Math 881 **Homework 1** **Due Wed Jan 17**

You are encouraged to discuss homework problems with other students, but you must write out solutions in your own words. LaTeX is encouraged, but not required. If you get significant help from a reference or a person, give explicit credit.

1. The octahedron is the Platonic solid with 8 triangular faces, made from two four-sided pyramids stuck together along their square bases. Let G be its edge-graph, with vertices $V = \{v_1, \ldots, v_6\}$, with 4 edges each ($d(v) = 4$), and 12 edges total.

a. Show that G could not contain any subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$: thus, Kuratowski's Theorem guarantees that G is a planar graph. (Recall that in a subdivision of a graph, each edge may be replaced by a path.) Sketch a planar drawing of G (with no edge crossings).

b. Determine the chromatic number $\chi(G) = \ell$, the minimum number of colors needed in a proper coloring ($c : V \to [\ell]$ with $c(x) \neq c(y)$ for $xy \in E$). Show that $\chi(G) \leqslant \ell$ by exhibting a proper ℓ-coloring; and that $\chi(G) \geqslant \ell$ by finding a subgraph $H \subset G$ (near Nevada) which clearly requires ℓ colors. (The Four Color Theorem guarantees $\chi(G) \leqslant 4$ for any map, i.e. any planar graph.)

c. Determine the connectivity $\kappa(G) = k$ by exhibiting a k-element cutset $S \subset V$ (with $G - S$ disconnected), and show that there is no smaller cutset.

d. Find a 6-element group Γ with generators $a, b, c \in \Gamma$, such that G is the Cayley graph (ignoring orientation and labeling of edges). Recall that the Cayley graph has vertices $V = \Gamma$, and edges of the form $g \to ga$, $g \to gb$, $g \to gc$.

e. Find the eigenvalues and eigenvectors of the adjacency matrix $A = (a_{ij})_{i,j=1}^{6}$, where $a_{ij} = 1$ when $v_i v_j \in E$, and $a_{ij} = 0$ otherwise. (You may use computer help.) Verify in this case the general theorem that $d \geqslant \kappa(G) \geqslant d - \lambda_2$, where G is a d-regular graph (all $d(v) = d$) with n vertices, and $d = \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ are the eigenvlaues of A.

2. For arbitrary $k < n$, does there always exist some connected k-regular graph G with n vertices? If not, classify the (k, n) for which there do exist such graphs. Hint: Experiment with small examples of (k,n). To show your condition on (k, n) is necessary, use a simple result we proved in class; to show it is sufficient, construct a graph for each allowed value (k, n).

3. (Bollobás Ex I.1) Prove that either a graph G or its complement $\bar{G}$ is connected, possibly both. (By definition, $\bar{G}$ has the same vertices as G, and $xy \in E(\bar{G})$ whenever $xy \notin E(G)$.)

4. (Bollobás Ex I.5) Show the following conditions are equivalent for a graph G with at least 3 vertices:

(i) $\kappa(G) \geqslant 2$, meaning G is connected with no cutvertex

(ii) any two vertices lie on a cycle

(iii) any two edges lie on a cycle, and there are no isolated vertices ($d(v) > 0$)

(iv) for any vertices x, y, z, there is a path from x to y to z.