Math 881 **Homework 5** **Due Mon Feb 19**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or person, give explicit credit.

1. Prove that for any G, there is an ordering of the vertices $V = \{v_1, \ldots, v_n\}$ such that the Greedy Algorithm produces a minimal coloring.

2a. Show that the United States map requires 4 colors; that is, if G is the graph whose vertices are the 48 contiguous states, with an edge between each pair of states which share a border (not just a corner), then $\chi(G) = 4$. *Hint:* Look at the subgraph of states surrounding Nevada.

b. For every $k \geqslant 3$, find a graph G_k with $\chi(G_k) = k$, but $K_k \not\subset G_k$.

c. *Extra Credit:* Find a graph G with $\chi(G) = k$ but $K_{k-1} \not\subset G$, or even $K_{k-2} \not\subset G$.

3. Recall Kempe's proof of the Five Color Theorem ([B] Ch V.3, Thm 8): For a vertex with $\deg(v) \leqslant 5$, we inductively assume a 5-coloring of $G{-}v$. If all colors 1,...,5 are already assigned to the neighbors of v, then we construct a new 5-coloring of $G{-}v$ by switching colors 1↔3 or 2↔4 in certain colored components (Kempe chains); this frees up one of the 5 colors to assign to v.

Why can't we modify this argument to prove the Four Color Theorem (as Kempe believed)? Namely, assume that $G{-}v$ has a 4-coloring, and construct a 4-coloring of G by color-switching as before.

Construct an example of $G{-}v$ with a coloring for which no possible color-switching in one or more Kempe chains will free up any of the 4 colors around v.

4. Draw an embedding of the graph K_7 in the torus T, the genus 1 closed surface. You can realize T as a quotient of $\mathbb{R}^2$ by a group of translations $\Gamma = \mathbb{Z}\vec{v} \oplus \mathbb{Z}\vec{w}$, where $\vec{v}, \vec{w}$ are linearly independent vectors. Then draw the torus as a fundamental domain F of the translations: e.g. the parallelogram $F = \{x\vec{v} + y\vec{w} \mid x, y \in [0,1]\}$, with the top and bottom edges identified, as well as the left and right edges.
Note: The answer is easy to look up, but try to find it yourself.

5. For a graph G, recall that the line graph LG has a vertex v_e for each edge e of G, i.e. $V(LG) = E(G)$; and there is an edge $v_e v_{e'}$ whenever e, e' share a vertex in G.

a. Show that if LG is planar, then every vertex $x \in G$ has either: $\deg(x) \leqslant 3$; or $\deg(x) = 4$ and x is a cut-vertex of G. *Hint:* Show that if $\deg(x) \geqslant 5$ or if $\deg(x) = 4$ and $G{-}x$ is connected, then $L(G)$ has a IK_5 minor.

b. Show that if LG is planar, then $\chi(LG) \leqslant 4$. That is, the Four Color Theorem holds for line graphs. *Hint:* Use Vizing's Theorem on edge-coloring, and the equivalence between edge-coloring of G and vertex-coloring of LG.

c. *Extra Credit:* Is it true that if LG is planar, then $\chi(G) \leqslant 4$?