Math 881 **Homework 5 Solutions** **Due Mon Feb 19**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or person, give explicit credit.

1. PROPOSITION: For a graph $G = (V, E)$, let $c : V \to [k]$ be a minimal proper coloring, and label $V = \{1, \ldots, n\}$ so that all vertices of color 1 are listed first, then all vertices of color 2, etc: $c(i) \leqslant c(j)$ for all $i < j$. Then the Greedy Algorithm with the given ordering of V also produces a minimal coloring.

Informal proof. Writing $V_r = c^{-1}(r)$, the vertices of color r, we perform the Greedy Algorithm to assign a color $g(v)$ successively to $v \in V_1, \ldots, V_k$. For $v \in V_1$, we set $g(v) = 1$, which does not conflict since v is not adjacent to any other $w \in V_1$. For $v \in V_2$, we set $g(v) = 1$ if v is not adjacent to any $w \in V_1$; and otherwise $g(v) = 2$, not conflicting with any other $w \in V_2$.

Continuing in this way, a vertex $v \in V_r$ may at worst conflict with earlier vertices of colors $1, \ldots, r{-}1$, so we set $g(v) \leqslant r$. Completing the coloring up to V_k, we see g uses at most k colors. Since k is minimal, g is a minimal coloring.

Formal proof. Letting $g : V \to \mathbb{N}_+$ denote the greedy coloring, we will show by induction that $g(j) \leqslant c(j)$ for $j = 1, \ldots, n$. Clearly $g(1) = c(1) = 1$. For $j > 1$, suppose by induction that $g(i) \leqslant c(i)$ for $i = 1, \ldots, j{-}1$. Let $i < j$ be the neighbor of j which has the *largest* color $g(i)$ so far. Now, $c(i) \neq c(j)$ since c is proper, and $c(i) \leqslant c(j)$, so $g(i) \leqslant c(i) < c(j)$. The Greedy Algorithm sets $g(j)$ to be *at most* $g(i){+}1$, so $g(j) \leqslant g(i){+}1 \leqslant c(j)$, and the induction proceeds.

We conclude that $g(j) \leqslant c(j) \leqslant c(n) = k$ for all j, so g uses at most k colors. Since k is minimal, g is a minimal coloring.

Note: From the first proof, we see the greedy coloring may be different from c.