Math 881 **Homework 3** **Due Mon Jan 29**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

Given a bipartite graph G with vertices $A \sqcup B$ and all edges between A and B, a *matching* is a set edges $M \subset E$ which are disjoint; that is, every vertex lies on at most one edge of M. A *complete matching* from A to B is a matching with $|M| = |A|$; that is, every $a \in A$ lies on *exactly* one edge of M. The *matching problem* is to find M of maximal size (ideally a complete matching).

The Hall Matching Theorem states that a complete matching exists on bipartite G iff every set of vertices in A has at least as many neighbors as the size of the set: $|\Gamma(S)| \geqslant |S|$ for all $S \subset A$.

We transform the matching problem into a max flow problem on a network. That is, we let $\vec{G}$ have the same vertices as G, so that each edge $ab \in G$ becomes a directed edge $a{\to}b \in \vec{G}$; and we add source and sink vertices s, t with edges $s{\to}a$ for each $a \in A$ and $b{\to}t$ for each $b \in B$. We set all capacities $c(x, y) = 1$. Then a flow f of volume k on $\vec{G}$ naturally corresponds to a matching M with k edges.

1. Transform the Augmenting Path Algorithm on $\vec{G}$ into an algorithm to find a maximal mathing on G. Translate so your algorithm does not refer to $\vec{G}$.

2. Bollobás (p. 76). Given a finite group and its subgroup H, we wish to find a set of representatives for all the left cosets $g_1H, \ldots, g_kH$ which are simultaneously representatives for all right cosets $Hg_1, \ldots, Hg_k$. (Note that H is a normal subgroup when $gH = Hg$, i.e. when *any* set of representatives are simultaneous left and right representatives.)

a. Transform this problem into the complete matching problem on a bipartite graph constructed from the group and subgroup.

b. Apply the Hall Matching Theorem to the bipartite graph from (a), and obtain a criterion for the existence of simultaneous representatives $g_1, \ldots, g_k$. (Translate so your criterion does not refer to the graph.)

c. *Extra Credit:* Show that your criterion always holds, so that any subgroup has simultaneous representatives; or find a counterexample.