Math 881 **Homework 7** **Due Mon Apr 2**

1. Given $p \in [0,1]$, recall the probability distribution $\mathcal{G}(n,p)$, the set of all graphs G on the vertices $V = \{v_1, \ldots, v_n\}$, endowed with probability measure $\mathbb{P}(G) := p^m q^{N-m}$, where $q = 1-p$, $m = \#E(G)$, and $N = \binom{n}{2}$. Prove the following key properties.

a. A fixed edge $e = v_i v_j$ has probability p of being present: $\mathbb{P}(e \in G) = p$.
Here we mean $\mathbb{P}(e \in G) = \sum_G \mathbb{P}(G)$, summing over G with $e \in G$.

b. The presence of distinct edges e, e' are independent events:

$$\mathbb{P}(e, e' \in G) \;=\; \mathbb{P}(e \in G)\, \mathbb{P}(e' \in G) \;=\; p^2.$$

2. We proved Erdös' Theorem that for any r, there exists a graph H with chromatic number $\chi(H) > r$ and shortest cycle girth$(H) > r$. In this problem, consider $r = 3$.

Briefly justify each of the following statements about $G \in \mathcal{G}(n,p)$, where n is arbitary, $0 < p < 1$, $r = 3$, and $s = \lceil \frac{n}{2r} \rceil = \lceil \frac{n}{6} \rceil$.

a. Let $\alpha(G)$ be the size of the largest set of independent vertices of G. Then:

$$\mathbb{P}(\alpha(G) \geqslant s) \;\leqslant\; \binom{n}{s}(1-p)^{\binom{s}{2}}.$$

b. Let $\mathrm{tri}(G)$ be the number of 3-cycles in G. Then:

$$\mathbb{P}(\mathrm{tri}(G) \geqslant \tfrac{n}{2}) \;\leqslant\; \frac{\mathbb{E}(\mathrm{tri})}{n/2} \;\leqslant\; \tfrac{1}{3}(n-1)(n-2)p^3.$$

c. If G has $\alpha(G) < s$, $\mathrm{tri}(G) < \frac{n}{2}$, then one can construct a subgraph $H \subset G$ with $\lceil \frac{n}{2} \rceil$ vertices such that $\chi(H) > 3$ and girth$(H) > 3$.

d. We have: $\mathbb{P}(\alpha(G) < s \text{ and } \mathrm{tri}(G) < \frac{n}{2}) \;\geqslant\; 1 - \binom{n}{s}(1-p)^{\binom{s}{2}} - \frac{1}{3}(n-1)(n-2)p^3 \overset{\text{def}}{=} P(n,p)$.
Do computer experiments to find n as small as you can such that $P(n,p) > 0$ for some p. As in part (c), what is the smallest H thus produced? Hint: Guess n, and maximize the lower bound over p. Adapt Stirling's asymptotic formula $n! \sim \sqrt{2\pi n}(\frac{n}{e})^n$ to approximate the binomial coefficient $\binom{n}{an}$ for a constant $0 < a < 1$ and $n \to \infty$.

e. Extra Credit: Give an explicit graph H of the type guaranteed in (c).

3. Recall that for fixed $0 < p < 1$ and $n \to \infty$, a graph $G \in \mathcal{G}(n,p)$ is almost certainly connected. In fact, with probability approaching 1, G is connected by 2-edge paths: for every pair of vertices a, b, there is some c such that $ac, bc \in E(G)$. As always, we prove this by bounding the probability of the negation, and showing it approaches zero.

a. Show that for fixed $0 < p < 1$ and $n \to \infty$, a graph $G \in \mathcal{G}(n,p)$ is almost certainly k-connected. In fact, for any vertices $S = \{v_1, \ldots, v_k\}$, we have $G-S$ connected by 2-edge paths.

b. Strengthen the result in (a) by finding a function $p = p(n)$ such that $G \in \mathcal{G}(n, p(n))$ is almost certainly k-connected, but $p(n) \to 0$ as $n \to \infty$, making G somewhat sparse. Hint: Find the smallest $p(n)$ which works with your estimate in part (a).