In a graph G, we let $a\mathcal{P}b$ denote a path $\mathcal{P}$ with end vertices a and b. An *ab-separator* of size k is a set S of k vertices (not containing a or b) such that $G{-}S$ contains no ab-path. An *ab-connector* of size k is a union of ab-paths $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_k$ which share no internal vertices, so that that $\mathcal{P}_i \cap \mathcal{P}_j = \{a, b\}$.

THEOREM 1: The minimum size of an ab-separator is equal to the maximum size of an ab-connector: that is, if no $k{-}1$ vertices can separate a from b, then there exist k internally disjoint paths which connect a to b.

It is convenient to prove an alternate form of the theorem. For sets of vertices $A, B \subset G$, an *AB-path* is an $a'\mathcal{P}b'$ with $a' \in A$ and $b' \in B$. An *AB-separator* of size k is a set S of k vertices (which may intersect A and B) such that $G{-}S$ contains no AB-path. An *AB-connector* of size k is a disjoint union of AB-paths $\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_k$, sharing no vertices at all.

THEOREM 2: The minimum size of an AB-separator is equal to the maximum size of an AB-connector.

The two statements are equivalent: for $a, b \in G$ in the first theorem, apply the second theorem to $G{-}\{a, b\}$ with $A = \Gamma(a)$, $B = \Gamma(b)$, the neighboring vertices of a, b. Then an ab-separator is the same thing as an AB-separator, and removing the end vertices in an ab-connector gives an AB-connector.

Proof of Theorem 2: (F. Göring[1]) Induction on the number of edges in G. For G with no edges, we have $S = A \cap B$, itself an AB-connector with 1-vertex paths.

For G having an edge $e = xy$, and minimal AB-separator of size k, we may assume by induction that the Theorem holds for $G{-}e$. If $G{-}e$ has a minimal AB-separator of size k, then there is an AB-connector of size k in $G{-}e$, and hence in G.

Otherwise, let $S = \{v_1, \ldots, v_{k-1}\}$ be a minimal AB-separator of $G{-}e$, so that every AB-path in G contains a vertex of S or the edge $e = xy$. (There cannot be a smaller S, since $S \cup \{x\}$ is an AB-separator of G.) Now, let $S' = S \cup \{x\}$, and consider an AS'-separator T in $G{-}e$. This means there is no AS'-path in $G{-}e{-}T$, so there is no AB-path in $G{-}T$: that is, T is an AB-separator of G, and hence has size at least k. By induction, $G{-}e$ contains an AS'-connector of size k:

$$a_1\mathcal{P}'_1v_1 \sqcup \cdots \sqcup a_{k-1}\mathcal{P}'_1v_{k-1} \sqcup a_k\mathcal{P}'_kx\,.$$

Similarly, letting $S'' = S \cup \{y\}$, an $S''B$-separator has minimal size k, and there is an $S''B$-connector of size k:

$$v_1\mathcal{P}''_1b_1 \sqcup \cdots \sqcup v_{k-1}\mathcal{P}''_1b_{k-1} \sqcup y\mathcal{P}''_kb_k\,.$$

Furthermore, since S disconnects $G{-}e$, every $\mathcal{P}'_i$ is internally disjoint from every $\mathcal{P}''_j$, and we can define an AB-connector of size k in G by concatenating paths: $\mathcal{P}_i = a_i\mathcal{P}'_iv_i\mathcal{P}''_ib_i$ for $i = 1, \ldots, k{-}1$, and $\mathcal{P}_k = a_k\mathcal{P}'_kxy\mathcal{P}''_kb_k$. $\square$

Note: Everything above works without change for a directed multi-graph G, provided we take *path* to mean directed path.

[1] F. Göring, *Short Proof of Menger's Theorem*, Discrete Mathematics **219** (2000) 295-296.