Math 881 Heawood's Theorem Spring 2018

Let G be a graph embedded on a closed surface S (a compact boundaryless 2-manifold). We will derive some constraints on G involving the topology of S.

Suppose G has n vertices and m edges. We can add some edges to G to get a graph $\hat{G}$ which is maximal on S. In fact, $\hat{G}$ defines a triangulation of S with n vertices, $\hat{m}\geq m$ edges, and $\hat{l}$ trianglular faces, so that $3\hat{l} = 2\hat{m}$. Then we have the Euler characteristic:

$$\mathrm{ch} = \mathrm{ch}(S) := n - \hat{m} + \hat{l} = n - \tfrac{1}{3}\hat{m}\,,$$

and

$$m \;\leq\; \hat{m} \;=\; 3n - 3\mathrm{ch}\,.$$

Recall that $\mathrm{ch} = 2 - 2g$, where g is the genus, the number of holes or handles of the surface S.

Now let us consider the chromatic number $\chi(G)$. The following easy upper bound for $\chi(G)$ was proved by Heawood in 1890.

Theorem Let S be a surface with Euler characteristic $\mathrm{ch} \leq 0$, i.e. genus $g \geq 1$. If a graph G is embeddable on S, then its chromatic number is at most:

$$\chi(G) \leq \frac{7 + \sqrt{49 - 24\,\mathrm{ch}}}{2}\,.$$

In particular, if S is the orientable surface of genus $g \geq 1$, then

$$\chi(G) \leq \frac{7 + \sqrt{1 + 48g}}{2}\,.$$

Proof. Let $h = \chi(G)$. Recall that by the greedy algorithm, we can find in G a subgraph H, with n' vertices and m' edges, such that:

$$h \leq 1 + \delta(H) \leq n'\,.$$

Since $\mathrm{ch} \leq 0$, we have

$$-\frac{\mathrm{ch}}{n'} \leq -\frac{\mathrm{ch}}{h}\,.$$

Thus:

$$\begin{aligned} h \;&\leq\; 1 + \delta(H) \;\leq\; 1 + d(H) \;=\; 1 + \frac{2m'}{n'} \\ &\leq\; 1 + \frac{2\,(3\,n' - 3\,\mathrm{ch})}{n'} \;=\; 7 - 6\,\frac{\mathrm{ch}}{n'} \;\leq\; 7 - 6\,\frac{\mathrm{ch}}{h} \end{aligned}$$

Hence:

$$h^2 - 7h + 6\,\mathrm{ch} \leq 0\,,$$

which yields the first inequality of the theorem. The second inequality results from substituting $\mathrm{ch} = 2 - 2g$. QED

A graph can be embedded in a sphere (genus $g = 0$) if and only if it is planar. Thus, the Four Color Theorem for planar graphs is equivalent to the second inequality for $g = 0$. It indicates the delicacy of coloring problems that this case is vastly more difficult to prove than $g \geq 1$.

Now, consider the complete graph $G = K_n$, and let $g(G)$ be the minimal genus in which G can be embedded. For example $g(K_4) = 0$ since K_4 is planar. Then we get $n = \chi(K_n) \leq \frac{1}{2}(\,7 + \sqrt{1 + 48g(K_n)}\,)$, or

$$g(K_n) \geq \frac{1}{12}n^2 - \frac{7}{12}n + 1\,.$$

Furthermore, any G with m edges can be embedded in genus $m - n$, so $g(G) \leq m - n$. (Proof: A spanning tree $T \subset K_n$ with $n - 1$ edges is planar, and so is $T + e$; and we can embed the remaining $m - n$ edges by adding a handle for each one.) Hence:

$$g(K_n) \leq \binom{n}{2} - n \;=\; \frac{1}{2}n^2 - \frac{3}{2}n\,.$$