Vectors in $\mathbb{R}^2$

1. Vectors in $\mathbb{R}^2$

**Definition 1 (Vector)** A vector $\mathbf{v} = \langle a, b \rangle$ in $\mathbb{R}^2$ is an ordered pair of real numbers. We call $a$ and $b$ the components of the vector $\mathbf{v}$.

Vectors are geometrically represented by directed line segments in the cartesian plane:

**Definition 2 (Equality)** The two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are equal if $u_1 = v_1$ and $u_2 = v_2$.

**Definition 3 (Addition)** The sum of two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is the vector $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.

**Definition 4 (Scalar Multiplication)** If $\mathbf{u} = \langle u_1, u_2 \rangle$ and $c$ is a real number, then the scalar multiple $c\mathbf{u}$ is the vector $c\mathbf{u} = \langle cu_1, cu_2 \rangle$.

*Ex.* If $\mathbf{u} = \langle 3, 5 \rangle$ and $\mathbf{v} = \langle -4, 4 \rangle$, find $\mathbf{u} + \mathbf{v}$, $2\mathbf{u}$, and $\mathbf{u} - \mathbf{v}$ (that is, $\mathbf{u} + (-1)\mathbf{v}$).

**Definition 5 (Length)** The length of $\mathbf{v} = \langle a, b \rangle$ is denoted $|\mathbf{v}|$ and is defined as $|\mathbf{v}| = |\langle a, b \rangle| = \sqrt{a^2 + b^2}$.

*Ex.* Find the length of $\mathbf{v} = \langle 3, 5 \rangle$

**Definition 6 (Unit Vectors)** A unit vector is a vector of length 1. If $\mathbf{a} = \langle a_1, a_2 \rangle \neq 0$, then $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$ is the unit vector in the same direction as $\mathbf{a}$.
Ex. Find a unit vector in the same direction as $\langle 3, 5 \rangle$.

**Ex.** Two unit vectors, $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$, are often used as an alternative way to represent vectors. Write $\langle 3, 5 \rangle$ as a sum of $\mathbf{i}$ and $\mathbf{j}$.

**Board Ex.** Show that the line segment joining the midpoints of two sides of a triangle is parallel to and half the length of its third side.

2. Vectors in $\mathbb{R}^3$

**Definition 7 (Vector)** A vector $\mathbf{v} = \langle a, b, c \rangle$ in $\mathbb{R}^3$ is an ordered triple of real numbers.

Vectors in $\mathbb{R}^3$ are geometrically represented by directed line segments in three dimensional Euclidean space. Addition, scalar multiplication, length, and unit vectors are all defined in the same way as for vectors in $\mathbb{R}^2$, but now we have *three* basic unit vectors, $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.

**Definition 8 (Distance)** The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = |\overrightarrow{PQ}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

**Ex.** Find the distance between the points $A(1, 2, 3)$ and $B(3, -2, 5)$. 
Definition 9 (Dot Product) The dot product of the two vectors 
\[ \mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k} \quad \text{and} \quad \mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k} \]
is the scalar defined as 
\[ \mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \]

Ex. Calculate the dot product of \( \langle 1, 2, 10 \rangle \) and \( \langle -2, 3, 4 \rangle \).

The following properties of the dot product are useful for working with the dot product.
(i) \( \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \)
(ii) \( \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} \)
(iii) \( \mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} \)
(iv) \( (r \mathbf{a}) \cdot \mathbf{b} = r(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (r \mathbf{b}) \)

Does \( (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c} \) make sense? What about \( (\mathbf{a} \cdot \mathbf{b}) \mathbf{c} \)?

3. Interpretation of the Dot Product

Theorem 10 If \( \theta \) is the angle between the vectors \( \mathbf{a} \) and \( \mathbf{b} \) in \( \mathbb{R}^3 \), then 
\[ \mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \| \mathbf{b} \| \cos \theta. \]

Board Ex. Prove Thm 10:
Corollary 11  The two nonzero vectors \( \mathbf{a} \) and \( \mathbf{b} \) are perpendicular if and only if \( \mathbf{a} \cdot \mathbf{b} = 0 \).

Ex. Show that the vectors \( 3\mathbf{i} + 5\mathbf{j} \) and \(-4\mathbf{i} + 4\mathbf{j} \) are not perpendicular using the dot product.

Ex. For what value \( k \) would the vectors \( \langle 3, 5, 1 \rangle \) and \( \langle -4, 4, k \rangle \) be perpendicular?

4. Projections

Board Ex. Given \( \mathbf{a} = \langle 4, -5, 3 \rangle \) and \( \mathbf{b} = \langle 2, 1, -2 \rangle \), find the projection of \( \mathbf{a} \) in the direction of \( \mathbf{b} \).
5. **Planes**

How can we define a plane in space using the dot product if we are given a point \( P_0(x_0, y_0, z_0) \) on the plane and a direction \( \mathbf{n} \) perpendicular to the plane?

*Board Ex.* Given a point \((2, 3, 1)\) and a plane \(2x + 3y - z = 5\), find the distance from the point to the plane.