Math 881 **Homework 6** **Due Wed Mar 14**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or person, give explicit credit.

Recall our disussion of Extremal Graph Theory, cf. Bollobás Ch. IV.2, IV.4.

- The *Turán graph* $T_r(n)$ is the complete r-partite graph with n vertices distributed as evenly as possible among $V = V_1 \sqcup \cdots \sqcup V_r$: no edges within any V_i, all possible edges between each V_i and V_j, and $|n_i - n_j| \leqslant 1$ for $n_i = |V_i|$. We let $t_r(n) = e(T_r(n))$ be the number of edges.
- Turán Theorem: $T_{r-1}(n)$ is the unique edge-maximal graph on n vertices which contains no K_r subgraph. Thus, any graph G with n vertices and $e(G) \geqslant t_{r-1}(n)$ edges has $K_r \subset G$, unless $G = T_{r-1}(n)$.
- $K_r(t)$ is the complete r-partite graph with $|V_i| = t$ for all i.
- Erdös-Stone Theorem: For fixed r and any $\epsilon > 0$, there is n_0 such that any graph G with $n \geqslant n_0$ vertices and $e(G) \geqslant \frac{1}{2}\left(1 - \frac{1}{r-1} + \epsilon\right) n^2$ edges has $K_r(t_n) \subset G$, where $t_n \to \infty$ as $n \to \infty$.
- The extremal number $\mathrm{ex}(F, n)$ of a graph F is the maximum possible number of edges in an n-vertex graph which does *not* contain a copy of F. For example, $\mathrm{ex}(K_r, n) = t_{r-1}(n)$.
- Erdös-Simonovits Theorem: A graph F with chromatic number $\chi(F) = r$ has extremal number $\mathrm{ex}(F, n) = \frac{1}{2}(1 - \frac{1}{r-1})n^2 + \mathrm{o}(n^2)$ as $n \to \infty$.
- In Landau little-o notation, an error term $\mathrm{o}(n^2)$ means that for any $\epsilon > 0$, the error is smaller than ϵn^2 for sufficiently large n.

1a. Show that $t_r(n) \geqslant \left(1 - \frac{1}{r}\right)\binom{n}{2}$ for all r, n
b. Show that $t_r(n) = \frac{1}{2}\left(1 - \frac{1}{r}\right)n^2 + \mathrm{o}(n^2)$, where r is fixed and $n \to \infty$.

2. The *density* of an n-vertex graph H is $\text{dens}(H) = e(H)/\binom{n}{2}$, the ratio of edges in H to edges in K_n. The *upper density* of an infinite graph G is the supremum of the densities of arbitrarily large finite subgraphs $H \subset G$: that is, for $a_n = \max\{\text{dens}(H) \text{ for } n\text{-vertex } H \subset G\}$, we set: $\overline{\text{dens}}(G) = \limsup a_n$ as $n \to \infty$; this is the smallest D such that $\text{dens}(H) \leqslant D$ for all but finitely many $H \subset G$.

Amazingly, the upper density of an arbitrary infinite G does not range over the entire real interval $[0, 1]$, but must lie in the countable set

$$\overline{\text{dens}}(G) \in \{0, \tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \ldots, 1\}.$$

Prove this using the Erdös-Stone Theorem. *Hint:* If G has density $D > 1 - \frac{1}{r-1}$, find an infinite family of special $H \subset G$ to show $D \geqslant 1 - \frac{1}{r}$.

3. Give a careful proof of the Erdös-Simonovits Theorem as a corollary of the Erdös-Stone Theorem. Justify each of the inequalities in the proof given in Bollobás Ch IV.4 Corollary 24, p. 123. (Correct any typos.)

4. Let Q be the 8-element group of unit quaternions. Guess a minimal set of generators and relations for Q, and verify it by carefully constructing the Cayley graph of your presentation.

5. Recall that for a group A with generators $\{a, b, \ldots\}$, and a subgroup $B \subset A$, the Schreier graph $G(A, B)$ has vertices $V = \{Bg \mid g \in A\}$ and arrows $Bg \xrightarrow{a} Bga$ colored by each generator a, so that multi-edges and loops may occur.

a. Find a graph-theoretic criterion on the directed, edge-colored multi-graph $G(A, B)$ which determines whether B is a normal subgroup, $B \lhd A$.
Hint: B is normal $\iff G(A, B) = G(A/B)$ is a Cayley graph, possibly with redundant generators. (Existence of automorphisms, translation of cycles)

b. Could there be a graph-theoretic criterion which detects whether B is a central subgroup, $B \subset Z(A)$? (Nope: see solns)

** NEXT TIME ** When is a colored directed graph a Cayley graph, or a Schreier graph?