Let $G = (V, E)$ be a connected graph with $n$ vertices $V = \{v_1, \ldots, v_n\}$ and $m$ edges. The $n{\times}n$ adjacency matrix $A = (a_{ij})$ has $a_{ij} = 1$ if $v_i v_j \in E$, and $a_{ij} = 0$ otherwise. Since $A$ is a real symmetric matrix, $\mathbb{R}^n$ has an orthogonal basis of eigenvectors, with real eigenvalues $\mu_1 \geq \cdots \geq \mu_n$. The matrix acts on $f \in \mathbb{R}^n$, written in the functional notation $f : V \to \mathbb{R}$, by summing over neighboring values: $Af(v) = \sum_{w \in \Gamma(v)} f(w)$.

We always have $\mu_1 \leq \Delta$, the maximal vertex degree, with equality if and only if $G$ is $\Delta$-regular. Indeed, if $Af_1 = \mu_1 f_1$, normalized so that $f_1(v) = 1$ for a certain $v$ and $f_1(w) \leq 1$ for all $w \in V$, then $\mu_1 = \sum_{w \in \Gamma(v)} f_1(w) \leq \Delta$, with equality if and only if $f_1(w) = 1$ for all neighbors and $\mu_1 = \deg(v) = \Delta$. Iterating this, we find $\deg(w) = \Delta$ and $f_1(w) = 1$ for all $w \in V$, so that $G$ is $\Delta$-regular with a unique $\mu_1$-eigenvector. Similarly, $\mu_n = -\Delta$ if and only if $G$ is bipartite and $\Delta$-regular. The spectrum gives bounds for the chromatic number: $1 + |\frac{\mu_1}{\mu_n}| \leq \chi(G) \leq 1 + \mu_1$.

The Laplacian is $L = D - A$ for $D = \mathrm{diag}(\deg(v_1), \ldots, \deg(v_n))$, a diagonal matrix. It has eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$, with 0-eigenvector again given by the constant function $f_1(v) = 1$, even if $G$ is not regular. We have $L = B^T B$, where the $m \times n$ coboundary matrix $B = (b_{ij})$ is defined by arbitrarily choosing an orientation for each edge and setting $b_{ij} = 1$ if $v_j$ is the start of $e_i$; $b_{ij} = -1$ if $v_j$ is the end of $e_i$; and $b_{ij} = 0$ otherwise. This guarantees that $L$ is a positive semi-definite symmetric matrix.

We define the numerical range:

$$W(A) \;=\; \{\langle Af, f\rangle \text{ for } f \in \mathbb{R}^n,\, |f| = 1\} \;=\; \left\{\tfrac{\langle Af,f\rangle}{\langle f,f\rangle} \text{ for } f \in \mathbb{R}^n{-}0\right\}.$$

In fact, $W(A)$ is the real interval $[\mu_n, \mu_1]$. To work with the numerical range $W(L)$, we may use: $\langle Lf, f\rangle = \langle Bf, Bf\rangle = \sum_{vw \in E} (f(v){-}f(w))^2$.

The Laplacian spectrum gives bounds for the connectivity, $\kappa(G) \geq \lambda_2$, and for the edge expansion, $h(G) \geq \frac{1}{2}\lambda_2$. These are shown using the variational characterization of the second-smallest eigenvalue,

$$\lambda_2 \;=\; \min\left\{\tfrac{\langle Lf,f\rangle}{\langle f,f\rangle} \text{ for } f \in \mathbb{R}^n{-}0 \text{ with } \langle f, f_1\rangle = 0\right\},$$

and computing $\frac{\langle Lf,f\rangle}{\langle f,f\rangle} = \frac{\langle Bf,Bf\rangle}{\langle f,f\rangle}$ for an appropriate almost-constant $f$.

**1.** Show that $\lambda_n \leq 2\Delta$, with equality if and only if $G$ is bipartite and $\Delta$-regular.

**2.** Determine the spectra $\{\mu_i\}$ and $\{\lambda_i\}$ for the following graphs:
**a.** The $n$-cycle $C_n$ **b.** The complete bipartite graph $K_{n_1,n_2}$
Explain how the $A$ and $L$ spectra are related. Also verify that the spectra satisfy all the conditions and inequalities mentioned above.

**3.** The vertex expansion constant (or vertex Cheeger constant) is:

$$g(G) \;=\; \min\Big\{\tfrac{|\Gamma(U)-U|}{|U|} \;\text{ for }\; U\subset V,\; |U|\le \tfrac{n}{2}\Big\}.$$

Here $\Gamma(U){-}U$ means the neighbors of $U$ which do not lie in $U$.

Show the following spectral lower bound for the vertex expansion:

$$g(G) \ge \frac{2\lambda_2}{\Delta + 2\lambda_2}.$$

*Hints:*

- If you cannot show the full inequality, at least show it for restricted values of $g(G)$ or $n$, or give a weaker equality in general.
- Model your proof on Bollobas Ch VIII.2 Thm 12 & 13 (p. 269), finding an upper bound for $\lambda_2$ using the variational characterization.
- If you get an upper bound for $\lambda_2$ containing $n = |V|$ or $\ell = |U|$, substitute the values of $n, \ell$ which maximize the expression.
- OOPS: Seems false! (Thanks to Theo) ~~Prove and use the following fact: If~~ $\cancel{U}$ ~~realizes the expansion constant, so that~~ $\cancel{g(G) = \frac{|\Gamma(U)-U|}{|U|}}$ ~~with~~ $\cancel{|U| \le \frac{n}{2}}$~~, then any vertex~~ $\cancel{v \in \Gamma(U)-U}$ ~~has at least as many neighbors in~~ $\cancel{V-U}$ ~~as in~~ $\cancel{U}$~~.~~