2. Let G be a connected n-vertex graph with largest vertex degree Δ, with adjacency matrix A having eigenvalues $\mu_1 > \mu_2 \geq \cdots \geq \mu_n$, and with Laplacian $L = D - A$ having eigenvalues $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n$.

a. Take $G = C_n$ the n-cycle, which we may think of as the union of a directed n-cycle and its opposite. We may thus decompose $A = C + C^{\mathrm{T}}$, where C is the n-cycle permutation matrix, and C^{T} is its transpose.[1] A vector $\vec{a} = (a_1, \ldots, a_n)$ is an eigenvector with $C\vec{a} = \lambda\vec{a}$ whenever $a_i = \lambda_i a_1$ for $i = 2, \ldots, n$, and $a_1 = \lambda^n a_1$. Thus, the eigenvectors are $\vec{a} = (1, \lambda, \lambda^2, \ldots, \lambda^{n-1})$ for any n^{th} root of unity $\lambda = \zeta^j$ for $\zeta = e^{2\pi i/n}$. These are also eigenvectors for $C^{-1} = C^{\mathrm{T}}$, but with eigenvalue ζ^{-j}. Hence the eigenvalues of A are: $\mu_1 = 2\cos(0) = 2$;

$$\mu_{2j} \;=\; \mu_{2j+1} \;=\; \zeta^j + \zeta^{-j} \;=\; 2\cos(\tfrac{2\pi j}{n}) \quad \text{for } j = 1, \ldots, \lceil \tfrac{n}{2} \rceil - 1;$$

and also $\mu_n = 2\cos(\frac{2\pi(n/2)}{n}) = -2$ if n is even. Note that $\mu_1 = 2$, as expected for a 2-regular graph, and $\mu_j = -\mu_{n+1-j}$ exactly when n is even and G is bipartite. Finally, since G is 2-regular, we have $\lambda_j = 2 - \mu_j$.

b. Take $G = K_{n_1,n_2}$ a complete bipartite graph with vertices $V = V_1 \sqcup V_2$. We may immediately give eigenvectors, expressed as functions $f : V \to \mathbb{C}$. The function:

$$f_1(v) = \begin{cases} \sqrt{n_2} & \text{for } v \in V_1 \\ \sqrt{n_1} & \text{for for } v \in V_2 \end{cases}$$

has $Af_1 = \sqrt{n_1 n_2} f_1$. Similarly:

$$f_n(v) = \begin{cases} \sqrt{n_2} & \text{for } v \in V_1 \\ -\sqrt{n_1} & \text{for for } v \in V_2 \end{cases}$$

has $Af_n = -\sqrt{n_1 n_2} f_n$. Finally, we have the $n-2$ dimensional null-space of A, consisting of functions with $\sum_{v \in V_1} f(v) = \sum_{v \in V_2} f(v) = 0$. Thus the eigenvalues of A are: $\pm\sqrt{n_1 n_2}$ with multiplicty 1, and 0 with multiplcity $n-2$. Note that G is regular whenever $n_1 = n_2 = \sqrt{n_1 n 2}$.

The Laplacian spectrum can be given similarly, but it can also be done using a trick. The complement of a regular bipartite graph is a disjoint union of complete graphs: $\overline{K}_{n_1,n_2} = K_{n_1} \sqcup K_{n_2}$.

[1] The directed n-cycle is the Cayley graph of the cyclic group $\Gamma = \langle \sigma \mid \sigma^n = 1 \rangle$, and the adjacency matrix C acts on the space of functions $f : \Gamma \to \mathbb{C}$ via right translation by the sum of the generators, that is $(Cf)(v_i) = f(v_i\sigma) = f(v_{i+1})$.

Claim: Let G be an n-vertex graph. If $\vec{a}$ is a non-null eigenvector of the Laplacian with $L_G(\vec{a}) = \lambda\vec{a} \neq 0$, then $\vec{a}$ is also an eigenvector of the complementary Laplacian, and $L_{\overline{G}}(\vec{a}) = (n{-}\lambda)\vec{a}$.

Proof: Let J be $n{\times}n$ matrix with every entry equal to 1, and I the identity matrix. The vector $\vec{j} = (1,\ldots,1)$ is always a nullvector of L_G, and is also an eigenvector with $J\vec{j} = n\vec{j}$. Let $\vec{a}$ be any eigenvector with $L_G\vec{a} = \lambda\vec{a} \neq 0$. Then $\langle \vec{a}, \vec{j}\rangle = 0$, and $J\vec{a} = 0$. Now, $L_G + L_{\overline{G}} = L_{K_n} = nI - J$, so:

$$L_{\overline{G}}(\vec{a}) \;=\; (nI{-}J{-}L_G)(\vec{a}) \;=\; (n{-}0{-}\lambda)\vec{a}.$$

This proves the Claim.

Now, recall that the eigenvalues of K_{n_1} are 0 (once) and n_1 ($n_1{-}1$ times). Thus $G = K_{n_1,n_2}$ has eigenvalues $n_1{+}n_2{-}n_1 = n_2$ with multiplicity n_1; and similarly n_1 with multiplicity n_2. Finally, since G is connected, it has only one nullvector, so L_G must have one more non-zero eigenvalue λ with eigenvector $\vec{a}$; this must be one of the remaining eigenvectors of $L_{\overline{G}}$, namely a nullvector, and $n_1{+}n_2{-}\lambda = 0$, so that $\lambda = n_1{+}n_2$.

That is, the eigenvalues of L_G are: 0 with multiplicity 1; n_1 with multiplicity n_2; and $n_1{+}n_2$ with multiplcity 1.

3. Consider a connected graph $G = (V,E)$ with $|V| = n$ vertices and maximum vertex degree Δ. Define the vertex expansion constant:

$$g(G) = \min\left\{ \tfrac{|\Gamma(U)-U|}{|U|} \text{ for } U{\subset}V,\ |U|{\le}\tfrac{n}{2} \right\},$$

where $\Gamma(U){-}U$ denotes the neighbors of U which do not lie in U. Let A be the adjacency matrix, D the diagonal matrix of vertex degrees, and $L = D - A$ the Laplacian, having eigenvalues $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le 2\Delta$.

Proposition: $g(G) \;\ge\; \dfrac{2\lambda_2}{\Delta + 2\lambda_2}$.

Proof: Let $g = g(G)$. The inequality is trivial if $g \ge 1$, so we assume $g < 1$. Solving the inequality for λ_2 under the condition $1{-}g > 0$, we see the Proposition is equivalent to:

$$\lambda_2 \;\stackrel{?}{\le}\; \frac{g\Delta}{2(1{-}g)}.$$

Now, the zero-eigenvector of L is the constant function $f_1(x) = 1$, so the variational definition of eigenvalues implies:

$$\lambda_2 \;=\; \min\left\{ \tfrac{\langle Lf,f\rangle}{\langle f,f\rangle} \;\text{ for } f \neq 0 \text{ with } \langle f, f_1\rangle = 0 \right\}.$$

However, $L = BB^{\mathrm{T}}$ for the edge-vertex incidence matrix B, giving $\langle Lf, f\rangle = |Bf|^2 = \sum_{xy\in E}(f(x)-f(y))^2$ for any function $f : V \to \mathbb{R}$. Therefore:

$$\lambda_2 \;\leq\; \frac{\sum_{xy\in E}(f(x)-f(y))^2}{\sum_{x\in V} f(x)^2}$$

for any $f \neq 0$ with $\sum_x f(x) = 0$.

Next consider a set U of $u \leq \frac{n}{2}$ vertices which realizes the expansion constant, so that the separating set $S = \Gamma(U)-U$ has gu vertices, and the complement $W = V-U-S$ has $w = n-u-gu$ vertices. Define a step function:

$$\tilde{f}(x) \;=\; \begin{cases} w & \text{if } x \in U \\ \frac{1}{2}(w-u) & \text{if } x \in S \\ -u & \text{if } x \in W, \end{cases}$$

where the function drops by $\frac{1}{2}(w+u)$ at each step. The mean is $\mu = \frac{1}{n}\sum_x \tilde{f}(x) = \frac{gu(w-u)}{2n}$, and the shifted function $f = \tilde{f} - \mu$ has $\sum_x f(x) = 0$. The square norm may be estimated as:

$$\sum_{x\in V} f(x)^2 \;\geq\; \sum_{x\in U\cup W} f(x)^2 \;=\; u(w-\mu)^2+w(u+\mu)^2 \;\geq\; uw^2+wu^2 \;=\; uw(w+u),$$

since the quadratic function $\phi(\mu) = u(w-\mu)^2 + w(u+\mu)^2$ has its minimum at $\mu = 0$. Lastly, we estimate the quadratic form $\langle Lf, f\rangle$ as:

$$\sum_{xy\in E}(f(x)-f(y))^2 \;=\; E(S,\overline{S})\left(\frac{w+u}{2}\right)^2 \;\leq\; \Delta gu\,\frac{(w+u)^2}{4},$$

since each of the gu vertices of S has at most Δ edges to $\overline{S}$.

Now we apply the variational inequality to f:

$$\begin{aligned}
\lambda_2 \;&\leq\; \frac{\sum_{xy\in E}(f(x)-f(y))^2}{\sum_{x\in V} f(x)^2} \\
&\leq\; \frac{\Delta gu(w+u)^2}{4uw(w+u)} \;=\; \frac{\Delta g(w+u)}{4w} \;=\; \frac{\Delta g(n-gu)}{4(n-u-gu)} \\
&\leq\; \frac{\Delta gn}{4(n-u-gu)} \;=\; \frac{\Delta g}{4(1-\frac{u}{n}(1+g))} \\
&\leq\; \frac{\Delta g}{4(1-\frac{1}{2}(1+g))} \;=\; \frac{\Delta g}{2(1-g)},
\end{aligned}$$

where the last inequality follows from $\frac{u}{n} \leq \frac{1}{2}$. This concludes the proof.

Notes

- The above function $f(x)$ gives such a good upper bound on λ_2 because it breaks up the step $w+u$, with its bond energy $(w+u)^2$, into two equal steps, minimizing the total bond energy $2(\frac{w+u}{2})^2 = \frac{1}{2}(w+u)^2$.

- Other step functions give generally weaker bounds. Again assuming U with u vertices realizes the expansion constant g, the function:

$$f(x) \;=\; \begin{cases} n-u & \text{if } x \in U \\ -u & \text{if } x \in \overline{U}, \end{cases}$$

  leads to $\lambda_2 \leq 2\Delta g$, which is smaller than $\frac{\Delta g}{2(1-g)}$ when $\frac{3}{4} < g < 1$.

- It might be possible to balance our step function more subtly, choosing the step locations to minimize boundary edges. Again taking the vertex boundary $S = \Gamma(U)-U$ with gu vertices, suppose we could find $S' \subset S$ so that the partition $V = U' \sqcup W'$, with $U' = U \cup S'$ and $W' = \overline{U'}$, satisfies $|\Gamma(x) \cap U'| \geq |\Gamma(x) \cap W'|$ for $x \in U'$, and $|\Gamma(x) \cap W'| \geq |\Gamma(x) \cap U'|$ for $x \in W'$. Then the edge boundary has size $|E(U', W')| \leq \frac{\Delta}{2}|S| = \frac{1}{2}\Delta g u$. Defining:

$$f(x) \;=\; \begin{cases} w' & \text{if } x \in U' \\ -u' & \text{if } x \in \overline{U}, \end{cases}$$

  for $u' = |U'|$ and $w' = |W'|$, we obtain $\lambda_2 \leq \frac{\Delta g}{1-g^2}$. Unfortunantely, this is larger than $\frac{\Delta g}{2(1-g)}$ for all g, so it is not really worth trying to find $V = U' \sqcup W'$, though it is an intriguing problem in itself.