Math 881 **Vizing's Theorem** **Spring 2018**

Recall that for a graph $G = (V, E)$, the edge-chromatic number $\chi'(G)$ is the minimum k such that there exists an edge-coloring $c : E \to [k]$ which is proper: $c(xy) \neq c(xz)$ whenever $xy, xz \in E$. The maximum degree $\Delta(G)$ is the largest number of neighbors $|\Gamma(v)|$ for $v \in V$.

Theorem: $\Delta(G) \leq \chi'(G) \leq \Delta(G){+}1$.

Proof (cf Diestel Thm 5.3.2). The first inequality is obvious. We prove the second by induction on the number of edges of G. A few edges is OK.

Now, for a graph G with edge xy, assume by induction that $H = G{-}xy$ (removing the edge, but not the end vertices) has a proper edge-coloring c with $\Delta(H){+}1 \leq \Delta(G){+}1$ available colors. We define a new proper coloring c' of G, including $c'(xy)$, by changing some of the colors of c, while leaving $c'(uv) = c(uv)$ unless otherwise specified.

Clearly c must have at least one color missing at each vertex v, and we let $\hat{c}(v)$ denote an arbitrary choice of such a color, with $\hat{c}(x) = a$. If $\hat{c}(y) = a$, we may let $c'(xy) = a$. Otherwise, we take a maximal sequence of distinct edges $xy = xy_0, xy_1, \ldots, xy_k$ such that $c(xy_i) = \hat{c}(y_{i-1})$ for $i = 1, \ldots, k$, and $\hat{c}(y_k) = b \neq c(xw)$ for $w \in \Gamma_G(x){-}\{y_0, \ldots, y_k\}$.

We shift the colors $c(xy_i)$ backwards: $c'(xy_{i-1}) = c(xy_i)$ for $i = 1, \ldots, k$. To define $c'(xy_k)$, we consider a maximal path $y_k z_1 \cdots z_\ell \subset G$, $\ell \geq 0$, such that:

$$c(y_k z_1) = a\,, c(z_1 z_2) = b, c(z_2 z_3) = a, \ldots.$$

Lastly, we shift the colors on this path backwards:

$$c'(xy_k) = a\,,\ c'(y_k z_1) = b\,,\ c'(z_1 z_2) = a\,,\ c'(z_2 z_3) = b, \ldots.$$

We easily see that c' is a proper edge-coloring of G except in two special cases.

In the first special case, the ab-path ends at $z_{\ell-1} z_\ell = y_j x$ with $c(z_{\ell-1} z_\ell) = b$; then we specify $c'(z_{\ell-1} z_\ell) = c'(y_j x) = c(xy_{j+1})$, not a.

In the second special case, the ab-path ends at $z_\ell = y_j$ with $c(z_{\ell-1} z_\ell) = c(z_{\ell-1} y_{j-1}) = a$, and we have $c(xy_{j+1}) = b$. We shift the colors $c(xy_i)$ backwards only up to $c(xy_{j-1})$: that is, $c'(xy_{i-1}) = c(xy_i)$ for $i = 1, \ldots, j$. Also:

$$c'(xy_k) = a\,,\ c'(y_k z_1) = b\,,\ c'(z_1 z_2) = a, \ldots, c'(z_{\ell-1} z_\ell) = b\,,\ c'(z_\ell x) = c'(y_j x) = a.$$