TECHNICAL NOTE: A MINOR CORRECTION OF THEOREM 1.3 FROM [1]

MARK IWEN, FELIX KRAHMER, AND ADITYA VISWANATHAN

Abstract. In this short note we correct and simplify the proof of equation (1.8) in Theorem 1.3 of [1]. In the new version (see Theorem 1 below) the noise-to-signal ratio need not be smaller than an absolute constant in order for the stated error guarantee to hold. In the process of proving Theorem 1 we also correct a small (re)normalization issue with Corollary 4 of [2].

We wish to reconstruct a given vector $x_0 \in \mathbb{C}^n$, up to a global phase factor, from magnitude measurements of the form

\begin{equation}
\label{equation_1}
b_i := |\langle p_i, x_0 \rangle|^2 + n_i,
\end{equation}

where $p_i \in \mathbb{C}^n$ and $n_i \in \mathbb{R}$ for $i = 1, \ldots, m$. Vectorizing (1) yields

\begin{equation}
\label{equation_2}
b := |\mathcal{P}x_0|^2 + n,
\end{equation}

where $b, n \in \mathbb{R}^m$, $\mathcal{P} \in \mathbb{C}^{m \times n}$, and $|\cdot|^2 : \mathbb{C}^m \to \mathbb{R}^m$ computes the component-wise squared magnitude of each vector entry. We aim to prove the following corrected version of equation (1.8) in Theorem 1.3 from [1] concerning this problem.

**Theorem 1.** Let $\mathcal{P} \in \mathbb{C}^{m \times n}$ have its $m$ rows be independently drawn either uniformly at random from the sphere of radius $\sqrt{n}$ in $\mathbb{C}^n$, or else as complex normal random vectors from $\mathcal{N}(0, \mathbb{I}_n/2) + i\mathcal{N}(0, \mathbb{I}_n/2)$. Then, $\exists$ universal constants $\tilde{B}, \tilde{C}, \tilde{D} \in \mathbb{R}^+$ such that the PhaseLift procedure $\Phi_{\mathcal{P}} : \mathbb{R}^m \to \mathbb{C}^n$ satisfies

\begin{equation}
\label{equation_3}
\min_{\theta \in [0, 2\pi]} \left\| \Phi_{\mathcal{P}}(b) - e^{i\theta}x_0 \right\|_2 \leq \tilde{C} \cdot \frac{\|n\|_1}{m} \frac{\|x_0\|_2}{m}
\end{equation}

for all $x \in \mathbb{C}^n$ with probability $1 - O(e^{-\tilde{B}m})$, provided that $m \geq \tilde{D}n$. Here $b, n \in \mathbb{R}^m$ are as in (2).

Our proof relies on a modified version of Corollary 4 from [2]. It reads:

\begin{itemize}
\item [	extit{Date}: April 23, 2015.]
\item M.A. Iwen: Department of Mathematics and Department of ECE, Michigan State University (markiwen@math.msu.edu).
\item F. Krahmer: Department of Mathematics, Technische Universität München, (felix.krahmer@tum.de).
\item A. Viswanathan: Department of Mathematics, Michigan State University (aditya@math.msu.edu).
\end{itemize}
Corollary 1. Let \( x_0 \in \mathbb{C}^n \), set \( X_0 = x_0 x_0^* \), and let \( X \succeq 0 \) be such that \( \| X - X_0 \|_F \leq \eta \| X_0 \|_F = \eta \| x_0 \|_2^2 \) for some \( \eta > 0 \), where \( \| \cdot \|_F \) denotes the Frobenius norm. Furthermore, let \( \lambda_i \) be the \( i \)-th largest eigenvalue of \( X \) and \( v_i \) an associated eigenvector, such that the \( v_i \) form an orthonormal eigenbasis. Then

\[
\min_{\theta \in [0, 2\pi]} \| e^{i\theta} x_0 - \sqrt{\lambda_1} v_1 \|_2 \leq (1 + 2\sqrt{2}) \eta \| x_0 \|_2.
\]

See Section 1 for the proof of Corollary 1.

We can now use Corollary 1 to prove Theorem 1:

Proof. Beginning with Equation (1.7) in Theorem 1.3 of [1], we have that

\[
\| X - x_0 x_0^* \|_F \leq C_0 \cdot \left( \frac{\| n \|_1}{m \| x_0 \|_2^2} \right) \| x_0 \|_2^2,
\]

where \( X \succeq 0 \) is the solution to (1.6) in [1], and \( C_0 \in \mathbb{R}^+ \) is a universal constant. Returning the leading eigenvector of \( X \) reweighed by the square root of its associated eigenvalue now establishes the desired error bound by Corollary 1. \( \square \)

Please note that no assumptions need to be made concerning the magnitude of the noise vector, \( n \), in Theorem 1.

1. PROOF OF COROLLARY 1

Proof. Note that by construction, the rank one matrix \( X_0 \) has one eigenvalue \( \nu := \| x_0 \|_2^2 \) and all other eigenvalues 0. By Weyl’s inequality,

\[
\max\{ |\nu - \lambda_1|, \lambda_2, \ldots, \lambda_n \} \leq \eta \nu.
\]

By orthonormality of the \( v_i \), the spectral norm of the matrix \( X - \nu v_1 v_1^* \) satisfies

\[
\| X - \nu v_1 v_1^* \| = \left\| (\lambda_1 - \nu) v_1 v_1^* + \sum_{j=2}^{n} \lambda_j v_j v_j^* \right\| \leq \eta \nu,
\]

where the last inequality uses (4). Consequently, by the triangle inequality,

\[
\| X_0 - \nu v_1 v_1^* \| \leq \| X_0 - X \| + \| X - \nu v_1 v_1^* \| \leq 2\eta \nu.
\]

Thus, we can see that

\[
\nu^2 - \nu \langle x_0, v_1 \rangle \leq \frac{1}{2} \| X_0 - \nu v_1 v_1^* \|_F^2 \leq \| X_0 - \nu v_1 v_1^* \|^2 \leq 4\eta^2 \nu^2,
\]

where the second to last inequality follows from the fact that \( X_0 - \nu v_1 v_1^* \) is at most rank 2.
We next choose $\phi \in [0, 2\pi]$ such that $\langle e^{i\phi}x_0, v_1 \rangle = |\langle x_0, v_1 \rangle|$, and then note that
\begin{align*}
\|e^{i\phi}x_0 - \sqrt{\nu}v_1\|_2^2 &= 2\nu - 2\sqrt{\nu}\langle e^{i\phi}x_0, v_1 \rangle = 2\nu - 2\sqrt{\nu} \cdot |\langle x_0, v_1 \rangle| \\
&\leq (2\nu - 2\sqrt{\nu} \cdot |\langle x_0, v_1 \rangle|) \left(\frac{\nu + \sqrt{\nu} \cdot |\langle x_0, v_1 \rangle|}{\nu}\right) \\
&= \frac{2}{\nu} \left(\nu^2 - \nu |\langle x_0, v_1 \rangle|^2\right) \leq 8\eta^2\nu,
\end{align*}
where the last inequality follows from (5). Finally, by the triangle inequality, (4), and (6), we have
\begin{align*}
\|e^{i\phi}x_0 - \sqrt{\lambda_1}v_1\|_2 &\leq \|e^{i\phi}x_0 - \sqrt{\nu}v_1\|_2 + \|\sqrt{\nu}v_1 - \sqrt{\lambda_1}v_1\|_2 \\
&\leq 2\sqrt{2}\eta\sqrt{\nu} + \left|\sqrt{\nu} - \sqrt{\lambda_1}\right| \\
&\leq 2\sqrt{2}\eta\sqrt{\nu} + \frac{|\nu - \lambda_1|}{\sqrt{\nu} + \sqrt{\lambda_1}} \\
&\leq 2\sqrt{2}\eta\sqrt{\nu} + \frac{\eta\nu}{\sqrt{\nu} + \sqrt{\lambda_1}} \\
&\leq 2\sqrt{2}\eta\sqrt{\nu} + \eta\sqrt{\nu} = (1 + 2\sqrt{2})\eta\sqrt{\nu}.
\end{align*}
The desired result now follows.

\[\square\]

References
