1a. PROP: The Turan graph numbers satisfy $t_r(n) \geqslant (1-\frac{1}{r})\binom{n}{2}$ for all r, n.
Proof: The Turan graph $T_r(n)$ is the complete r-partite graph on n vertices with smallest part of order $\lfloor \frac{n}{r} \rfloor$, and largest part of order $\lceil \frac{n}{r} \rceil$. We construct $T_r(n{+}1)$ by adding a vertex to the smallest part, with new edges to all other parts, so the number of edges is $t_r(n{+}1) = t_r(n) + n - \lfloor \frac{n}{r} \rfloor$.

We prove the Proposition by induction on n, with $n = 1$ being trivial. Assuming the Proposition for n vertices, $t_r(n) \geqslant (1-\frac{1}{r})\binom{n}{2}$, gives:

$$\begin{aligned} t_r(n{+}1) &= t_r(n) + n - \lfloor \tfrac{n}{r} \rfloor \\ &\geqslant (1-\tfrac{1}{r})\tbinom{n}{2} + n - \tfrac{n}{r} \\ &= (1-\tfrac{1}{r})(\tbinom{n}{2} + n) \;=\; (1-\tfrac{1}{r})\tbinom{n+1}{2}. \end{aligned}$$

Thus, the Proposition is true for $n{+}1$ vertices.

1b. PROP: $t_r(n) = \frac{1}{2}\big(1-\frac{1}{r}\big)n^2 + \mathrm{o}(n^2)$, where r is fixed and $n \to \infty$.
Proof: Let the parts of $T_r(n)$ have $n_1, \ldots, n_r$ vertices, where $n_1 + \cdots + n_r = n$. The number of edges is half the sum of vertex degrees, and each vertex neighbors all vertices outside its own part, so:

$$t_r(n) \;=\; \frac{1}{2}\sum_{i=1}^{r} n_i(n{-}n_i) \;=\; \tfrac{1}{2}(n\textstyle\sum_i n_i - \sum_i n_i^2) \;=\; \tfrac{1}{2}(n^2 - \textstyle\sum_i n_i^2).$$

Now, the function $f(x_1, \ldots, x_n) = \frac{1}{2}(n^2 - \sum_i x_i^2)$, constrained to the hyperplane $g(x_1, \ldots, x_n) = x_1 + \cdots + x_r = n$, is concave-down with a unique maximum point given by the Lagrange multiplier equation $\nabla f = \lambda \nabla g$, i.e. $x_i = \lambda = \frac{n}{r}$ for all i. Thus:

$$t_r(n) \;\leqslant\; f(\tfrac{n}{r}, \ldots, \tfrac{n}{r}) \;=\; \tfrac{1}{2}(n^2 - r(\tfrac{n}{r})^2) \;=\; \tfrac{1}{2}(1-\tfrac{1}{r})n^2.$$

Part (a) implies:

$$t_r(n) \;\geqslant\; (1-\tfrac{1}{r})\tbinom{n}{2} \;=\; \tfrac{1}{2}(1-\tfrac{1}{r})n^2 - \tfrac{1}{2}(1-\tfrac{1}{r})n.$$

These upper and lower bounds clearly imply the asymptotic error bound:

$$\frac{\big|t_r(n) - \frac{1}{2}(1-\frac{1}{r})n^2\big|}{n^2} \to 0 \;\text{ as }\; n \to \infty,$$

which is the meaning of the Proposition.

3. THEOREM: (Erdös-Simonovits) A graph F with chromatic number $\chi(F) = r$ has extremal edge number $\mathrm{ex}(F,n) = \frac{1}{2}(1-\frac{1}{r-1})n^2 + \mathrm{o}(n^2)$ as $n \to \infty$.
Proof: We define $\mathrm{ex}(F,n)$ as the smallest number satisfying, for all n-vertex graphs G_n:

$$\mathrm{ex}(F,n) < e(G_n) \;\Rightarrow\; F \subset G_n,$$

or contrapositively:

$$F \not\subset G_n \;\Rightarrow\; \mathrm{ex}(F,n) \geqslant e(G_n).$$

Since F is not $(r-1)$-colorable, we know $F \not\subset T_{r-1}(n)$, so $\mathrm{ex}(F,n) \geqslant e(T_{r-1}(n))$, and by Prob 1a, $e(T_{r-1}(n)) = t_{r-1}(n) \geqslant \frac{1}{2}(1-\frac{1}{r-1})n^2 - cn$ for a constant c.

On the other hand, F is r-colorable, so $F \subset K_r(s)$ for sufficiently large s. By the Erdös-Stone Theorem, for any $\epsilon > 0$, sufficiently large n, and any G_n,

$$\tfrac{1}{2}(1+\tfrac{1}{r-1}+\epsilon)n^2 < e(G_n) \;\Rightarrow\; K_r(s_n) \subset G_n \;\Rightarrow\; F \subset G_n,$$

taking n large enough that $s_n \geqslant s$. Since $\frac{1}{2}(1+\frac{1}{r-1}+\epsilon)n^2$ satisfies the first form of the definition, we get $\mathrm{ex}(F,n) \leqslant \frac{1}{2}(1+\frac{1}{r-1}+\epsilon)n^2$.

Restating these bounds, we have, for any $\epsilon > 0$ and sufficiently large n,

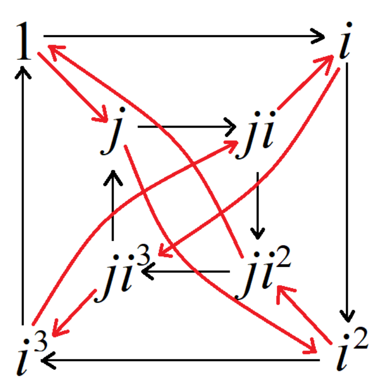
$$\tfrac{1}{2}(1+\tfrac{1}{r-1})n^2 - cn \;\leqslant\; \mathrm{ex}(F,n) \;\leqslant\; \tfrac{1}{2}(1+\tfrac{1}{r-1})n^2 + \epsilon n^2.$$

This clearly implies the conclusion.

4. A minimal presentation for the 8-element group of unit quaternions is:

$$Q \;=\; \langle i, j \mid i^4 = 1, iji = j, jij = i\rangle.$$

To see this, draw a Cayley diagram starting with the j-edge $1 \xrightarrow{j} j$. Using the relation $i^4 = 1$, draw two 4-cycles of i-edges. Using $iji = j$, follow the path $i^3 \xrightarrow{i} 1 \xrightarrow{j} j \xrightarrow{i} ji$ to draw the arrow $i^3 \xrightarrow{j} ji$. Repeat this to draw two more j-arrows. Finally, using $jij = i$, or $iji^{-1} = j^{-1}$, follow the path $i^3 \xrightarrow{i} 1 \xrightarrow{j} j \xleftarrow{i} ji^3$ to draw the arrow $i^3 \xleftarrow{j} ji^3$. Repeat this to draw the final three j-arrows. The result has 8 i-arrows (black, thin) and 8 j-arrows (red, thick):



Neglecting orientation and coloring, this is the edge-graph of a cube with 4 extra edges connecting opposite corners. As for minimality, the non-cyclic group Q requires at least two generators, and all three relations are clearly necessary. (Can we rule out a presentation with only two relations??)

5a. For a finite group A with generators $P = \{a, b, \ldots\}$ and a subgroup $B \subset A$, recall the Schreier graph $G(A, B)$ whose vertices are the left cosets Bg for $g \in A$, and whose edges $Bg \xrightarrow{p} Bgp$ are directed and colored by generators $p \in P$.

Prop: The subgroup B is normal in A if and only if there is an an automorphism of colored graphs $\phi_q : G(A, B) \to G(A, B)$ with $\phi_q(B) = Bq$, for each generator $q \in P$. Informally, the colored graph looks the same from each vertex Bq as from the base point B.

Proof: If B is normal, meaning $gB = Bg$ for all $g \in A$, then define $\phi_q(Bg) = qBg = Bqg$. If $Bg \xrightarrow{p} Bgp$ is an edge, then $\phi_q(Bg) = Bqg \xrightarrow{p} Bqgp = \phi_q(Bgp)$ is also an edge, so ϕ_q is indeed an automorphism of colored graphs.

Conversely, if there exist such automorphisms ϕ_q for $q \in P$, write any $b \in B$ as a product of generators: $b = p_1 \cdots p_\ell$ for p_i or $p_i^{-1} \in P$, where we allow a backward step along $Bgp \xleftarrow{p} Bg$, equivalent to $Bgp \xrightarrow{p^{-1}} Bg$. Then the cycle

$$B \xrightarrow{p_1} Bp_1 \xrightarrow{p_2} \cdots \xrightarrow{p_\ell} Bb = B$$

is taken by ϕ_q to a cycle

$$Bq \xrightarrow{p_1} Bqp_1 \xrightarrow{p_2} \cdots \xrightarrow{p_\ell} Bqb = Bq.$$

That is, $qb \in Bq$ for all $b \in B$, so that $qB \subset Bq$. This implies $qB = Bq$ for all generators q, and hence $gB = Bg$ for all $g \in A$, and B is normal.

Notes: If A is not finite, but we assume further automorphisms $\phi_{q^{-1}}$ with $\phi_{q^{-1}}(B) = Bq^{-1}$, then we have $qB \subset Bq$ and $q^{-1}B \subset Bq^{-1}$, so that $qB = Bq$; but I am not sure the extra automorphisms are really needed.

Some further equivalent conditions:

- B is normal in A
- $G(A, B) = G(A/B)$, the Cayley graph of the quotient group
- For each $q \in P$, there is a colored graph automorphism with $\phi_q(B) = Bq$
- For each $g \in A$, there is a colored graph automorphism with $\phi_g(B) = Bg$
- Each cycle $B \xrightarrow{p_1} \cdots \xrightarrow{p_\ell} B$ corresponds to a cycle $Bq \xrightarrow{p_1} \cdots \xrightarrow{p_\ell} Bq$, $\forall q \in P$
- Each cycle $B \xrightarrow{p_1} \cdots \xrightarrow{p_\ell} B$ corresponds to a cycle $Bg \xrightarrow{p_1} \cdots \xrightarrow{p_\ell} Bg$, $\forall g \in A$.

Finally, note that a colored graph automorphism ϕ is uniquely determined by $\phi(B) = Bg$, since if $h = p_1 \cdots p_\ell$, then $\phi(Bh)$ is obtained by following the path $Bg \xrightarrow{p_1} \cdots \xrightarrow{p_\ell} Bgh$. The automorphism exists if this procedure is well-defined, independent of the chosen representative $Bh = Bbh$. For any Cayley graph, the mapping $G(A) \to \operatorname{Aut} G(A)$, $g \mapsto \phi_g$, is an injective group homomorphism.

5b. The Cayley graph of any group C can be obtained as $G(C) = G(C \times B, B)$ for any group B, where we have the normal subgroup $B \cong 1 \times B \lhd C \times B$. The subgroup B is central in $C \times B$ if and only if B is abelian. Thus, any Cayley graph can be obtained as a Schreier graph both of a central and a non-central subgroup, and no criterion on the graph can distinguish these cases.