derivatives and Planes

1. Partial Derivatives

**Definition 1 (Partial Derivatives)** The partial derivatives (with respect to $x$ and with respect to $y$) of the function $f(x, y)$ are the two functions defined by

\[
\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
\]

\[
\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}
\]

whenever these limits exist.

**Ex. (1)** Compute the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for $f(x, y) = \cos(x^2y) + y^3$.

**Theorem 2 (Componentwise Differentiation)** Suppose that

\[
c(t) = (x(t), y(t), z(t)) = x(t)i + y(t)j + z(t)k
\]

is a curve, where $x(t)$, $y(t)$, and $z(t)$ are differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Then

\[
c'(t) = (x'(t), y'(t), z'(t)) = x'(t)i + y'(t)j + z'(t)k.
\]

The derivative vector $c'(t)$ is tangent to the curve $c(t)$ at $(x(t), y(t), z(t))$.

**Ex. (2)** Find a vector that is tangent to the curve

\[
x(t) = t^2, \quad y(t) = t, \quad z(t) = \cos(t)
\]

at the point $P(0, 0, 1)$.
2. Tangent Planes

Let \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). The **plane tangent** to the surface \( z = f(x, y) \) at the point \( P(a, b, f(a, b)) \) is the plane through \( P \) that contains the tangent lines of both the curve

\[
x(t) = t, \quad y(t) = b, \quad z(t) = f(t, b)
\]

and the curve

\[
x(t) = a, \quad y(t) = t, \quad z(t) = f(a, t).
\]
at \( P \).
Board Ex. Consider the surface \( z = f(x, y) = x^2 + 2xy^2 - y^3 \). It contains the point \( P(1, 1, f(1,1)) = P(1,1,2) \). Write the equation of the line that is tangent to the curve
\[
x(t) = t, \quad y(t) = 1, \quad z(t) = f(t, 1),
\]
and sketch the curve inside the graph of \( f \).

Board Ex. Consider the surface \( z = f(x, y) = x^2 + 2xy^2 - y^3 \). It contains the point \( P(1,1,f(1,1)) = P(1,1,2) \). Write the equation of the line that is tangent to the curve
\[
x(t) = 1, \quad y(t) = t, \quad z(t) = f(1, t),
\]
and sketch the curve inside the graph of \( f \).

Board Ex. Write the equation of the plane tangent to the surface \( z = f(x, y) = x^2 + 2xy^2 - y^3 \) at the point \( P(1, 1, 2) \).
Board Ex. Consider the tangent plane 

\[-4x - y + z = -3\]

to the surface \( z = f(x, y) = x^2 + 2xy^2 - y^3 \) at the point \( P(1, 1, 2) \) that you found in the last problem. Show that the tangent plane formula on page 110 gives the same answer, i.e., that

\[
z = f(1, 1) + \left[ \frac{\partial f}{\partial x}(1, 1) \right] (x - 1) + \left[ \frac{\partial f}{\partial y}(1, 1) \right] (y - 1)
\]

\[= f(1, 1) + \mathbf{D}f(1, 1) \cdot (x - 1, y - 1).\]

Challenge Ex. Show that \( \mathbf{D}f(1, 1) \) from above satisfies the definition of the derivative of \( f(x, y) = x^2 + 2xy^2 - y^3 \) at \((1, 1)\) on page 111 by using the polar coordinate transform

\[ x = r \cos \theta + 1, \ y = r \sin \theta + 1 \]

in order to show that

\[
\lim_{(x,y) \to (1,1)} \frac{|f(x,y) - f(1,1) - \mathbf{D}f(1,1) \cdot (x-1, y-1)|}{\|(x,y) - (1,1)\|} = 0.
\]