Math 881 **Homework 4 Solutions** **Wed Feb 7**

1. PROPOSITION: The following are equivalent for a planar graph G with $n \geqslant 3$ vertices.

(a) G has the maximum number of edges, $m = 3n - 6$.

(b) G is edge-maximal: that is, $G{+}xy$ is not planar for any edge $xy \notin E(G)$.

(c) Every face of G, including the infinite face, is trianglular (bounded by a 3-cycle).

Proof. (a) $\Rightarrow$ (b): Any planar G with $n \geqslant 3$ vertices has: m edges each on the boundary of at most 2 faces; ℓ faces each with at least 3 boundary edges; and k connected components. Then Euler's formula $n - m + \ell = k + 1$ and the edge-region inequality $2m \geqslant \sum \deg(F) \geqslant 3\ell$ imply $m \leqslant 3n - 3(k{+}1) \leqslant 3n - 6$. Now, if G has the maximum number of edges $m = 3n - 6$, adding one more edge makes $G{+}xy$ non-planar.

(b) $\Rightarrow$ (c): Suppose G is planar but any $G{+}xy$ is non-planar. If a face F had non-adjacent vertices x, y in its boundary, then we could connect these with an edge across the interior of F, and get a planar embedding of $G{+}xy$. Thus, the boundary of F must be a complete graph, but it cannot contain K_4, which is impossible to draw with one region touching all four vertices. Thus the boundary of F is a 3-cycle.

(c) $\Rightarrow$ (a): Suppose every face of G is bounded by a 3-cycle. If G were disconnected, then the boundary of the infinite face would also be disconnected, not a 3-cycle; thus G must be connected. Now, each edge is contained in the 3-cycle boundary of a face, and in the boundary of another face outside the first 3-cycle. Counting pairs (e, F), where e is in the boundary of F, gives $2m = 3\ell$, so that Euler's formula reduces to $m = 3n - 6$.

2. PROPOSITION: If K is a graph with all $\deg(v) \leqslant 3$, then any general minor of K contains a topological minor: $TK \subset IK$.

First proof: We transform K into IK by repeated inflation: replacing a vertex v with an edge xy, and replacing each incident edge va with one or two edges xa and/or ya.

We induct on the number of inflation operations, based on the trivial case of no inflations. Suppose by induction that $TK \subset IK$, and inflate $v \in IK$ into $xy \subset IK'$. If $v \notin TK$, then $TK \subset IK'$.

Otherwise $v \in TK$, and by hypothesis v has at most three neighbors $a, b, c \in TK$. Up to relabeling, all but one of these must be neighbors of x, say $xa, xb \subset IK'$. If the last neighbor of v is also a neighbor of x, say $xc \subset IK'$, we may define the K-topological minor $TK' \subset IK'$ by relabeling $v \in TK$ as $x \in TK'$. If instead $yc \subset IK'$, we may transform $TK \subset IK$ into $TK' \subset IK'$ by replacing vertex v with x; edges va, vb with xa, xb; and the edge vc with the path xyc.

Second proof: We transform K into IK by inflating each vertex $v_i \in K$ to a connected subgraph $H_i \subset IK$, and for each edge $v_iv_j \subset K$, adding at least one edge from H_i to H_j in IK; let us specify one such edge $x_{ij}x_{ji} \subset IK$ with $x_{ij} \in H_i$, $x_{ji} \in H_j$. Inside H_i, we will choose a center vertex y_i and independent paths P_{ij} from y_i to each x_{ij}. All the P_{ij} together with the edges $x_{ij}x_{ji}$ will clearly form a $TK \subset IK$.

If v_i has at most two neighbors, we choose $y_i = x_{ij}$ for some j, $P_{ij} = x_{ij}$ the one-point path, and P_{ik} any path from x_{ij} to the other x_{ik}, if any. If v_i has three neighbors, we have $x_{ij}, x_{ik}, x_{i\ell} \in H_i$, not necessarily distinct. Consider paths $x_{ij}Px_{i\ell}, x_{ik}Qx_{i\ell} \subset H_i$. Let y_i be the vertex of $P \cap Q$ which is closest to x_{ij}, so that $P = x_{ij}P'y_iP''x_{i\ell}$ and $Q = x_{ik}Q'y_iQ''x_{i\ell}$ with $P' \cap Q = y_i$. Then we set $P_{ij} = P'$, $P_{ik} = Q'$, $P_{i\ell} = Q''$, all independent paths.