Math 881 **Homework 1** **Due Wed Jan 17**

1. The octahedron is the Platonic solid with 8 triangular faces, made from two four-sided pyramids stuck together along their square bases. Let G be its edge-graph, with vertices $V = \{v_1, \ldots, v_6\}$, all with 4 edges ($d(v) = 4$), and 12 edges.

a. Show that G could not contain any subdivision of the complete graph K_5 or the complete bipartite graph $K_{3,3}$: thus, Kuratowski's Theorem guarantees that G is a planar graph. (Recall that in a subdivision of a graph, each edge may be subdivided into a path.) Sketch a planar drawing of G (with no edge crossings).

Solution: (Emmet Harrington) Clearly G has no K_5 subgraph. The 6-vertex subdivision of K_5 has 11 edges and one vertex of degree 2, but deleting 1 of the 12 edges of G gives mimimum degree 3, so we can't produce the subdivision as a subgraph. Similarly, $K_{3,3}$ has 9 edges, but removing any 3 of the 12 edges of G leaves at least $8 - 2(3) = 2$ triangle K_3's in G, so we can't produce the bipartite $K_{3,3}$ as a subgraph.

c. Determine the connectivity $\kappa(G) = k$ by exhibiting a k-element cutset $S \subset V$ (with $G{-}S$ disconnected), and show that there is no smaller cutset.

Solution: Removing any 3 vertices leaves 3 vertices forming a path or a triangle, both connected. Removing two non-adjacent vertices leaves a 4-cycle, and removing two adjacent vertices leaves a 4-cycle with a chord, both connected. Removing a single vertex leaves a connected 5-vertex wheel graph. Thus $\kappa(G) > 3$, and we can remove a 4-cycle along the equator of the octahedron to disconnect the top and bottom vertices.

d. Find a 6-element group Γ with generators $a, b, c \in \Gamma$, such that G is the Cayley graph (ignoring orientation and labeling of edges). Recall that the Cayley graph has vertices $V = \Gamma$, and edges of the form $g \to ga$, $g \to gb$, $g \to gc$.

Solution 1: $\Gamma = S_3$ with generators $a = (123)$, $b = (12)$, $c = (13)$, obeying $a^3 = b^2 = c^2 = abc = 1$. The 3 generators produce a regular directed graph with in-degree and out-degree 3, but the two generators of order 2 each produce a single undirected edge at each vertex, so ignoring orientation gives the 4-regular graph G.

Solution 2: (Mohit Bansil) Take $G = C_6 = \mathbb{Z}/6\mathbb{Z}$ with generators $a = 1$, $b = 2$, making a 4-regular graph with 6 vertices (see #2 below). This can only be the octahedron graph, though it is not obvious the octahedron has a 6-cycle.

2. For arbitrary $k < n$, does there always exist some connected k-regular graph G with n vertices? If not, classify the (k, n) for which there do exist such graphs. Hint: Experiment with small examples of (k,n). To show your condition on (k, n) is necessary, use a simple result we proved in class; to show it is sufficient, construct a graph for each allowed value (k, n).

Solution: We know $kn = \sum d(v) = 2m$ must be even. To show existence, take the Cayley graph of $\Gamma = C_n$ with generators $\{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$, and in case of odd k, also the generator $\frac{n}{2}$.

3. (Bollobás Ex I.1) Prove that either a graph G or its complement $\bar{G}$ is connected, possibly both. (By definition, $\bar{G}$ has the same vertices as G, and $xy \in E(\bar{G})$ whenever $xy \notin E(G)$.)

Solution: If G is not connected, consider any vertices x, y. If x, y are in different connected components of G, then they are neighbors in $\bar{G}$. Otherwise, take v in another connected component of (disconneced) G, so that v is not a neighbor of x or y in G; then xvy is a path in $\bar{G}$. Either way, x, y are connected in $\bar{G}$, and $\bar{G}$ is connected.

4. (Bollobás Ex I.5) Show the following conditions are equivalent for a graph G with at least 3 vertices:

(i) $\kappa(G) \geqslant 2$, meaning G is connected with no cutvertex
(ii) any two vertices lie on a cycle
(iii) any two edges lie on a cycle, and there are no isolated vertices ($d(v) > 0$)
(iv) for any vertices x, y, z, there is a path from x to y to z.

Solution: We will prove (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iii) $\Leftarrow$ (iv) $\Leftarrow$ (i).

(i) $\Rightarrow$ (ii). (Jihye Hwang) Let G be a graph with no cut-vertex, and fix a vertex x. We will prove that for any vertex y, there is a cycle containing x, y, by induction on $\mathrm{dist}(x, y)$.

Suppose $\mathrm{dist}(x, y) = 1$. Since y is not a cut-vertex, x has another neighbor z. Since x is not a cut-vertex, there is some path zPy not containing x, and $xzPyx$ is a cycle containing x, y.

Suppose $\mathrm{dist}(x, y) > 1$. Letting $x \cdots py$ be a minimal path, by induction there is a cycle $xPpP'x$. If $y \in PP'$, we have our desired cycle, so suppose $y \notin PP'$. Since p is not a cut-vertex, there is a path xQy not containing p. Now, let q be the vertex of $Q \cap PP'$ which is closest to y along Q (possibly $q = x$), so that $xQy = xQ_1qQ_2y$. Since P, P' have symmetric roles, we may assume that $q \in P$, and $xPz = xP_1qP_2z$.

Now we have the cycle $xP_1qQ_2ypP'x$. Indeed, $xP_1q \cap ypP'x = \{x\}$ since $xPpP'x$ is a cycle and $y \notin PP'$; and $qQ_2y \cap ypP'x = \{y\}$ by the definition of $q \in Q$. Again x, y lie on a cycle, so the induction proceeds, and ultimately every x, y lie on a cycle.

(iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are easy.

(iv) $\Rightarrow$ (iii). Suppose G has a path between any three vertices, and let xy, pq be disjoint edges of G. Take paths $xPpP'y$ and $yQqQ'p$, and let r be the vertex of $Q \cap PP'$ which is closest to y along Q. Since P, P' have symmetric roles, we may assume that $y \in Q$, and $yQq = yQ_1rQ_2q$.

Now we have the cycle $xPrQ_2qpP'yx$. Indeed, $xPr \cap pP'yx = \{x\}$ since $xPpP'x$ is a path; and $rQ_2qp \cap pP'yx = \{p\}$ by the definition of $r \in Q$, and since $p \notin Q$. Thus the edges xy, pq lie on a cycle.

Also, for a pair of incident edges xy, yp, we can take a path $pPxP'y$, so that $xypPx$ is again a cycle containing these edges.

(i) $\Rightarrow$ (iv). Take vertices $x, y, z \in G$, a graph with no cut-vertex. Produce a larger graph G' by adding a vertex v with edges vx, vz. Now, removing a vertex of G' cannot separate v from both x and z, nor x and z from the other vertices of G, so G' has no cutvertex. From our proof of (i) $\Rightarrow$ (ii), we know v, y lie on a cycle $vxPyP'zv$ in G', which includes the path $xPyP'z$ in G.