2e. Mycielski gave an iterative construction of a triangle-free graph of arbitrarily high chromatic number. Given a triangle-free G with n vertices and $\chi(G) = k$, we define a new triangle-free graph M with $2n{+}1$ vertices and $\chi(M) = k{+}1$.

Let M contain a copy of G with vertices $\{v_1, \ldots, v_n\}$, as well as a star graph $K_{1,n}$ with vertices $\{u_0, u_1, \ldots, u_n\}$ and edges u_0u_i. Keep each edge v_iv_j in G, and add two edges v_iu_j, v_ju_i in M.

The only possible triangle in M is of the form $v_iv_ju_k$, but this would occur only if $v_iv_jv_k$ were a triangle in G.

To see that the construction increases the chromatic number $\chi(G) = k$, consider a proper k-coloring of $M{-}\{u_0\}$; that is, a mapping $c : \{v_1, \ldots, v_n, u_1, \ldots, u_n\} \to \{1, 2, \ldots, k\}$ with $c(x) \neq c(y)$ for adjacent vertices x, y. If we had $c(u_i) \subset \{1, 2, \ldots, k{-}1\}$ for all i, then we could define a proper $(k{-}1)$-coloring of G by $c'(v_i) = c(u_i)$ when $c(v_i) = k$, and $c'(v_i) = c(v_i)$ otherwise. But this is impossible for $\chi(G) = k$, so c must use all k colors for $\{u_1, \ldots, u_n\}$, and any proper coloring of the last vertex u_0 must use an extra color. That is, $\chi(M) = k{+}1$.

3. Proposition: For fixed k and $p(n) = \frac{\log(n)}{\sqrt{n}}$, a graph $G \in \mathcal{G}(n, p(n))$ is almost certainly k-connected (in fact, connected by 2-edge paths).

Proof: To show $\mathbb{P}\big(G \in \mathcal{G}(n, p(n)) \text{ is } k\text{-connected}\big) \to 1$ as $n \to \infty$, we will bound the probability that G *fails* to be k-connected. Writing $G = (V, E)$ with $|V| = n$ vertices, we ask if there is a k-element vertex set $S \subset V$ for which $G{-}S$ is not connected by 2-edge paths, so that there are $a, b \in V{-}S$ such that no path acb lies in G, for $c \in V{-}S{-}\{a, b\}$:

$$\begin{aligned} \mathbb{P}(\text{FAIL}) &\leq \mathbb{P}\big(\exists S\, \exists a,b\, \forall c : ac \notin E \text{ or } bc \notin E\big) \\ &\leq \sum_S \sum_{a,b} \mathbb{P}\big(\forall c : \text{NOT}(ac, bc \in E)\big)\,. \end{aligned}$$

Now, for $c \neq c'$, the edges ac, bc, ac', bc' are distinct, so the events $ac, bc \in G$ and $ac', bc' \in G$ are independent. Hence:

$$\begin{aligned} \mathbb{P}(\text{FAIL}) &\leq \sum_S \sum_{a,b} \prod_c \big(1 - \mathbb{P}(ac, bc \in G)\big) \\ &= \binom{n}{k}\binom{n-k}{2}(1{-}p^2)^{n-k-2} \\ &\leq An^{k+2}(1{-}p^2)^n \;=\; An^{k+2}\Big(1{-}\frac{\log^2(n)}{n}\Big)^n, \end{aligned}$$

where $A = A(k)$ is a constant. We must show this upper bound tends to zero as $n \to \infty$.

Taking the logarithm and using $\log(1{-}x) \leq -x$ gives:

$$\log(A) + (k{+}2)\log(n) + n\log\Big(1{-}\frac{\log^2(n)}{n}\Big) \;\leq\; \log(A) + (k{+}2)\log(n) - n\Big(\frac{\log^2(n)}{n}\Big)\,.$$

Since $\log^2(n) \gg (k{+}2)\log(n)$, it is clear that $\lim_{n\to\infty} \log(An^{k+2}(1{-}p^2)^n) = -\infty$, and:

$$\mathbb{P}(\text{FAIL}) \;\leq\; An^{k+2}(1{-}p^2)^n \to 0 \;\text{ as }\; n \to \infty.$$

Hence, the complementary probability that G is k-connected tends to 1.

Note that this argument works for any $p(n) = \frac{\sqrt{\log(n)}}{\sqrt{n}} f(n)$ with $f(n) \to \infty$.