Math 881 **Homework 2** **Due Mon Jan 22**

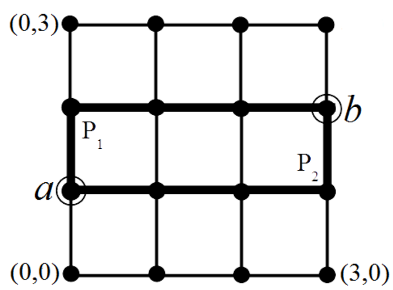
You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

1. The vertex form of Menger's Theorem states: For a graph $G = (V, E)$ and non-adjacent vertices a, b, the minimum number of vertices $S \subset V - \{a\, b\}$ which disconnect a from b is equal to the maximum number of independent ab-paths (paths from a to b with no other common vertices).

Prove this from the Max Flow/Min Cut Theorem. That is, transform the graph G into an appropriate network: a directed graph $\widetilde{G}$ with source s, sink t, and edge-capacity function $c(x, y)$. Any vertex cut $S \subset V$ in G should correspond to an edge-cut $\tilde{S} \sqcup \tilde{T} = \tilde{V}$ in $\widetilde{G}$; and any set of k independent ab-paths in G should correspond to a flow of volume k in $\widetilde{G}$.

Hint: The difficulty is to ensure that the paths corresponding to a flow are *vertex disjoint*, not just edge-disjoint. Thus, you need to limit the capacity of each G-vertex to a flow of 1, by an appropriate edge-capacity restriction on $\widetilde{G}$.

2. Consider the 4×4 grid graph G whose vertices are the integer points $V = [0, 3] \times [0, 3]$ for $[0, 3] = \{0, 1, 2, 3\}$; with edges of the form $e = (i, j)(i, j{+}1)$ or $(i, j)(i{+}1, j)$ for all i, j. Let $a = (0, 1)$ and $b = (3, 2)$; and let P_1, P_2 be two "greedy" independent paths (vertex-disjoint except for endpoints), connecting a and b in the shortest way: $P_1 = (0, 1)(0, 2), (1, 2), (2, 2), (3, 2)$ and $P_2 = (0, 1), (1, 1), (2, 1), (3, 1)(3, 2)$.



These greedy paths block any third independent path, and they are not a maximal family of independent ab-paths.

a. Apply the Augmenting Path Algorithm (translated from the associated network $\widetilde{G}$ from Prob. 1) to modify the above paths into a family of 3 independent ab-paths.

b. Apply the algorithm one more time to find a disconnecting set $\{v_1, v_2, v_3\}$.

c. What is the vertex connectivity $\kappa(G)$? How can you be sure?